ISSN 1066-369X, Russian Mathematics, 2018, Vol. 62, No. 1, pp. 36–49. © Allerton Press, Inc., 2018.
Original Russian Text © S.P. Kuznetsov, V.V. Mochalov, V.P. Chuev, 2018, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2018, No. 1, pp. 42–56.

On Riemann Boundary-Value Problem for Regular Functions in Clifford Algebras

S. P. Kuznetsov*, **V. V. Mochalov****, and **V. P. Chuev*****

Chuvash State University
Moskovskii pr. 15, Cheboksary, 428010 Russia

Received October 11, 2016

Abstract—We pose and investigate the Riemann boundary-value problem for regular and strongly regular functions in Clifford algebras. The posed problem is reduced to the matrix problem for analytical functions in one and two complex variables and we give its solution. We carry out the boundary-value problems in special cases.

DOI: 10.3103/S1066369X18010061

Keywords: *Clifford algebra, regular functions, Riemann boundary-value problem.*

The monographs of N. I. Muskhelishvili ([1], pp. 416–446) and N. P. Vekua ([2], pp. 11–59) contain the linear conjugation problem for several unknown functions. The problem is to find a piecewise holomorphic vector $\Phi(z) = (\Phi_1, \Phi_2, \dots, \Phi_n)$ with a jump line L and having a finite order at infinity by the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \;\; t \in L, \tag{1}$$

where $G(t)$ is a matrix of class H defined on L and is not singular anywhere in L, and $g(t)$ is a vector of class H on L. We construct and study the canonical system of solutions to the homogeneous Riemann problem. We also give the solution to homogeneous and inhomogeneous Riemann problem (1), and obtain the conditions for the solvability of the inhomogeneous problem with a negative index.

V. A. Kakichev ([3], pp. 4–56) set and carried out an investigation of the Riemann boundary-value problem for analytic functions of two complex variables. The two-dimensional Riemann problem is to find the four functions $\Phi^{\pm\pm}(z, w)$ analytic in the domains $D^{\pm\pm} = D_1^\pm \times D_2^\pm$, respectively, by the boundary condition

$$A(t,\omega)\Phi^{++}(t,\omega) + B(t,\omega)\Phi^{-+}(t,\omega)+C(t,\omega)\Phi^{+-}(t,\omega) + D(t,\omega)\Phi^{--}(t,\omega) = F(t,\omega), \tag{2}$$

$$\Phi^{\pm=}(z_1,\infty) = 0, \;\; z_1 \in D_1^\pm, \;\; \Phi^{=\pm}(\infty, z_2) = 0, \;\; z_2 \in D_2^\pm,$$

here the variables $(t,\omega) \in L^2 = L_1 \times L_2$, $L_1 = \partial D_1$, $L_2 = \partial D_2$, and the coefficients $A(t,\omega)$, $B(t,\omega)$, $C(t,\omega)$, $D(t,\omega)$, $F(t,\omega)$ are of class $H(L^2)$. In the general case, there is no solution to problem (2). If $A(t,\omega) = B(t,\omega), C(t,\omega) = D(t,\omega)$ or $A(t,\omega) = C(t,\omega)), B(t,\omega) = D(t,\omega)$, then we obtain the degenerate Riemann problems of the first kind that reduce to Riemann boundary-value problems for one variable with the second variable as a parameter.

The index $\chi_L(A)$ of a function $A(t,\omega)$ continuous on L^2 is defined ([3], pp. 10–11) as a change in its argument passing along $L : \chi_L = \frac{1}{2\pi i}\int\limits_L d\arg A(t,\omega)$. Assume the notation $\chi_1(A) = \chi_{L_1}(A)$ and $\chi_2 = \chi_{L_2}(A)$. It is proved in [4, 5] that if A, B, C, D, F are summable with degree $p > 1$ on the skeleton L^2 formed by simple closed Lyapunov curves, then conditions

1) $A \neq 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$ on L^2,

*E-mail: chevchenka@mail.ru.
**E-mail: m622573@gmail.com.
***E-mail: 570065@mail.ru.

2) $\chi_1(A) = \chi_1(B)$, $\chi_1(C) = \chi_1(D)$, $\chi_2(A) = \chi_2(C)$, $\chi_2(B) = \chi_2(D)$
are necessary and sufficient for problem (2) to have a finite index.

In [6] they posed and considered the Riemann boundary-value problem for regular (monogenic) left functions in the complex Clifford algebra $C_{0,n} = R_{0,n} \oplus \mathbb{C}$. The Riemann problem reduces to a singular integral equation, which is investigated by the operator method.

In [7–11] the authors study the jump problem for monogenic functions in the set $\overline{\Omega} \subset \mathbb{R}^m$ whose boundary Γ is compact topological surface in the Clifford algebra $R_{0,n}$. They found the equivalent monogeneity conditions from the left (from the right), the two-sided monogeneity condition for the function in $\overline{\Omega}$. Also we know when the jump problem has a solution. In [6–8], the study is conducted for the Dirac operator $\overline{D} = \sum\limits_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$ of an incomplete variable $\sum\limits_{i=1}^{n} x_i e_i$.

In [12–14] the authors consider the Riemann boundary-value problem for weakly regular left functions in the Clifford algebras $R_{0,2}$, $R_{2,0}$, $R_{3,0}$. Clifford-valued constants there were considered as coefficients of boundary-value problem (2). The presented problems were reduced to matrix problems for regular functions of two complex variables, which then split into Riemann boundary-value problems for analytic functions of one variable.

In this paper we present and study the Riemann boundary-value problem for regular and strongly regular functions in the Clifford algebras $R_{0,2}$, $R_{2,0}$, $R_{3,0}$ with arbitrary coefficients. We reduce the problem to a matrix problem for analytic functions of one and two complex variables, and give its solution. Then we carry out the study of boundary-value problems in special cases and indicate a method for solving the Riemann boundary-value problem in an arbitrary Clifford algebra.

In contrast with [6–11], we consider the Riemann boundary-value problem for regular functions $f(w)$ meeting the equation $\overline{D} \cdot f = 0$, here $\overline{D} = \frac{1}{2^n} \sum\limits_{\alpha \in \Gamma_n} e_\alpha \frac{\partial}{\partial x_\alpha}$, and for strongly regular functions.

