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Bifurcations in the Generalized Korteweg–de Vries Equation

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Abstract—We study the generalized Korteweg–de Vries (KdV) equation and the Korteweg–de Vries–Burgers (KdVB) equation with periodic in the spatial variable boundary conditions. For various values of parameters, in a sufficiently small neighborhood of the zero equilibrium state we construct asymptotics of periodic solutions and invariant tori. Separately we consider the case when the stability spectrum of the zero solution contains a countable number of roots of the characteristic equation. In this case we state a special nonlinear boundary-value problem which plays the role of a normal form and determines the dynamics of the initial problem.

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INTRODUCTION

Consider the equation

$$u_t + u_{xxx} + \Phi(u)u_x = \gamma u_{xx} + F(u) \ \ (\gamma > 0) \tag{1}$$

with periodic boundary value conditions

$$u(t, x + T) = u(t, x). \tag{2}$$

Assume that

$$\Phi(u) = \delta + \alpha u + \beta u^2 + O(|u|^3), \ F(u) = au + bu^2 + cu^3 + O(|u|^4).$$

Let us study the behavior of all solutions to the boundary-value problem (1), (2) with sufficiently small (in the norm of W_3^2) periodic in the spatial variable initial conditions. An important role is played here by the location of roots of the characteristic (wave) equation

$$\lambda_k = ik^3 + a. \tag{3}$$

In view of formula (2) the variable k in (3) takes on only a countable number of values $k = 2\pi n/T$ ($n \in \mathbb{Z}$). With $a < 0$ it holds that $\operatorname{Re}\lambda_k = a < 0$, therefore all solutions to (1), (2), that are located in some sufficiently small neighborhood of the zero equilibrium state, tend to zero as $t \longrightarrow \infty$. But if $a > 0$, then $\operatorname{Re}\lambda_k > 0$ and the problem is nonlocal. Here we assume that parameters a and γ are sufficiently small, i.e.,

$$a = \varepsilon a_0, \ \ \gamma = \varepsilon\gamma_0, \ \ 0 < \varepsilon \ll 1. \tag{4}$$

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Note also that we can exclude the term δu_x in (1), performing the replacement $x \longrightarrow x - \delta t$, therefore in what follows we put $\delta = 0$. With the zero right-hand side in (1) and with $\Phi(u) = \alpha u$ we get the classic Korteweg–de Vries equation [1], and with $\Phi(u) = \alpha u + \beta u^2$ we do the so-called [2] modified Korteweg–de Vries equation, and if, in addition, $\gamma \neq 0$, then we get the Korteweg–de Vries–Burgers equation [2–4].

Let us first put $\gamma = 0$ and study the question of the construction of periodic solutions and invariant tori of various dimensions in some sufficiently small neighborhood of the zero equilibrium state. Condition (4) of smallness of a allows us to use methods of bifurcation analysis [5–7]. With $\varepsilon \to 0$ infinitely many roots of the characteristic equation (3) tend to the imaginary axis, and in the problem of the stability of the zero solution to (1), (2) we get the critical case of infinite dimension. Recall that one can study the local dynamics of equations in a finite-dimensional critical case by well-known methods of local invariant integral manifolds (e.g., [8–10] and methods of normal forms [11, 12]. These methods allow one to study the equation by reducing it to a specific finite-dimensional nonlinear system of ordinary differential equations (ODE). Though it is impossible to reduce the boundary-value problem (1), (2) (subject to (4)) to a finite-dimensional system of ODE, in what follows we prove the existence (in a small neighborhood of zero) of periodic solutions and invariant tori of various dimensions by applying the formalism of the method of normal forms.

In Sections 1 and 2 we consider the simplest case, when $b = 0$ and the function u satisfies the condition

$$M(u) = 0, \tag{5}$$

where

$$M(u) = \frac{1}{T}\int_0^T u(t,x)dx. \tag{6}$$

Note that in Section 1 we consider periodic solutions, and in Section 2 we construct an asymptotic for tori. In Section 3 we disclaim the condition that the mean value of the function u should equal zero, and in Section 4 we consider a general case when $b \neq 0$. In Section 5 we consider the Korteweg–de Vries–Burgers equation, namely, the case when $\gamma \neq 0$ and the function u has the zero mean value, i.e., correlation (5) is valid.

1. CONSTRUCTION OF ASYMPTOTICALLY PERIODIC SOLUTIONS TO THE BOUNDARY-VALUE PROBLEM (1), (2) UNDER CONDITIONS (4) AND $\gamma = b = 0$

Fix arbitrary natural $n_0 \neq 0$ and put $k_0 = 2\pi n_0/T$. Then the linear boundary-value problem

$$u_t + u_{xxx} = 0,\ u(t, x+T) = u(t,x) \tag{7}$$

has the following periodic solution:

$$u = \xi \exp(ik_0x + ik_0^3t) + \overline{\xi}\exp(-ik_0x - ik_0^3t),$$

where the symbol ξ stands for an arbitrary complex-valued constant.

Consider the simplest case. Assume, in addition, that solutions to (1), (2) satisfy condition (5).

We seek for periodic solutions to the nonlinear boundary value problem (1), (2), (5) in the form of the formal series

$$u = \sqrt{\varepsilon}\big(\xi(\tau)\exp(ik_0x + ik_0^3t) + \overline{\xi}(\tau)\exp(-ik_0x - ik_0^3t)\big) + \varepsilon u_2(t,x,\tau) + \varepsilon^{3/2}u_3(t,x,\tau) + \cdots, \tag{8}$$

where $\tau = \varepsilon t$ is a slow time, while functions $u_j(t,x,\tau)$ are periodic in the first two arguments. Substitute (8) in (1) and equate coefficients at the same degrees of ε. Then on the second step for the function $u_2(t,x,\tau)$ we get the equality

$$u_2(t,x,\tau) = u_{20}(\tau)\exp(ik_0x + ik_0^3t) + c\overline{c},$$

where $u_{20}(\tau) = \xi^2(\tau)\alpha/6k_0^2$. Here and below, we denote by $c\overline{c}$ the term which is complex conjugate to the previous one.