1. The Clifford algebra. Detailed information on the structure and properties of the Clifford algebra can be found in the monograph [15] (pp. 82–150).

Let $R_{p,q}$ be a Clifford algebra of dimension $m = 2^n$ $(n = p + q)$ with the basis $e_\alpha = e_{i_1} \dots e_{i_k}$, $1 \le i_1 < \dots < i_k \le n$, where the multi-index $\alpha = i_1 \dots i_k$ runs through all subsets in the set $\{1, \dots, n\}$, whose collection we denote by Γ_n. Let $e_\phi = e_0 = 1, e_1, \dots, e_n$ be the canonical basis, $e_{12,\dots,n} = e_\tau$ and multiplication in $R_{p,q}$ be defined by the relation

$$e_i e_j + e_j e_i = 2\delta_{ij}\varepsilon_i,$$

here $\varepsilon_i = e_i^2 = 1$, $i = 1, \dots, p$, $\varepsilon_i = e_i^2 = -1$, $i = p + 1, \dots, p + q$.

The arbitrary and conjugate elements of the Clifford algebra are representable in the form $w = \sum\limits_{\alpha \in \Gamma_n} x_\alpha e_\alpha$, $\overline{w} = \sum\limits_{\alpha \in \Gamma_n} x_\alpha \varepsilon_\alpha e_\alpha$. Denote by $f(w) = \sum\limits_{\alpha \in \Gamma_n} f_\alpha(w) e_\alpha$, $f_\alpha(w) : \Omega \to \mathbb{R}$ a function with values in the Clifford algebra and defined in the domain $\Omega \subset \mathbb{R}^m$, and by $\overline{D} = \frac{1}{2^n} \sum\limits_{\alpha \in \Gamma_n} e_\alpha \frac{\partial}{\partial x_\alpha}$ the differential operator. The function f belongs to $F_{p,q}^{(k)}(\Omega)$ if its components f_α lie in $C^{(k)}(\Omega)$.

B-set is the set of basis elements $B = \{e_\alpha\}_{\alpha \in B} = \{e_{\alpha_i}, \dots, e_{\alpha_k}\}$, with the following property: For any $e_{\alpha_i}, e_{\alpha_j} \in B$, $\alpha_i \ne \alpha_j$, $\varepsilon_{\alpha_i} + a_{\alpha_i \alpha_j} \varepsilon_{\alpha_j} = 0$, here $a_{\alpha_i \alpha_j}$ are commutation coefficients defined by $e_{\alpha_i} e_{\alpha_j} = a_{\alpha_i \alpha_j} e_{\alpha_j} e_{\alpha_i}$. In [16] the author proved a theorem on the basis $R_{p,q}$ decomposition into B-sets consisting of two elements. In this case an arbitrary element of the algebra, the Clifford-valued function, and the differential operator can be represented in the form

$$w = \sum_{\alpha \in \Gamma_{n-1}} e_\alpha u_{B_0}^\alpha, \;\; f(w) = \sum_{\alpha \in \Gamma_{n-1}} \widetilde{g}_{B_0^\alpha} e_\alpha, \;\; \overline{D} = \frac{1}{2^n} \sum_{\alpha \in \Gamma_{n-1}} e_\alpha D_{B_0^\alpha},$$

$$w_{B_0^\alpha} = \sum_{\beta \in B_0^\alpha} x_\beta^\alpha e_\beta, \;\; \widetilde{g}_{B_0^\alpha} = \sum_{\beta \in B_0^\alpha} f_\beta^\alpha e_\beta, \;\; \overline{D} = \sum_{\beta \in B_0^\alpha} e_\beta \frac{\partial}{\partial x_\beta^\alpha}.$$

Note that $w_{B_0^\alpha}$ are complex numbers formed on the set B_0 containing e_0. In the Clifford algebra, we can introduce various classes of regular functions. A function $f \in F^1_{p,q}(\Omega)$ is called [17–19]) weakly regular on the left (from the right) in $\Omega \in \mathbb{R}^m$ if

$$\overline{D}\cdot f = 0 \;\; (f \cdot \overline{D} = 0), \;\; \Omega \in \mathbb{R}^m,$$

is strongly regular on the left (from the right) if for all $\nu \in \Gamma_{n-1}$

$${}^{\nu}\overline{D} \cdot f = 0, \;\; (f \cdot {}^{\nu}\overline{D} = 0), \;\; {}^{\nu}\overline{D} = \varepsilon_\nu e_\nu \overline{D} e_\nu, \;\; \Omega \in \mathbb{R}^m.$$

In [18] it was proved that the conditions of strong regularity are equivalent to the regularity conditions by B-sets, consisting of two elements $\overline{D_{B_0^\alpha}}\cdot f = 0$, $\alpha \in \Gamma_{n-1}$, which are equivalent to the conditions $\overline{D_{B_0^\alpha}}\cdot \widetilde{g}_{B_0^\beta} = 0$, $\alpha, \beta \in \Gamma_{n-1}$.

2. Quaternion algebra. Quaternion algebra $R_{0,2}$ is a real associative noncommutative algebra of dimension $m = 4$ generated by elements e_1, e_2. The basis of the algebra is formed by the elements $\{e_0, e_1, e_2, e_{12}\}$, here e_0 is the algebra unit, $e_{12} = e_1e_2$, and the elements e_1, e_2, e_{12} meet the relations $e_1^2 = e_2^2 = e_{12}^2 = -e_0$, $e_i \cdot e_j + e_j \cdot e_i = 0$, $i \neq j, i,\; j = 1, 2, 12$. These relations define the operation of multiplication in $R_{0,2}$.

An arbitrary element of the algebra can be represented in real and complex form

$$w = x_0e_0 + x_1e_1 + x_2e_2 + x_{12}e_{12} = z_1e_0 + z_2e_1,$$

here $z_1 = x_0e_0 + x_{12}e_{12}$, $z_2 = x_1e_0 + x_2e_{12}$ are complex numbers (the role of the imaginary unit plays e_{12}). The conjugate element is $\overline{w} = x_0e_0 - x_1e_1 - x_2e_2 - x_{12}e_{12} = e_0\overline{z_1} - e_1\overline{z_2}$, here $\overline{z_1} = x_0e_0 - x_{12}e_{12}$, $\overline{z_2} = x_1e_0 - x_2e_{12}$. Note that any nonzero element w has the inverse one $w^{-1} = \frac{\overline{w}}{\|w\|^2} = \frac{x_0e_0 - x_1e_1 - x_2e_2 - x_{12}e_{12}}{x_0^2 + x_1^2 + x_2^2 + x_{12}^2}$, so $R_{0,2}$ does not contain zero divisors.

Denote by $f(w) = f_0(w)e_0 + f_1(w)e_1 + f_2(w)e_2 + f_{12}(w)e_{12}$ a function with values in algebra $R_{0,2}$ and defined in the domain $\Omega \subset \mathbb{R}^4$, and by

$$\overline{D} = \frac{1}{4}\left(e_0\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_{12}\frac{\partial}{\partial x_{12}}\right)$$

the differential operator. The quaternion-valued function $f(w)$ and the differential operator $\overline{D}$ can be represented in the complex form:

$$f(w) = \Phi_1(z_{1,}, z_2)e_0 + \Phi_2(z_1, z_2)e_1, \;\; \text{here} \;\; \Phi_1(z_1, z_2) = f_0e_0 + f_{12}e_{12}, \;\; \Phi_2(z_1, z_2) = f_1e_0 + f_2e_{12};$$

$$\overline{D} = \frac{1}{2}\left(e_0\frac{\partial}{\partial \overline{z_1}} + e_1\frac{\partial}{\partial \overline{z_2}}\right), \;\; \text{here} \;\; \frac{\partial}{\partial \overline{z_1}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_0} + e_{12}\frac{\partial}{\partial x_{12}}\right), \;\; \frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_1} - e_{12}\frac{\partial}{\partial x_2}\right).$$

Let ${}^{12}\overline{D} = \frac{1}{2}(e_0\frac{\partial}{\partial \overline{z_1}} - e_1\frac{\partial}{\partial \overline{z_2}})$ be the differential operator. The function $f(w) \in F^1_{0,2}(\Omega)$ is called [19] strongly regular on the left if

$$\overline{D} \cdot f = 0, {}^{12}\overline{D} \cdot f = 0, \;\; \Omega \subset \mathbb{R}^4.$$

The strong regularity conditions [18] are equivalent to the equalities

$$\frac{\partial \Phi_1}{\partial \overline{z_1}} = \frac{\partial \Phi_1}{\partial \overline{z_2}} = \frac{\partial \Phi_2}{\partial \overline{z_1}} = \frac{\partial \Phi_2}{\partial \overline{z_2}} = 0,$$

which mean that the functions $\Phi_1(z_1, z_2)$, $\Phi_2(z_1, z_2)$ are analytic in the variables z_1, z_2.

2.1. First we consider the case of the function $f(w)$ dependent only on one complex variable z_k:

$$f(w) = \Phi_1(z_k)e_0 + \Phi_2(z_k)e_1, \;\; \Phi_1(z_k) = f_0e_0 + f_{12}e_{12}, \;\; \Phi_2(z_k) = f_1e_0 + f_2e_{12},$$

$\frac{\partial \Phi_1}{\partial \overline{z}_k} = 0$, $\frac{\partial \Phi_2}{\partial \overline{z}_k} = 0$. The latter equalities mean that the functions $\Phi_1(z_k)$, $\Phi_2(z_k)$ are analytic in the variable z_k.

Let L_k be a simple smooth closed contour bounding in the plane of the complex variable the domain $D_k^+ \subset \overline{C}$, D_k^- completes $D_k^+ \bigcup L_k$ to the extended complex plane $\overline{C}$.

Statement of the Riemann problem: Find the regular function $f(w) = \Phi_1(z_k)e_0 + \Phi_2(z_k)e_1$ by the conditions

$$f^+(t)A(t) + f^-(t)C(t) = F(t), \;\; t \in L_k, \tag{3}$$

$$f^-(\infty) = 0,$$

here the coefficients $A(t)$, $C(t)$, $F(t)$ are quaternion-valued functions that do not vanish on L_k, and the components of which belong to the class $H(L_k)$. The functions $A(t)$, $C(t)$, $F(t)$ are representable in the complex form: $A(t) = A_1(t)e_0 + A_2(t)e_1$, $C(t) = C_1(t)e_0 + C_2(t)e_1$, $F(t) = F_1(t)e_0 + F_2(t)e_1$.

By the equalities $A_k e_1 = e_1\overline{A}_k$, $C_k e_1 = e_1\overline{C}_k$ boundary condition (3) takes the form

$$\Phi_1^+(t)A_1(t) - \Phi_2^+(t)\overline{A_2(t)} + \Phi_1^-(t)C_1(t) - \Phi_2^-(t)\overline{C_2(t)} = F_1(t),$$

$$\Phi_1^+(t)A_2(t) - \Phi_2^+(t)\overline{A_1(t)} + \Phi_1^-(t)C_2(t) - \Phi_2^-(t)\overline{C_1(t)} = F_2(t), \tag{4}$$

$$\Phi_1^-(\infty) = \Phi_2^-(\infty) = 0.$$

We represent boundary conditions (4) in the matrix form

$$R(t)\Phi^+(t) + G(t)\Phi^-(t) = F(t), \;\; t \in L_k, \tag{5}$$

$$R(t) = \begin{pmatrix} A_1(t) & -\overline{A_2(t)} \\ A_2(t) & \overline{A_1(t)} \end{pmatrix}, \;\; G(t) = \begin{pmatrix} C_1(t) & -\overline{C_2(t)} \\ C_2(t) & \overline{C_1(t)} \end{pmatrix},$$

$$\Phi^+(t) = \begin{pmatrix} \Phi_1^+(t) \\ \Phi_2^+(t) \end{pmatrix}, \;\; \Phi^-(t) = \begin{pmatrix} \Phi_1^-(t) \\ \Phi_2^-(t) \end{pmatrix}, \;\; F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix},$$

$$\Phi_1^-(\infty) = \Phi_2^-(\infty) = 0.$$

Note that the determinants of the matrices $\det R(t) = |A_1(t)|^2 + |A_2(t)|^2$, $\det G(t) = |C_1(t)|^2 + |C_2(t)|^2$ are non-zero and are real functions. Multiplying equality (5) by the inverse matrix $R^{-1}(t)$, we obtain the inhomogeneous Riemann boundary-value problem for a vector-valued function

$$\Phi^+(t) - G_1(t)\Phi^-(t) = F^*(t), \;\; t \in L_k, \tag{6}$$

$$\Phi^-(\infty) = 0,$$

here the matrix $G_1(t) = -R^{-1}(t)G(t)$ is of class H and $\det G_1(t) = \frac{\det G(t)}{\det R(t)} \neq 0$ on L_k, $F^*(t) = R^{-1}(t)F(t)$. Since the determinant of the matrix $\det G_1(t)$ is a real function $\chi = \operatorname{Ind} \det G_1(t) = 0$.