Furthermore, collecting coefficients at $\varepsilon^{3/2}$, we get an equation with respect to u_3. Since by condition it is solvable in the mentioned function class, we get the following equation with respect to the unknown amplitude $\xi = \xi(\tau)$:

$$\frac{d\xi}{d\tau} = a_0\xi + \Delta\xi|\xi|^2, \tag{9}$$

where $\Delta = 3c - i\beta k_0 - i\alpha^2/6k_0$. Note that $\operatorname{Re}\Delta = 3c$.

Equation (9) describes the behavior of solutions to the boundary-value problem (1), (2), (5) in a sufficiently small neighborhood of the solution $u \equiv 0$. For example, under the condition

$$a_0 \operatorname{Re}\Delta < 0 \tag{10}$$

Eq. (9) has the periodic solution $\xi_*(\tau) = \rho_* \exp(i\varphi_*\tau)$, where $\rho_* = \sqrt{-a/\operatorname{Re}\Delta}$; $\varphi_* = \operatorname{Im}\Delta$. This solution for (9) is stable with $a_0 > 0$ and nonstable with $a_0 < 0$.

One can prove the following assertion by standard considerations.

Theorem 1. *Under conditions* $b = 0$ *and* (10) *Eq.* (1) *has the following asymptotically orbitally stable periodic solution*:

$$u_*(t, x, \varepsilon) = \sqrt{\varepsilon}\rho_* \exp\Big(ik_0x + i\big(k_0^3 + \varepsilon\varphi_* + o(\varepsilon)\big)t\Big) + c\overline{c} + O(\varepsilon^{3/2}). \tag{11}$$

One can prove this assertion in two ways. The first one is based on the following scheme. Firstly, one linearizes the initial boundary-value problem (1), (2) subject to (4), (5) on the approximate periodic solution (11); then one performs certain smooth periodic replacements in the obtained linear boundary value problem so as to fulfill the well-known averaging principle. The second approach is connected with the reduction of the problem under consideration to a third-order ODE and the use of well-known Andronov–Hopf bifurcation method under condition (5).

2. TORI IN EQUATION (1) UNDER CONDITIONS (4) AND $\gamma = b = 0$

Let us generalize the above formalism for constructing periodic solutions in a more difficult case.

2.1. *Construction of the asymptotic of two-dimensional tori.* Fix arbitrarily two natural numbers n_1 and n_2 ($n_1 \neq n_2$) and put $k_1 = 2\pi n_1/T$, $k_2 = 2\pi n_2/T$. Introduce the formal series

$$u(t, x, \varepsilon) = \sqrt{\varepsilon}\Big(\sum_{j=1}^{2} \xi_j(\tau)\exp(ik_jx + ik_j^3t) + \sum_{j=1}^{2} \overline{\xi}_j(\tau)\exp(-ik_jx - ik_j^3t)\Big) + \varepsilon u_2(t, x, \tau) + \varepsilon^{3/2}u_3(t, x, \tau) + \cdots . \tag{12}$$

Here $\tau = \varepsilon t$, $\xi_{1,2}(\tau)$ are (unknown) complex-valued "amplitudes", functions $u_j(t, x, \tau)$ $(j = 2, 3, \dots)$ are trigonometric polynomials in first two arguments, and

$$M(u_j) = 0, \;\; j = 2, 3, \dots, \tag{13}$$

where

$$M(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t, x, \tau)dx.$$

Substitute (12) in (1). By standard considerations we first find

$$u_2(t, x, \tau) = \sum_{j=1}^{2} \big(u_{2j}\xi_j^2 \exp(2ik_jx + 2ik_j^3t) + c\overline{c}\big) + u_{23}\xi_1\xi_2 \exp\big(i(k_1 + k_2)x + i(k_1^3 + k_2^3)t\big) + c\overline{c} + u_{24}\xi_1\overline{\xi}_2 \exp\big(i(k_1 - k_2)x + i(k_1^3 - k_2^3)t\big) + c\overline{c}. \tag{14}$$

For u_{2j} $(j = 1, \dots, 4)$ we get

$$u_{2j} = \alpha/(6k_j^2),\ \ j = 1, 2;\ \ u_{23} = u_{24} = \alpha/(3k_1k_2).$$

Then, collecting coefficients at $\varepsilon^{3/2}$, from the condition of the solvability (in the indicated function class) of the obtained equation with respect to u_3 we get the following equations with respect to $\xi_{1,2}(\tau)$:

$$\begin{aligned} d\xi_1/d\tau &= a_0\xi_1 + \xi_1\bigl(\Delta_1|\xi_1|^2 + |\Delta_2|\xi_2|^2\bigr), \\ d\xi_2/d\tau &= a_0\xi_2 + \xi_2\bigl(\Delta_3|\xi_1|^2 + \Delta_4|\xi_2|^2\bigr). \end{aligned} \tag{15}$$

For Δ_j $(j = 1, \dots, 4)$ we get equalities

$$\Delta_1 = 3c - (\alpha k_1 u_{21} + \beta k_1)i,\ \ \Delta_2 = 6c - 2(\alpha k_1 u_{23} + \beta k_1)i,$$

$$\Delta_3 = 6c - 2(\alpha k_2 u_{23} + \beta k_2)i,\ \ \Delta_4 = 3c - (\alpha k_2 u_{22} + \beta k_2)i.$$

The answer to the question of the existence of the simplest two-dimensional invariant torus in Eq. (1) is connected with the question of the existence of a solution to system (15) in the form $\xi_j = \rho_j \exp(i\varphi_j\tau)$ $(\tau > 0)$. Then from (15) we get the following system of two real equations with respect to nonzero ρ_1 and ρ_2:

$$a_0 + 3c\rho_1^2 + 6c\rho_2^2 = 0,\ \ a_0 + 6c\rho_1^2 + 3c\rho_1^2 = 0. \tag{16}$$

Hence we find the solution

$$\rho_{1*} = \rho_{2*} = \frac{1}{3}\sqrt{-\frac{a_0}{c}}. \tag{17}$$

Then for φ_j we get

$$\varphi_{1*} = \rho_{1*}^2(\operatorname{Im}\Delta_1 + \operatorname{Im}\Delta_2),\ \ \varphi_{2*} = \rho_{2*}^2(\operatorname{Im}\Delta_3 + \operatorname{Im}\Delta_4),$$

whence

$$\varphi_{1*} = \frac{a_0k_1}{9c}(\alpha u_{21} + 2\alpha u_{23} + \beta),\ \ \varphi_{2*} = \frac{a_0k_2}{9c}(\alpha u_{22} + 2\alpha u_{23} + \beta). \tag{18}$$