Let $\mathrm{X}^\beta(z_k) = (\mathrm{X}_1^\beta(z_k), \mathrm{X}_2^\beta(z_k))$, $\beta = 1, 2$, be the canonical system of solutions of the homogeneous problem, and $\mathrm{X}(z_k)$ is the corresponding canonical matrix ([1], pp. 427–430):

$$\mathrm{X}(z_k) = \|\mathrm{X}_\alpha^\beta(z_k)\| = \begin{pmatrix} \mathrm{X}_1^1(z_k) & \mathrm{X}_1^2(z_k) \\ \mathrm{X}_2^1(z_k) & \mathrm{X}_2^2(z_k) \end{pmatrix}.$$

The canonical system has the following properties:

1) The canonical matrix is normal, i.e., its determinant $\Delta(z_k) = \det \mathrm{X}(z_k)$ does not vanish anywhere in the finite part of the plane;

2) let $(-\chi_\beta)$ be the solution $\mathrm{X}^\beta(z_k)$ order at infinity, then the determinant $\Delta^0(z_k) = \det \|z_k^{\chi_\beta} \mathrm{X}_\alpha^\beta(z_k)\| = z_k^{\chi_1+\chi_2} \Delta(z_k)$ is non-zero for $z_k = \infty$;

3) $\mathrm{X}^+(t) = G_1(t)\mathrm{X}^-(t)$.

Putting $G_1(\mathrm{t})$ into (6) we obtain

$$[\mathrm{X}^+(t)]^{-1}\Phi^+(t) = [\mathrm{X}^-(t)]^{-1}\Phi^-(t) + [\mathrm{X}^+(t)]^{-1}F^*(t).$$

Since the index of the problem $\chi = 0$, the solution to the problem under consideration exists and is given by formula

$$\Phi(z_k) = \frac{\mathrm{X}(z_k)}{2\pi e_{12}} \int_{L_k} \frac{[\mathrm{X}^+(t)]^{-1} F^*(t) dt}{t - z_k}. \tag{7}$$

2.2. *The spatial Riemann boundary-value problem in* $R_{0,2}$. Let $L^2 = L_1 \times L_2$ denote the common frame of four regions $D^{\pm\pm} = D_1^\pm \times D_2^\pm$, and $H(L^2)$ be the set of continuous functions on L^2 subject to Hölder condition. A function $\Phi(t,\omega) \in H(L^2)$, if there exist constants M_k and powers α_k $(0 < \alpha_k < 1)$, $k = 1, 2$, such that for any pair of points (t,ω), (t^1,ω^1) from L^2 we have the inequality

$$|\Phi(t,\omega) - \Phi(t^1,\omega^1)| \le M_1|t - t^1|^{\alpha_1} + M_2|\omega - \omega^1|^{\alpha_2}.$$

Statement of the Riemann problem: Find a strongly regular function $f(z_1,z_2) = \Phi_1(z_1,z_2)e_0 + \Phi_2(z_1,z_2)e_1$ by conditions

$$f^{++}(t,\omega)A(t,\omega) + f^{-+}(t,\omega)B(t,\omega) + f^{+-}(t,\omega)C(t,\omega) + f^{--}(t,\omega)D(t,\omega) = F(t,\omega), \tag{8}$$

$$f^{\pm=}(z_1,\infty) = 0, \;\; z_1 \in D_1^\pm, f^{=\pm}(\infty,z_2) = 0, \;\; z_2 \in D_2^\pm, \tag{9}$$

here the variables $(t,\omega) \in L^2 = L_1 \times L_2$, $L_1 = \partial D_1, L_2 = \partial D_2$, and the coefficients $A(t,\omega)$, $B(t,\omega)$, $C(t,\omega)$, $D(t,\omega)$ are non-zero on L^2 quaternion-valued functions whose components belong to the class $H(L^2)$, $F(t,\omega) \in H(L^2)$.

We consider the degenerate case of problem (8), (9), for $A(t,\omega) = B(t,\omega)$, $C(t,\omega) = D(t,\omega)$. We introduce the notation

$$\Phi_1^{++}(t,\omega) + \Phi_1^{-+}(t,\omega) = \phi_{1t}^+(\omega), \;\; \Phi_2^{++}(t,\omega) + \Phi_2^{-+}(t,\omega) = \phi_{2t}^+(\omega),$$

$$\Phi_1^{+-}(t,\omega) + \Phi_1^{--}(t,\omega) = \phi_{1t}^-(\omega), \;\; \Phi_2^{+-}(t,\omega) + \Phi_2^{--}(t,\omega) = \phi_{2t}^-(\omega), \tag{10}$$

here $t \in L_1$ is the parameter and $\omega \in L_2$ is the variable. Equalities (9) yield

$$\Phi_k^{-+}(\infty,z_2) = 0, \;\; \Phi_k^{--}(\infty,z_2) = 0, \;\; z_2 \in D_2^\pm, \;\; \Phi_k^{+-}(z_1,\infty) = 0, \;\; \Phi_k^{--}(z_1,\infty) = 0, \;\; z_1 \in D_1^\pm.$$

Let us represent the functions $A(t,\omega)$, $C(t,\omega)$, $F(t,\omega)$ in complex form: $A(t,\omega) = A_1(t,\omega)e_0 + A_2(t,\omega)e_1$, $C(t,\omega) = C_1(t,\omega)e_0 + C_2(t,\omega)e_1$, $F(t,\omega) = F_1(t,\omega)e_0 + F_2(t,\omega)e_1$. By the equalities $A_k e_1 = e_1 \overline{A_k}$, $C_k e_1 = e_1 \overline{C_k}$ we can rewrite boundary equations (8), (9) as

$$\phi_{1t}^+(\omega)A_1(t,\omega) - \phi_{2t}^+(\omega)\overline{A_2(t,\omega)} + \phi_{1t}^-(\omega)C_1(t,\omega) - \phi_{2t}^-(\omega)\overline{C_2(t,\omega)} = F_1(t,\omega),$$

$$\phi_{1t}^+(\omega)A_2(t,\omega) + \phi_{2t}^+(\omega)\overline{A_1(t,\omega)} + \phi_{1t}^-(\omega)C_2(t,\omega) + \phi_{2t}^-(\omega)\overline{C_1(t,\omega)} = F_2(t,\omega), \tag{11}$$

$$\phi_{1t}^-(\infty) = 0, \;\; \phi_{2t}^-(\infty) = 0.$$

We represent the boundary conditions (11) in the matrix form

$$R(t,\omega)\phi_t^+(\omega) + G(t,\omega)\phi_t^-(\omega) = F(t,\omega), \tag{12}$$

$$\phi_t^-(\infty) = 0,$$

here

$$R(t,\omega)=\begin{pmatrix} A_1(t,\omega) & -\overline{A_2(t,\omega)} \\ A_2(t,\omega) & \overline{A_1(t,\omega)} \end{pmatrix},\ G(t,\omega)=\begin{pmatrix} C_1(t,\omega) & -\overline{C_2(t,\omega)} \\ C_2(t,\omega) & \overline{C_1(t,\omega)} \end{pmatrix},$$

$$\phi_t^+(\omega)=\begin{pmatrix} \phi_{1t}^+(\omega) \\ \phi_{2t}^+(\omega) \end{pmatrix},\ \phi_t^-(\omega)=\begin{pmatrix} \phi_{1t}^-(\omega) \\ \phi_{2t}^-(\omega) \end{pmatrix},\ F(t,\omega)=\begin{pmatrix} F_1(t,\omega) \\ F_2(t,\omega) \end{pmatrix},$$

$$\phi_t^-(\infty)=(\phi_{1t}^-(\infty),\ \phi_{2t}^-(\infty)).$$

Let us find the solution to problem (12). Since $\det R(t,\omega)=|A_1(t,\omega)|^2+|A_2(t,\omega)|^2\neq 0$, multiplication of (12) with the inverse matrix $R^{-1}(t,\omega)$ provides us with inhomogeneous Riemann boundary-value problem for a vector-valued function

$$\phi_t^+(\omega)-G_1(t,\omega)\phi_t^-(\omega)=F^*(t,\omega),\ \omega\in L_2, \tag{13}$$

$$\phi_t^-(\infty)=0,$$

here the matrix $G_1(t,\omega)=-R^{-1}(t,\omega)G(t,\omega)$ is of class H and $\det G_1(t,\omega)=\frac{\det G(t,\omega)}{\det R(t,\omega)}\neq 0$ on L_1, $F^*(t,\omega)=R^{-1}(t,\omega)F(t,\omega)$. Since the determinant $\det G_1(t,\omega)\neq 0$ and is a real function for any values t, ω, we have $\chi=\operatorname{Ind}\det G_1(t,\omega)=0$. Let $\mathrm{X}_t^\beta(z_2)=(\mathrm{X}_{1t}^\beta(z_2),\mathrm{X}_{2t}^\beta(z_2))$, $\beta=1,2$, be the canonical system of the homogeneous problem solutions, and $\mathrm{X}_t(z_2)$ be the corresponding canonical matrix ([1], pp. 427–430) of type

$$\mathrm{X}_t(z_2)=\left\|\mathrm{X}_{\alpha t}^\beta(z_2)\right\|=\begin{pmatrix} \mathrm{X}_{1t}^1(z_2) & \mathrm{X}_{1t}^2(z_2) \\ \mathrm{X}_{2t}^1(z_2) & \mathrm{X}_{2t}^2(z_2) \end{pmatrix}.$$

The canonical system of solutions has the following properties:

1) The canonical matrix is normal, i.e., its determinant $\Delta_t(z_2)=\det \mathrm{X}_t(z_2)$ vanishes nowhere in the finite part of the plane z_2;

2) let $(-\chi_\beta)$ denote the order of the solution $\mathrm{X}_t^\beta(z_2)$ at infinity. Then the determinant $\Delta^0(z_2)=\det\left\|z_2^{\chi_\beta}\mathrm{X}_{t\alpha}^\beta(z_2)\right\|=z_2^{\chi_1+\chi_2}\Delta_t(z_2)$ is non-zero for $z_2=\infty$;

3) $\mathrm{X}_t^+(\omega)=G_1(t,\omega)\mathrm{X}_t^-(\omega)$.

Put $G_1(t,\omega)$ into (13) and obtain

$$[\mathrm{X}_t^+(\omega)]^{-1}\phi_t^+(\omega)=[\mathrm{X}_t^-(\omega)]^{-1}\phi_t^-(t)+[\mathrm{X}_t^+(\omega)]^{-1}F^*(t,\omega).$$

Since the index of the problem $\chi=0$, the solution of the problem under consideration exists and is given by

$$\phi_t(z_2)=\frac{\mathrm{X}_t(z_2)}{2\pi e_{12}}\int_{L_k}\frac{[\mathrm{X}_t^+(\omega)]^{-1}F^*(t,\omega)d\omega}{t-z_2}. \tag{14}$$

The solution to problem (8), (9) we derive from (10) by formulas

$$\Phi_k^{\pm\pm}(z_1,z_2)=\pm\frac{1}{2\pi e_{12}}\int_{L_1}\frac{\phi_{kt}^+(z_2)dt}{t-z_1},\ \Phi_k^{\pm\mp}(z_1,z_2)=\pm\frac{1}{2\pi e_{12}}\int_{L_1}\frac{\phi_{kt}^-(z_2)dt}{t-z_1}. \tag{15}$$

3. Algebra $R_{2,0}$. The algebra $R_{2,0}$ is a real associative noncommutative algebra of dimension $m=4$, generated by vectors e_1, e_2. The basis of the algebra consists of elements e_0, e_1, e_2, e_{12}, here e_0 is the algebra unity, $e_{12} = e_1e_2$, and the elements e_1, e_2, e_{12} possess the properties: $e_1^2 = e_2^2 = e_0$, $e_{12}^2 = -e_0$, $e_i \cdot e_j + e_j \cdot e_i = 0$, $i \neq j$. These relations define the multiplication operation in $R_{2,0}$.