Theorem 2. *Let ρ_{1*} and ρ_{2*} be real-valued simple roots of system* (16). *Then with all sufficiently small ε, Eq.* (1) *has a two-dimensional invariant torus, namely,*

$$\begin{aligned} u_0(t, x, \tau) = \sqrt{\varepsilon}\rho_{1*} \exp\Bigl(ik_1x + i\bigl(k_1^3 + \varepsilon\varphi_{1*} + o(\varepsilon)\bigr)t\Bigr) + c\overline{c} \\ + \sqrt{\varepsilon}\rho_{2*} \exp\Bigl(ik_2x + i\bigl(k_2^3 + \varepsilon\varphi_{2*} + o(\varepsilon)\bigr)t\Bigr) + c\overline{c} + \varepsilon u_2 + O(\varepsilon^{3/2}), \end{aligned}$$

where ρ_{j*}, φ_{j*} $(j = 1, 2)$ obey equalities (17) *and* (18)*, while the function u_2 does* (14).

Certainly, one can study the expansion of tori into asymptotic series with respect to degrees of $\sqrt{\varepsilon}$ with any order of accuracy.

2.2. *Construction of families of multidimensional tori.* Fix arbitrarily a natural number N $(N > 2)$ and a collection of various natural numbers $n_1, \dots, n_N$ and put $k_1 = 2\pi n_1/T, \dots, k_N = 2\pi n_N/T$. Introduce the following formal series:

$$u(t, x, \varepsilon) = \sqrt{\varepsilon}\sum_{j=1}^{N}\bigl(\xi_i(\tau)\exp(ik_jx + ik_j^3t) + c\overline{c}\bigr) + \varepsilon u_2(t, x, \tau) + \varepsilon^{3/2}u_3(t, x, \tau) + \cdots, \tag{19}$$

where $\tau = \varepsilon t$, while functions $u_j(t, x, \tau)$ are trigonometric polynomials in first two arguments, and in view of (13) conditions $M\bigl(u_j(t, x, \tau)\bigr) = 0$ are fulfilled. Substituting (19) in (1), applying standard techniques, we first find

$$u_2(t,x,\tau) = \sum_{j,s=1}^{N} \Big(u_{1js}\xi_j\xi_s \exp\big(i(k_j+k_s)x + i(k_j^3+k_s^3)t\big) + c\overline{c}\Big)$$
$$+ \sum_{j,s=1,\ j\neq s}^{N} \Big(u_{2js}\xi_j\overline{\xi}_s \exp\big(i(k_j-k_s)x + i(k_j^3-k_s^3)t\big) + c\overline{c}\Big). \quad (20)$$

Then on the next step, we get the following system of equations with respect to $\xi_j(\tau)$:

$$\frac{d\xi_i}{d\tau} = a_0\xi_j + \xi_j\sum_{s=1}^{N}\Delta_{js}|\xi_s|^2,\ \ j=1,\dots,N. \quad (21)$$

We omit explicit formulas for u_{1jm}, u_{2jm}, and Δ_{jm}, because they are rather cumbersome. One can prove the next theorem analogously to Theorem 2.

Theorem 3. *Let system* (21) *have a solution in the form* $\xi_{j*} = \rho_{j*}\exp(i\varepsilon\varphi_{j*}t)$ $(j=1,\dots,N)$ *and* $\rho_{j*} > 0$. *Then with all sufficiently small* ε, *Eq.* (1) *has an* N*-dimensional invariant torus with the asymptotic*

$$u(t,x,\varepsilon) = \sqrt{\varepsilon}\sum_{j=1}^{N}\big(\xi_{j*}\exp(ik_jx + ik_j^3t) + c\overline{c}\big) + \varepsilon u_2 + O(\varepsilon^{3/2})$$

with indicated ξ_{j*}, *and the function* u_2 *obeys equality* (20).

Note that the question of the existence of solutions to system (21) in the form $\xi_j = \rho_j\exp(i\varphi_j\tau)$ is equivalent to the question of the solvability in the class of positive solutions of the system of equations

$$\rho_j(a_0 + \sum_{s=1}^{N}\operatorname{Re}\Delta_{js}\rho_s^2) = 0,\ \ j=1,\dots,N.$$

It is important to note that system (21) has a relatively simple form due to the absence of basic resonant correlations, i.e., two systems of equations

(1) $k_{j1}+k_{j2}=k_{j3}$, $k_{j1}^3+k_{j2}^3=k_{j3}^3$;

(2) $k_{j1}+k_{j2}+k_{j3}=k_{j4}$, $k_{j1}^3+k_{j2}^3+k_{j3}^3=k_{j4}^3$

are not solvable in the class of nontrivial solutions (with $k_j\neq k_s$ for $j\neq s$).

3. CONSTRUCTION OF PERIODIC SOLUTIONS AND INVARIANT TORI OF EQUATION (1) UNDER THE CONDITION $\gamma = b = 0$ AND WITHOUT CONDITION (13)

In Item 3.1 we first give some general well-known results, and then in Items 3.2, 3.3 and in Section 4 we apply these results for studying the boundary value problem (1), (2).

3.1. *The normal form for the critical case of one zero and two purely imaginary roots of the characteristic equation.* Let us recall some well-known results on the local dynamics (in a small neighborhood of the zero equilibrium state) of the following system of nonlinear equations:

$$\dot{w} = (A+\varepsilon B)w + \Phi(w). \quad (22)$$

Here $w\in R^n$, A and B are square $n\times n$-matrices, $0<\varepsilon\ll 1$ is a small parameter, $\Phi(w)$ is a sufficiently smooth nonlinear vector function whose order of smallness at zero is greater than one. Assume that the matrix A has the zero eigenvalue ($Ag_0=0$, $g_0\neq 0$) and two purely imaginary ones ($Ag_1 = i\omega g_1$, $\omega>0$). All the rest eigenvalues of this matrix have negative real parts. Then [12] with all sufficiently small ε, system (22) has a 3-dimensional stable local invariant integral manifold such that the behavior of solutions on it defines the behavior of all solutions to system (22) near the zero solution. One can write system (22) on this manifold (accurate to terms of a higher order of smallness) as a specific system of three ordinary differential equations, i.e., in the normal form [12]. It is convenient to write it for a real variable η and a complex one ξ, namely,

$$\dot{\eta} = \alpha_1\varepsilon\eta + \alpha_2\eta^2 + \alpha_3|\xi|^2 + \alpha_4\eta^3 + \alpha_5\eta|\xi|^2, \;\; \dot{\xi} = \beta_1\varepsilon\xi + \beta_2\eta\xi + \xi(\beta_3|\xi|^2 + \beta_4\eta^2). \tag{23}$$

Solutions to systems (22) and (23) are connected by the correlation

$$w = \eta(t,\varepsilon)g_0 + \xi(t,\varepsilon)g_1\exp(i\omega t) + \overline{cc} + \cdots, \tag{24}$$

where the symbol ... denotes a formal series in degrees of ε, η, and ξ, starting with quadratic terms, with the $(2\pi/\omega)$-periodic dependence on t. One can find all coefficients α_i and β_i by substituting (24) in (22) and by equating coefficients at the same degrees of ε, η, and ξ.

3.2. *Construction of periodic solutions to Eq.* (1) *subject to* $\gamma = b = 0$ *without condition* (13). The absence of boundary conditions (13) makes the structure of periodic solutions to (1) much more complex than that considered in Section 1. Fix, as above, an arbitrary natural number $n_0 \neq 0$ and put $k_0 = 2\pi n_0/T$. Note that the characteristic equation of the boundary-value problem (7) has the zero root and two purely imaginary ones $\pm ik_0^3$. In particular, Eq. (7) has the following $(2\pi/k_0)$-periodic solution:

$$u = \eta + \xi\exp(ik_0x + ik_0^3t) + \overline{cc}.$$

Formally we do not take into account other periodic solutions to Eq. (7).

Consider the formal series

$$u = \eta(t,\varepsilon) + \xi(t,\varepsilon)\exp(ik_0x + ik_0^3t) + \overline{cc} + u_2(\eta,\xi,x,t) + u_3(\eta,\xi,x,t) + \cdots, \tag{25}$$

where u_2 contains quadratic (with respect to ξ,η) terms, u_3 does cubic ones, etc., and the dependence of the function $u_j(\eta,\xi,x,t)$ on x and t is periodic.

Let us mention the main properties of functions $\eta = \eta(t,\varepsilon)$ and $\xi = \xi(t,\varepsilon)$. First, η and ξ are sufficiently small, therefore, one can consider series with respect to their degrees. Second, in accordance with (23), their derivatives in t are also small and one can express them in terms of small values η, ξ, and ε.

Let us replace u in formula (1) with expression (25). In the resulted formal identity we replace $\dot{\eta}$ and $\dot{\xi}$ in accordance with system (23) with unknown coefficients α_j and β_j. Then we collect coefficients at the same degrees of ε, η, and ξ and, as a result, obtain

$$\begin{gathered}\alpha_1 = \beta_1 = a_0, \;\; \alpha_2 = \alpha_3 = 0, \;\; \alpha_4 = c, \;\; \alpha_5 = 6c, \\ \beta_2 = -i\alpha k_0, \;\; \beta_3 = 3c - i\big(\alpha^2/(6k_0) + \beta k_0\big), \;\; \beta_4 = 3c - i\beta k_0.\end{gathered} \tag{26}$$

Substituting $\xi = \rho\exp(i\varphi)$ in (23) for determining periodic solutions and taking into account the calculated values of coefficients (26), we get the following amplitude system split off from (23):

$$\dot{\eta} = \eta(a_0\varepsilon + c\eta^2 + 6c\rho^2), \;\; \dot{\rho} = \rho(a_0\varepsilon + 3c\rho^2 + 3c\eta^2). \tag{27}$$

Here φ is connected with ρ and η by the formula

$$\dot{\varphi} = -\alpha k_0\eta + \big(\alpha^2/(6k_0) + \beta k_0\big)\rho^2 - \beta k_0\eta^2.$$

We are interested in equilibrium states of system (27). Therefore, we can refine the dependence of desired ρ and η on ε with the help of the system

$$\eta(a_0\varepsilon + c\eta^2 + 6c\rho^2) = 0, \;\; \rho(a_0\varepsilon + 3c\rho^2 + 3c\eta^2) = 0. \tag{28}$$

Hence we get the following four variants of solutions:

$$\begin{aligned} &\eta_*^{(1)} = 0, && \rho_*^{(1)} = \sqrt{\varepsilon}\sqrt{-\frac{a_0}{3c}}; \\ &\eta_\pm^{(2)} = \pm\sqrt{\varepsilon}\sqrt{-\frac{a_0}{5c}}, && \rho_*^{(2)} = \sqrt{\varepsilon}\sqrt{-\frac{2a_0}{15c}}; \\ &\eta_*^{(3)} = 0, && \rho_*^{(3)} = 0; \end{aligned}$$

$$\eta_{\pm}^{(4)} = \pm\sqrt{\varepsilon}\sqrt{-\frac{a_0}{c}}, \quad \rho_*^{(4)} = 0.$$

Note that with $a_0 > 0$ and $c < 0$ equilibrium states $(\eta_*^{(1)}, \rho_*^{(1)})$ and $(\eta_*^{(4)}, \rho_*^{(4)})$ are stable, while $(\eta_*^{(2)}, \rho_*^{(2)})$ and $(\eta_*^{(3)}, \rho_*^{(3)})$ are nonstable.

Therefore, with $a_0 > 0$ and $c < 0$ we get the following three distinct periodic solutions to system (23):

1) $\eta_*(t,\varepsilon) \equiv 0$, $\xi_*(t,\varepsilon) = \rho_*^{(1)} \exp(i\varphi_* t)$, where $\varphi_* = \varepsilon\big(\frac{\alpha^2}{6k_0} + \beta k_0\big)\frac{a_0}{3c}$;

2), 3) $\eta_{\pm}(t,\varepsilon) = \eta_{\pm}^{(2)}$, $\xi_{\pm}(t,\varepsilon) = \rho_*^{(2)} \exp(i\varphi_{\pm} t)$, where $\varphi_{\pm} = \mp\sqrt{\varepsilon}\alpha k_0\sqrt{-\frac{a_0}{5c}} + O(\varepsilon)$.