An arbitrary element of an algebra can be represented in real and complex form $w = x_0e_0 + x_1e_1 + x_2e_2 + x_{12}e_{12} = e_0z_1 + e_1z_2$, here $z_1 = x_0e_0 + x_{12}e_{12}$, $z_2 = x_1e_0 + x_2e_{12}$ are complex numbers. The conjugate element $\overline{w}$ takes the form $\overline{w} = x_0e_0 + x_1e_1 + x_2e_2 - x_{12}e_{12} = \overline{z_1}e_0 - \overline{z_2}e_1$, here $\overline{z_1} = x_0e_0 - x_{12}e_{12}$, $\overline{z_2} = x_1e_0 - x_2e_{12}$. Zero divisors in $R_{2,0}$ are defined either by the relation (see [20]) $|z_1|^2 = |z_2|^2$ or by $x_0^2 + x_{12}^2 = x_1^2 + x_2^2$.

Denote by $f(w) = f_0(w)e_0 + f_1(w)e_1 + f_2(w)e_2 + f_{12}(w)e_{12}$ a function with values in algebra $R_{2,0}$, defined in the domain $\Omega \subset \mathbb{R}^4$, and by

$$\overline{D} = \frac{1}{4}\left(e_0\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_{12}\frac{\partial}{\partial x_{12}}\right)$$

the differential operator. The function $f(w)$ and the differential operator $\overline{D}$ can be represented in the complex form: $f(w) = \Phi_1(z_{1,}, z_2)e_0 + \Phi_2(z_1, z_2)e_1$, here $\Phi_1(z_1, z_2) = f_0e_0 + f_{12}e_{12}, \Phi_2(z_1, z_2) = f_1e_0 - f_2e_{12}$; $\overline{D} = \frac{1}{2}(e_0\frac{\partial}{\partial \overline{z_1}} + e_1\frac{\partial}{\partial \overline{z_2}})$, here $\frac{\partial}{\partial \overline{z_1}} = \frac{1}{2}(e_0\frac{\partial}{\partial x_0} + e_{12}\frac{\partial}{\partial x_{12}}), \frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}(e_0\frac{\partial}{\partial x_1} + e_{12}\frac{\partial}{\partial x_2})$. Let ${}^{12}\overline{D} = \frac{1}{2}(e_0\frac{\partial}{\partial \overline{z_1}} - e_1\frac{\partial}{\partial \overline{z_2}})$ be the differential operator. A function $f(w) \in F^1_{2,0}(\Omega)$ is [19] strongly regular on the left if $\overline{D} \cdot f = 0$, ${}^{12}\overline{D} \cdot f = 0$, $\Omega \subset \mathbb{R}^4$. Conditions of strong regularity [19] are equivalent to the equalities

$$\frac{\partial \Phi_1}{\partial \overline{z_1}} = \frac{\partial \Phi_1}{\partial \overline{z_2}} = \frac{\partial \Phi_2}{\partial \overline{z_1}} = \frac{\partial \Phi_2}{\partial \overline{z_2}} = 0,$$

which mean that the functions $\Phi_1(z_1, z_2)$, $\Phi_2(z_1, z_2)$ are analytic in the variables z_1, z_2.

3.1. Riemann boundary-value problem (3) for a regular function $f(w) = \Phi_1(z_k)e_0 + \Phi_2(z_k)e_1$ with Clifford-valued coefficients is studied similarly to the previous situations. We consider the spatial case.

Let $L^2 = L_1 \times L_2$ be the common frame of four domains $D^{\pm\pm} = D_1^{\pm} \times D_2^{\pm}$. Find a strongly regular function $f(z_1, z_2) = \Phi_1(z_1, z_2)e_0 + \Phi_2(z_1, z_2)e_1$ by conditions

$$f^{++}(t,\omega)A(t,\omega) + f^{-+}(t,\omega)B(t,\omega) + f^{+-}(t,\omega)C(t,\omega) + f^{--}(t,\omega)D(t,\omega) = F(t,\omega), \tag{16}$$

$$f^{\pm=}(z_1,\infty) = 0,\ \ z_1 \in D_1^{\pm},\ \ f^{=\pm}(\infty, z_2) = 0,\ \ z_2 \in D_2^{\pm}, \tag{17}$$

here the variables $(t,\omega) \in L^2 = L_1 \times L_2$, $L_1 = \partial D_1$, $L_2 = \partial D_2$, and the coefficients $A(t,\omega)$, $B(t,\omega)$, $C(t,\omega)$, $D(t,\omega)$ are Clifford-valued functions that are non-zero on L^2, components of which belong to the class $H(L^2)$, $F(t,\omega) \in H(L^2)$.

Consider the degenerate case of the problem for $A(t,\omega) = B(t,\omega)$, $C(t,\omega) = D(t,\omega)$. Represent the functions $A(t,\omega)$, $C(t,\omega)$, $F(t,\omega)$ in complex form. By equalities (10) boundary condition (16), (17) turns into

$$\phi_{1t}^{+}(\omega)A_1(t,\omega) + \phi_{2t}^{+}(\omega)\overline{A_2(t,\omega)} + \phi_{1t}^{-}(\omega)C_1(t,\omega) + \phi_{2t}^{-}(\omega)\overline{C_2(t,\omega)} = F_1(t,\omega),$$

$$\phi_{1t}^{+}(\omega)A_2(t,\omega) + \phi_{2t}^{+}(\omega)\overline{A_1(t,\omega)} + \phi_{1t}^{-}(\omega)C_2(t,\omega) + \phi_{2t}^{-}(\omega)\overline{C_1(t,\omega)} = F_2(t,\omega), \tag{18}$$

$$\phi_{1t}^{-}(\infty) = 0,\ \ \phi_{2t}^{-}(\infty) = 0.$$

We represent boundary conditions (18) in the matrix form

$$R(t,\omega)\phi_t^{+}(\omega) + G(t,\omega)\phi_t^{-}(\omega) = F(t,\omega), \tag{19}$$