Then for Eq. (1) we get the asymptotic of two more periodic solutions $u_{\pm}(t,x,\varepsilon)$ different from solutions to (11), namely,

$$u_{\pm}(t,x,\varepsilon) = \eta_{\pm}(t,\varepsilon) + \xi_{\pm}(t,\varepsilon)\exp(ik_0x + ik_0^3 t) + c\overline{c} + O(\varepsilon).$$

Therefore, the rejection of boundary conditions (13) can lead to the appearance of two new periodic solutions.

3.3. *Tori for Eq.* (1) *under the condition* $\gamma = b = 0$ *and without condition* (13). We study the construction of nondegenerate tori for Eq. (1) as the formal series

$$u = \eta(t,\varepsilon) + \sum_{j=1}^{N}\big(\xi_j(t,\varepsilon)\exp(ik_jx + ik_j^3 t) + c\overline{c}\big) + \cdots, \tag{29}$$

where $k_j \neq 0$ and $k_j \neq k_s$ with $j \neq s$, the symbol $\ldots$ stands for quadratic, cubic, etc. terms with respect to ε, η, and ξ_j that are trigonometric polynomials in t and x. The corresponding normal form (analogous to (23)) is

$$\begin{gathered}\dot{\eta} = \alpha_1\varepsilon\eta + \alpha_2\eta^2 + \sum_{s=1}^{N}\alpha_{3s}|\xi_s|^2 + \alpha_4\eta^3 + \eta\sum_{s=1}^{N}\alpha_{5s}|\xi_s|^2,\\ \dot{\xi_j} = \beta_{1j}\varepsilon\xi_j + \beta_{2j}\eta\xi_j + \xi_j\bigg(\sum_{s=1}^{N}\beta_{3js}|\xi_s|^2 + \beta_{4j}\eta^2\bigg), \ \ j = 1,\ldots,N.\end{gathered} \tag{30}$$

Substituting (29) in (1) and performing some standard calculations, we sequentially find all coefficients of system (30). As a result, we get equalities ($j = 1,\ldots,N$)

$$\alpha_1 = \beta_{1j} = a_0, \ \ \alpha_2 = \alpha_{3j} = 0, \ \ \alpha_4 = c, \ \ \alpha_{5j} = 6c, \ \ \beta_{2j} = -i\alpha k_j,$$

$$\beta_{3js} = 6c - 2i\beta k_j \ \text{ with } \ s \neq j, \ \ \beta_{3jj} = 3c - i\big(\alpha^2/(6k_j) + \beta k_j\big), \ \ \beta_{4j} = 3c - i\beta k_j.$$

Thus, the question of the existence of solutions to system (30) in the form $\xi_j = \rho_j\exp(i\varphi_j t)$ is equivalent to the question of the existence and solvability of the system of equations

$$\begin{gathered}\eta\Big(a_0\varepsilon + 6c\sum_{s=1}^{N}\rho_s^2 + c\eta^2\Big) = 0,\\ \rho_j\Big(a_0\varepsilon + 6c\sum_{s=1}^{N}\rho_s^2 - 3c\rho_j^2 + 3c\eta^2\Big) = 0, \ \ j = 1,\ldots,N.\end{gathered} \tag{31}$$

Hence, assuming that $\rho_j > 0$, for system (30) with $a_0 > 0$ and $c < 0$ we find three periodic solutions:

1) $\eta_*(t,\varepsilon) \equiv 0$, $\xi_{j*}(t,\varepsilon) = \rho_{j*}\exp(i\varphi_{j*}t)$, where

$$\rho_{j*} = \sqrt{\varepsilon}\sqrt{-\frac{a_0}{3(2N-1)c}}, \ \ \varphi_{j*} = \varepsilon\bigg(\frac{\alpha^2}{6k_j} + (2N-1)\beta k_j\bigg)\frac{a_0}{3(2N-1)c};$$

2), 3) $\eta_\pm(t,\varepsilon)=\pm\sqrt{-\frac{a_0}{(4N+1)c}}$, $\xi_{j\pm}(t,\varepsilon)=\rho_{j*}\sqrt{\varepsilon}\exp(i\varphi_{j\pm}t)$, where

$$\rho_{j*}=\sqrt{\varepsilon}\sqrt{-\frac{2a_0}{3(4N+1)c}},\;\; \varphi_{j\pm}=\mp\sqrt{\varepsilon}\alpha k_j\sqrt{-\frac{a_0}{(4N+1)c}}+O(\varepsilon).$$

In particular, with $N=2$ we get

$$\begin{aligned}\eta\big(a_0\varepsilon+6c(\rho_1^2+\rho_2^2)+c\eta^2\big)&=0,\\ \rho_1(a_0\varepsilon+3c\rho_1^2+6c\rho_2^2+3c\eta^2)&=0,\\ \rho_2(a_0\varepsilon+6c\rho_1^2+3c\rho_2^2+3c\eta^2)&=0.\end{aligned}\tag{32}$$

With $\eta=0$ we get $\rho_j=\rho_{j0}\sqrt{\varepsilon}$, where ρ_{j0} coincides with the solution to system (16). With $\eta\neq 0$ we get $\eta=\pm\sqrt{\varepsilon}\sqrt{-\frac{a_0}{9c}}$, $\rho_1=\rho_2=\sqrt{\varepsilon}\sqrt{-\frac{2a_0}{27c}}$. The number of solutions of (32) exceeds that of system (16), which means that the number of two-dimensional tori in (1) can be much greater.

We understand a rough solution (satisfying the non-degeneracy condition) as a simple solution of the system for finding amplitudes.