$$\phi_t^{-}(\infty) = 0,$$

here

$$R(t,\omega) = \begin{pmatrix} A_1(t,\omega) & \overline{A_2(t,\omega)} \\ A_2(t,\omega) & \overline{A_1(t,\omega)} \end{pmatrix},\ \ G(t,\omega) = \begin{pmatrix} C_1(t,\omega) & \overline{C_2(t,\omega)} \\ C_2(t,\omega) & \overline{C_1(t,\omega)} \end{pmatrix},$$

$$\phi_t^+(\omega) = \begin{pmatrix} \phi_{1t}^+(\omega) \\ \phi_{2t}^+(\omega) \end{pmatrix}, \quad \phi_t^-(\omega) = \begin{pmatrix} \phi_{1t}^-(\omega) \\ \phi_{2t}^-(\omega) \end{pmatrix}, \quad F(t,\omega) = \begin{pmatrix} F_1(t,\omega) \\ F_2(t,\omega) \end{pmatrix},$$

$$\phi_t^-(\infty) = (\phi_{1t}^-(\infty), \phi_{2t}^-(\infty)).$$

Let us find the solution to problem (19). Determinants of matrices $R(t,\omega), G(t,\omega)$ equal $\det R(t,\omega) = |A_1(t,\omega)|^2 - |A_2(t,\omega)|^2$, $\det G(t,\omega) = |C_1(t,\omega)|^2 - |C_2(t,\omega)|^2$, respectively. Let us consider the case $\det R(t,\omega) \neq 0$, $\det G(t,\omega) \neq 0$.

Multiplying (19) by the inverse matrix $R^{-1}(t,\omega)$ we obtain an inhomogeneous Riemann boundary-value problem for a vector-valued function

$$\phi_t^+(\omega) - G_1(t,\omega)\phi_t^-(\omega) = F^*(t,\omega), \quad \omega \in L_2,$$

$$\phi_t^-(\infty) = 0,$$

here $G_1(t,\omega) = -R^{-1}(t,\omega)G(t,\omega)$ is of class H and $\det G_1(t,\omega) = \frac{\det G(t,\omega)}{\det R(t,\omega)} \neq 0$ on L_2 for any value of the parameter t, $F^*(t,\omega) = R^{-1}(t,\omega)F(t,\omega)$. Since the determinant $\det G_1(t,\omega)$ is a real function for any values t,ω, we have $\chi = \operatorname{Ind}\det G_1(t,\omega) = 0$. Degenerate Riemann problem (16), (17) has a solution given by formulas (14), (15).

3.2. *Special partial cases.* Consider problem (16), (17) for $\det R(t,\omega) = 0$, $\det G(t,\omega) = 0$. Then $|A_1(t,\omega)| = |A_2(t,\omega)|$, $|C_1(t,\omega)| = |C_2(t,\omega)|$.

Let $A(t,\omega)$, $C(t,\omega)$ be spinors. It is known that the spinor space V is a left ideal in $R_{2,0}$, i.e., multiplying an arbitrary element of the algebra $R_{2,0}$ by a spinor, we obtain an element of the space V. Thus, in order for the boundary-value problem(16), (17) to have a solution, a function $F(t,\omega)$ should be a spinor. The basis of the spinors in $R_{2,0}$ ([21], P. 15) is formed by the elements $f_0 = \frac{e_0+e_1}{2}$, $f_1 = \frac{e_2-e_{12}}{2}$. The coefficients $A(t,\omega)$, $C(t,\omega)$, $F(t,\omega)$ are representable in the form $A(t,\omega) = A_1(t,\omega)(e_0+e_1)$, $C(t,\omega) = C_1(t,\omega)(e_0+e_1)$, $F(t,\omega) = F_1(t,\omega)(e_0+e_1)$. Degenerate boundary-value problem (16), (17) is represented by the equalities

$$\phi_{1t}^+(\omega)A_1(t,\omega) + \phi_{2t}^+(\omega)\overline{A_1(t,\omega)} + \phi_{1t}^-(\omega)C_1(t,\omega) + \phi_{2t}^-(\omega)\overline{C_1(t,\omega)} = F_1(t,\omega), \quad \omega \in L_2, \tag{20}$$

$$\phi_{1t}^-(\infty) = 0, \phi_{2t}^-(\infty) = 0.$$

Problem (20) has an infinite set of solutions.

In the general case we have $A_1(t,\omega) = |A_1(t,\omega)|\, e^{e_{12}\theta_1}$, $A_2(t,\omega) = |A_1(t,\omega)|\, e^{e_{12}\theta_2}$, $C_1(t,\omega) = |C_1(t,\omega)|\, e^{e_{12}\theta_3}$, $C_2(t,\omega) = |C_1(t,\omega)|\, e^{e_{12}\theta_4}$. In what follows we assume that $\theta_1 = \theta_3$, $\theta_2 = \theta_4$. Equations (18) turn into

$$|A_1(t,\omega)|\, e^{e_{12}\frac{\theta_1-\theta_2}{2}} \left(\phi_{1t}^+(\omega)e^{e_{12}\frac{\theta_1+\theta_2}{2}} + \phi_{2t}^+(\omega)e^{-e_{12}\frac{\theta_1+\theta_2}{2}}\right) + |C_1(t,\omega)|\, e^{e_{12}\frac{\theta_1-\theta_2}{2}} \left(\phi_{1t}^-(\omega)e^{e_{12}\frac{\theta_1+\theta_2}{2}} + \phi_{2t}^-(\omega)e^{-e_{12}\frac{\theta_1+\theta_2}{2}}\right) = F_1(t,\omega),$$

$$|A_1(t,\omega)|\, e^{e_{12}\frac{\theta_2-\theta_1}{2}} \left(\phi_{1t}^+(\omega)e^{e_{12}\frac{\theta_1+\theta_2}{2}} + \phi_{2t}^+(\omega)e^{-e_{12}\frac{\theta_1+\theta_2}{2}}\right) + |C_1(t,\omega)|\, e^{e_{12}\frac{\theta_2-\theta_1}{2}} \left(\phi_{1t}^-(\omega)e^{e_{12}\frac{\theta_1+\theta_2}{2}} + \phi_{2t}^-(\omega)e^{-e_{12}\frac{\theta_1+\theta_2}{2}}\right) = F_2(t,\omega).$$