Theorem 4. *Let $b=0$ and let system* (31) *have a rough solution η_*, ρ_{j*} ($\rho_{j*}>0$, $j=1,\dots,N$). Then with all sufficiently small ε, Eq.* (1) *has a residual asymptotic N-dimensional invariant torus*

$$u_*=\eta_*+\sum_{j=1}^{N}\Big(\rho_{j*}\exp\big(ik_jx+i(k_j^3+\varphi_{j*})t\big)+c\overline{c}\Big)+O(\varepsilon).$$

4. CONSTRUCTION OF PERIODIC SOLUTIONS AND INVARIANT TORI OF EQUATION (1) SUBJECT TO $\gamma=0$ AND $b\neq 0$ BUT WITHOUT CONDITION (13)

4.1. *The asymptotic behavior of periodic solutions to Eq.* (1) *with $\gamma=0$ and $b\neq 0$.* Let us pay a special attention to the fact that the role of parameters α and β in the question of the existence of periodic solutions and tori consists only in the presence of the product $\alpha\beta$ in the coefficient at cubic terms. Below we prove that the role of the term bu^2 is more important.

Substitute the formal series (25) in (1) and equate coefficients at the same degrees of η, ξ, and ε. As a result, we get the following formulas for coefficients of the normal form (23):

$$\alpha_1=\beta_1=a_0,\;\;\alpha_2=b,\;\;\alpha_3=2b,\;\;\alpha_4=c,\;\;\alpha_5=6c,$$

$$\beta_2=2b-i\alpha k_0,\;\beta_3=3c+\frac{\alpha b}{2k_0^2}+i\Big(\frac{b^2}{3k_0^3}-\frac{\alpha^2}{6k_0}-k_0\beta\Big),\;\;\beta_4=3c-i\beta k_0.$$

As above, in the obtained normal form we proceed to polar coordinates in the variable $\xi=\rho\exp(i\varphi t)$. Therefore, we proceed to the following system for finding amplitudes which is analogous to (28):

$$\begin{aligned}\dot\eta&=a_0\varepsilon\eta+b\eta^2+2b\rho^2+c\eta^3+6c\eta\rho^2,\\ \dot\rho&=\rho\Big(a_0\varepsilon+2b\eta+\rho^2\big(3c+\alpha b/(2k_0^2)\big)+3c\eta^2\Big),\end{aligned}\tag{33}$$

whence with $a_0>0$ we get the rough solution

$$\rho_*=\frac{a_0\varepsilon}{2\sqrt{2}|b|}+O(\varepsilon^2),\;\;\eta_*=-\frac{a_0\varepsilon}{2b}+O(\varepsilon^2).\tag{34}$$

In addition, for φ_* we get

$$\varphi_*=\frac{\alpha k_0a_0\varepsilon}{2b}+O(\varepsilon^2).$$

Theorem 5. *Let system* (33) *have the rough solution* (34). *Then with all sufficiently small* ε, *Eq.* (1) *has the following residual asymptotically periodic solution*

$$u = \eta_* + \rho_* \exp\big(ik_0x + i(k_0^3 + \varphi_*)t\big) + c\overline{c} + O(\varepsilon^2).$$

Note that dynamic properties of system (23) were studied by many authors. See [12] for most complete results on this theme.

4.2. *The asymptotic of invariant tori of Eq.* (1) *with* $\gamma = 0$ *and* $b \neq 0$. Analogously to Item 3.3, for a fixed collection of positive pairwise distinct numbers $k_1 = 2\pi n_1/T, \ldots, k_N = 2\pi n_N/T$ ($n_1, \ldots, n_N$ are distinct natural numbers) we consider series (29). Substitute it in Eq. (1), assuming that $b \neq 0$, by standard calculations we find coefficients of the corresponding normal form (30):

$$\alpha_1 = \beta_{1j} = a_0,\ \ \alpha_2 = b,\ \ \alpha_{3j} = 2b,\ \ \alpha_4 = c,\ \ \alpha_{5j} = 6c,$$

$$\beta_{2j} = 2b - i\alpha k_j,\ \ \beta_{3jj} = 3c + \frac{\alpha b}{2k_j^2} + i\Big(\frac{b^2}{3k_j^3} - \frac{\alpha^2}{6k_j} - \beta k_j\Big),$$

$$\beta_{3js} = 6c - \frac{4b\alpha}{3(k_j^2 - k_s^2)} + i\Big(\frac{8b^2}{3k_j(k_j^2 - k_s^2)} - 2\beta k_j\Big) \ \text{ with } \ s \neq j,\ \ \beta_{4j} = 3c - i\beta k_j.$$

Therefore, the question of the existence of solutions to system (30) in the form $\xi_j = \rho_j \exp(i\varphi_j t)$ is reduced to the question of the solvability of the system of equations

$$\begin{aligned} &a_0\varepsilon\eta + 2b\sum_{s=1}^{N}\rho_s^2 + b\eta^2 + c\eta^3 + 6c\eta\sum_{s=1}^{N}\rho_s^2 = 0,\\ &\rho_j\Big(a_0\varepsilon + 2b\eta + \sum_{s=1}^{N}\operatorname{Re}\beta_{3js}\rho_s^2 + 3c\eta^2\Big) = 0,\ \ j = 1, \ldots, N. \end{aligned} \tag{35}$$

In particular, with $N = 2$ we get the system

$$\begin{aligned} &a_0\varepsilon\eta + 2b(\rho_1^2 + \rho_2^2) + b\eta^2 + c\eta^3 + 6c\eta(\rho_1^2 + \rho_2^2) = 0,\\ &\rho_1\big(a_0\varepsilon + 2b\eta + \operatorname{Re}\beta_{311}\rho_1^2 + \operatorname{Re}\beta_{312}\rho_2^2 + 3c\eta^2\big) = 0,\\ &\rho_2\big(a_0\varepsilon + 2b\eta + \operatorname{Re}\beta_{321}\rho_1^2 + \operatorname{Re}\beta_{322}\rho_2^2 + 3c\eta^2\big) = 0. \end{aligned} \tag{36}$$

Note that when η and ρ_j have the order ε, for solving this system we get equalities

$$\eta_* = -\frac{a_0}{2b}\varepsilon + O(\varepsilon^2),\ \ \rho_{1*} = \rho_{12*}\varepsilon + O(\varepsilon^2),\ \ \rho_{2*} = \rho_{21*}\varepsilon + O(\varepsilon^2), \tag{37}$$

where

$$\rho_{js*} = \frac{a_0}{2\sqrt{2}|b|}\sqrt{\frac{3c - 4\alpha b/\big(3(k_j^2 - k_s^2)\big) - \alpha b/(2k_s^2)}{6c - \alpha b/(2k_j^2) - \alpha b/(2k_s^2)}}.$$

For φ_j we get asymptotic equalities

$$\varphi_{1*} = \frac{\alpha k_1 a_0\varepsilon}{2b} + O(\varepsilon^2),\ \ \varphi_{2*} = \frac{\alpha k_2 a_0 \varepsilon}{2b} + O(\varepsilon^2).$$

Theorem 6. *Let system* (36) *have the rough solution* (37). *Then with all sufficiently small* ε, *Eq.* (1) *has a residual asymptotic accurate to* $O(\varepsilon^2)$ *two-dimensional invariant torus, namely,*

$$u_* = \eta_* + \sum_{j=1}^{2}\Big(\rho_{j*}\exp\big(ik_jx + i(k_j^3 + \varphi_{j*})t\big) + c\overline{c}\Big).$$

The dynamics of system (30) can be much more complex even with $N = 2$ [12]. Such solutions to system (30) allow one to write residual asymptotic solutions to the initial boundary value problem.

5. THE NORMAL FORM OF THE KORTEWEG–DE VRIES–BURGERS EQUATION ($\gamma \neq 0$)

The case, when solutions to (1), (2) contain no zero harmonic (in the expansion into the Fourier series in the spatial variable), is most interesting and important for applications. In this case it is necessary that in place of degrees u^j ($j = 2, 3$) for $F(u)$ there should be differences $\big(u^j - M(u^j)\big)$. Here $M(u)$, i.e., the mean value of the function $u(t, x)$ in the spatial variable, obeys equality (6).

It is convenient to perform normalizations $x \mapsto \frac{T}{2\pi}x$, $t \mapsto \big(\frac{T}{2\pi}\big)^3 t$ and to introduce new denotations for proceeding from the T-periodic boundary-value problem to the following 2π-periodic one:

$$u(t, x + 2\pi) \equiv u(t, x). \tag{38}$$

Thus, in what follows, we study a local in some sufficiently small (and independent of ε) neighborhood of the zero equilibrium state dynamic of the equation

$$u_t + u_{xxx} + \big(\alpha u + \beta u^2 + O(|u|^3)\big)u_x = \varepsilon(\gamma_0 u_{xx} + a_0 u) + b\big(u^2 - M(u^2)\big) + c\big(u^3 - M(u^3)\big) + O(|u|^4). \tag{39}$$

Here, naturally, we need to study Eq. (39) subject to the boundary condition (38) and condition (5) for the zero mean value of the function u.

An important role in the study of solutions located in a small neighborhood of zero is played by the behavior of solutions to the linear boundary-value problem

$$u_t + u_{xxx} = 0, \ \ u(t, x + 2\pi) \equiv u(t, x), \ \ M(u) = 0. \tag{40}$$

Solutions to problem (40) take the form

$$u(t, x) = \sum_{k=-\infty, k\neq 0}^{\infty} V_k \exp(ikx + ik^3 t), \ \ V_{-k} = \overline{V}_k, \ \ k = \pm 1, \pm 2, \dots\ . \tag{41}$$

Thus, in the problem on the local dynamic of (38)–(40) we get the case which is close (provided that $0 < \varepsilon \ll 1$) to the critical case of infinite dimension. Roots λ_k of the corresponding characteristic equation satisfy the equality $\lambda_k = ik^3$. Eigenfunctions equal $\exp(ikx)$. Hence, we conclude that minor resonances 1:1, 1:2, and 1:3 are absent.

Note that modified Korteweg–de Vries and Korteweg–de Vries–Burgers equations were studied by many authors [1, 2, 4, 13, 14]. In particular, they considered questions of integrability and construction (with certain values of coefficients) of exact solutions. Here we use the research technique proposed in [5, 15–18] for studying the local dynamics for infinite-dimensional critical cases. The corresponding formalism is based on the representation of the "main" part of solutions to (38)–(40) in the form (41), where $V_k = V_k(\tau)$ and $\tau = \varepsilon t$ is a "slow" time. Thus, we are interested in constructing the first approximation equations for defining slow amplitudes. Note also that the transfer from the initial equation (39) to equations for $V_k(\tau)$ is called the normalization, and the resulting equations for $V_k(\tau)$ are said to have the normal form (or the truncated normal form). Using well-known methods (e.g., [8, 10]) one can easily obtain the normal form, i.e., an infinite system of ordinary differential equations for $V_k(\tau)$. The main goal of this paper is to construct the normal form for the boundary-value problem (38)–(40) as one partial differential evolution equation.

Consider the formal series

$$u(t, x, \varepsilon) = \sqrt{\varepsilon} u_1(t, x, \tau) + \varepsilon u_2(t, x, \tau) + \varepsilon^{3/2} u_3(t, x, \tau) + \cdots, \tag{42}$$

where $\tau = \varepsilon t$,

$$u_1(t, x, \tau) = \sum_{k=-\infty,\ k\neq 0}^{\infty} V_k(\tau) \exp(ikx + ik^3 t). \tag{43}$$

Functions $u_2, u_3, \dots$ are 2π-periodic in t and x. Substitute (42) in (39). In the obtained formal identity we equate coefficients at the same degrees of ε. Thus, collecting coefficients at the first degree of ε, we get correlations

$$\frac{\partial u_2}{\partial t} + \frac{\partial^3 u_2}{\partial x^3} = \Big(b - \frac{\alpha}{2}\frac{\partial}{\partial x}\Big)\big(u_1^2 - M(u_1^2)\big).$$

Hence we find

$$u_2 = \sum_{\substack{k=-\infty,\, k\neq 0,\\ p=-\infty,\, p\neq 0,-k}}^{\infty} u_{2kp} \exp\big(i(k+p)x + i(k^3+p^3)t\big),$$

where

$$u_{2kp} = \frac{ib}{3}\frac{V_k V_p}{kp(k+p)} + \frac{\alpha}{6}\frac{V_k V_p}{kp} \ \text{ with } \ p, k \neq 0, \ \ p \neq -k.$$

Introduce some denotations. Let

$$V = \sum_{k=-\infty,\, k\neq 0}^{\infty} V_k \exp(ikx).$$

Denote by J the "integration" operator

$$J(V) = \sum_{k=-\infty,\, k\neq 0}^{\infty} \frac{V_k}{ik} \exp(ikx).$$

Using this operator, we write the expression for $u_2(0, x, \tau)$ in the compact form

$$u_2(0,\tau,x) = \frac{b}{3} J\Big(J^2(V) - M\big(J^2(V)\big)\Big) - \frac{\alpha}{6}\Big(J^2(V) - M\big(J^2(V)\big)\Big).$$

On the next step we collect coefficients at $\varepsilon^{3/2}$ and, as a result, obtain the equation

$$\frac{\partial u_3}{\partial t} + \frac{\partial^3 u_3}{\partial x^3} = -\frac{\partial u_1}{\partial \tau} + \gamma_0 \frac{\partial^2 u_1}{\partial x^2} + a_0 u_1 + \Big(2b - \alpha\frac{\partial}{\partial x}\Big)(u_1 u_2) + \Big(c - \frac{\beta}{3}\frac{\partial}{\partial x}\Big)u_1^3. \qquad (44)$$

This equation is solvable with respect to u_3 in the mentioned function class, provided that coefficients at all harmonics in the form $\exp(ikx + ik^3 t)$ in the right-hand side of (44) are equal zero. Hence we obtain the following infinite system of ordinary differential equations with respect to all Fourier coefficients $V_k(\tau)$ of the function $u_1(t, x, \tau)$:

$$\frac{\partial V_k}{\partial \tau} = (-\gamma_0 k^2 + a_0)V_k + \psi_k(V), \ \ k = \pm 1, \pm 2, \dots, \qquad (45)$$

where

$$\psi_k(V) = \frac{1}{2\pi}\int_0^{2\pi} \bigg(\Big(2b - \alpha\frac{\partial}{\partial x}\Big)\big(u_1 u_2 - M(u_1 u_2)\big) + \Big(c - \frac{\beta}{3}\frac{\partial}{\partial x}\Big)\big(u_1^3 - M(u_1^3)\big)\bigg)\exp(-ikx)dx.$$

As appeared, one can write system (45) as one (scalar) equation. To this end, we introduce one more operator $R_1(W)$ by the following rule: For the function

$$W(x) = \sum_{k=-\infty,\, k\neq 0}^{\infty} W_k \exp(ikx)dx$$

we put

$$R_1(W) = \big(\dots,\ W_{-1}\exp(-ix),\ 0,\ W_1\exp(ix),\ W_2\exp(2ix), \dots\big).$$

Consider the boundary-value problem

$$\frac{\partial W}{\partial \tau} = \gamma_0 \frac{\partial^2 W}{\partial x^2} + R(W),\ W(\tau, x+2\pi) \equiv W(\tau, x);\ \ M(W) = 0, \tag{46}$$

where

$$R(W) = a_0 W + \alpha^2 A(W) + \alpha b B(W) + b^2 C(W) + \beta G(W) + cH(W).$$

Here we use the following denotations (the vector multiplication below is coordinate-wise):

$$A(W) = -\frac{1}{3}\Big(R_1(W)J(W) - M\big(R_1(W)J(W)\big),\ \overline{R}_1\big(J(W)\big)\Big),$$

$$B(W) = -\frac{1}{3}\bigg(J\big(R_1(W)\big)J(W) - M\Big(J\big(R_1(W)\big)J(W)\Big),\ \overline{R}_1(W)\bigg) + \frac{1}{3}\bigg(J\Big(R_1(W)W - M\big(R_1(W)W\big)\Big),\ \ \overline{R}_1\big(J(W)\big)\bigg),$$

$$C(W) = -\frac{2}{3}\bigg(J\Big(R_1(W)J(W) - M\big(R_1(W)J(W)\big)\Big),\ \ \overline{R}_1\big(J(W)\big)\bigg),$$

$$G(W) = -\frac{\partial W}{\partial x} M(W^2),\ \ H(W) = 3WM(W^2).$$

For stating the main result, in formula (46) we put

$$W(\tau, x) = \sum_{k=-\infty,\ k\neq 0}^{\infty} W_k(\tau)\exp(ikx)dx$$

and collect coefficients at the same functions $\exp(ikx)$.

As a result, we get the following countable system of ordinary differential equations with respect to all $W_k(\tau)$:

$$\dot{W}_k = P_k(W_{\pm 1}, W_{\pm 2}, \dots),\ \ k = \pm 1, \pm 2, \dots \tag{47}$$

Theorem 7. *The system of equations* (47) *coincides with system* (45).

Therefore, we conclude that the boundary-value problem (46) plays the role of the normal form for the initial boundary-value problem (38)–(40). Fourier coefficients of solution (46) which are slow with respect to time t are also Fourier coefficients of residual asymptotic solutions to (38)–(40). In particular, $u(0, x, \tau) = \sqrt{\varepsilon}W(\tau, x) + O(\varepsilon)$. Hence and from (42), (43) we get the asymptotic of the function $u(t, x, \tau)$. Note that the boundary-value problem (46) can have a rather rich dynamics.

In the case of the modified Korteweg–de Vries equation we proceed from (46) to the boundary-value problem

$$\frac{\partial W}{\partial \tau} = \alpha^2 A(W) + \beta G(W),\ \ W(\tau, x+2\pi) = W(\tau, x),\ \ M(W) = 0. \tag{48}$$

All solutions to (48) are non-rough tori with time-constant amplitudes in the form $W(\tau, x) = W(0, x)\exp(i\psi(\tau, x))$. Performing some simple transformations, we reduce (48) to the following completely split infinite system of first-order equations:

$$\dot{W}_k = -ik^{-1} W_k |W_k|^2 (\alpha^2/3 - \beta),\ \ k = \pm 1,\ \pm 2, \dots$$

6. CONCLUSION

We study the local dynamics of the generalized Korteweg–de Vries equation. Using the classical methods of bifurcation analysis, we study the question of the existence, asymptotic behavior, and stability of periodic solutions and tori. We prove that if values of diffusion coefficients γ are close to zero, then in the question of the stability of the equilibrium state there occurs a critical case of infinite dimension. Using the idea of the normalization method, we have succeeded to reduce the initial boundary-value problem to a much simpler one, namely, to a specific nonlinear boundary value problem that defines dynamic properties of the initial generalized Korteweg–de Vries equation.

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