If $F_2(t,\omega) = F_1(t,\omega)e^{e_{12}(\theta_2-\theta_1)}$, then degenerate boundary-value problem (16), (17) is equivalent to the equalities

$$\phi_{1t}^+(\omega)A_1^*(t,\omega) + \phi_{2t}^+(\omega)\overline{A_1^*(t,\omega)} + \phi_{1t}^-(\omega)C_1^*(t,\omega) + \phi_{2t}^-(\omega)\overline{C_1^*(t,\omega)} = F_1^*(t,\omega), \quad \omega \in L_2,$$

$$\phi_{1t}^-(\infty) = 0, \phi_{2t}^-(\infty) = 0,$$

here $A_1^*(t,\omega) = |A_1(t,\omega)|\, e^{e_{12}\frac{\theta_1+\theta_2}{2}}$, $C_1^*(t,\omega) = |C_1(t,\omega)|\, e^{e_{12}\frac{\theta_1+\theta_2}{2}}$, $F_1^*(t,\omega) = |F_1(t,\omega)|\, e^{e_{12}\frac{\theta_1+\theta_2}{2}}$. This problem also has an infinite number of solutions.

4. The Pauli algebra $R_{3,0}$. The Pauli algebra $R_{3,0}$ is a real associative noncommutative algebra of dimension $m = 8$, generated by the vectors e_1, e_2, e_3. The basis of the algebra is formed by the elements $\{e_\alpha\}_{\alpha\in\Gamma_3} = \{e_0, e_1, e_2, e_{12}, e_3, e_{13}, e_{23}, e_{123}\}$, here Γ_3 is a collection of subsets of the set $\{1,2,3\}$, e_0 is the algebra unity, $e_{ij} = e_ie_j$, $e_{123} = e_1e_2e_3$, and the elements e_1, e_2, e_3 have the properties

$$e_1^2 = e_2^2 = e_3^2 = e_0, \;\; e_ie_j + e_je_i = 0, \;\; i \neq j, i, j = 1,2,3. \tag{21}$$

Relation (21) yields that $e_{12}^2 = e_{13}^2 = e_{23}^2 = e_{123}^2 = -e_0$, e_{123} commutes with all elements of the basis. The basis of the Pauli algebra can be decomposed into B-sets of two elements [16]: $\{e_\alpha\}_{\alpha\in\Gamma_3} = B_0 \sqcup B_1 \sqcup B_2 \sqcup B_3$, $B_0 = \{e_0, e_{123}\}$, $B_1 = \{e_1, e_{23}\}$, $B_2 = \{e_2, e_{23}\}$, $B_3 = \{e_3, e_{12}\}$.

An arbitrary element of an algebra can be represented in real and complex form $w = \sum\limits_{\alpha\in\Gamma_3} x_\alpha e_\alpha = z_0e_0 + z_1e_{12} + z_2e_{13} + z_3e_{23}$, here $z_0 = x_0e_0 + x_{123}e_{123}$, $z_1 = x_{12}e_0 - x_3e_{123}$, $z_2 = x_{13}e_0 + x_2e_{123}$, $z_3 = x_{23}e_0 - x_1e_{123}$ are complex numbers (e_{123} plays the part of the imaginary unit).

Denote by

$$f(w) = \Phi_0(w)e_0 + \Phi_1(w)e_{12} + \Phi_2(w)e_{13} + \Phi_3(w)e_{23},$$

$$\Phi_0(w) = f_0e_0 + f_{123}e_{123}, \;\; \Phi_1(w) = f_{12}e_0 - f_3e_{123}, \;\; \Phi_2(w) = f_{13}e_0 + f_2e_{123}, \;\; \Phi_3(w) = f_{23}e_0 - f_1e_{123}$$

a function with values in the algebra $R_{3,0}$, and defined in the domain $\Omega \subset \mathbb{R}^8$, and by

$$\overline{D} = \frac{1}{4}\left(e_0\frac{\partial}{\partial \overline{z}_0} + e_{12}\frac{\partial}{\partial \overline{z}_1} + e_{13}\frac{\partial}{\partial \overline{z}_2} + e_{23}\frac{\partial}{\partial \overline{z}_3}\right),$$

$$\frac{\partial}{\partial \overline{z_0}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_0} + e_{123}\frac{\partial}{\partial x_{123}}\right), \;\; \frac{\partial}{\partial \overline{z_1}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_{12}} - e_{123}\frac{\partial}{\partial x_3}\right),$$
$$\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_{13}} + e_{123}\frac{\partial}{\partial x_2}\right), \;\; \frac{\partial}{\partial \overline{z_3}} = \frac{1}{2}\left(e_0\frac{\partial}{\partial x_{23}} - e_{123}\frac{\partial}{\partial x_3}\right),$$

the differential operator.

A function $f(w) \in F_{3,0}^1(\Omega)$ is said to be strongly regular on the left [19] if for all $\nu \in \Gamma_2$

$${}^{\nu}\overline{D}\cdot f = 0, \;\; {}^{\nu}\overline{D} = \varepsilon_\nu e_\nu \overline{D} e_\nu.$$

Conditions of strong regularity [19] are equivalent to the equalities

$$\frac{\partial \Phi_i}{\partial \overline{z_j}} = 0, \;\; i, j = 0,1,2,3,$$

which mean that $\Phi_i(z_0, z_1, z_2, z_3)$ are analytic functions with respect to variables z_0, z_1, z_2, z_3.

4.1. The Riemann boundary-value problem in $R_{3,0}$. Consider the case of the Clifford-valued function $f(w)$ depending only on one complex variable z_k, i.e., $f(w) = \Phi_0(z_k)e_0 + \Phi_1(z_k)e_{12} + \Phi_2(z_k)e_{13} + \Phi_3(z_k)e_{23}$. Let L_k be a simple smooth closed contour bounding in the complex variable plane a domain $D_k^+ \subset \overline{\mathbb{C}}$, D_k^- completes $D_k^+ \bigcup L_k$ to the extended complex plane $\overline{\mathbb{C}}$.

Statement of the Riemann problem: Find a regular function $f(w)$ by conditions

$$f^+(t)A(t) + f^-(t)C(t) = F(t), \;\; t \in L_k, \tag{22}$$

$$f^-(\infty) = 0,$$

here $A(t)$, $C(t)$, $F(t)$ are Clifford-valued functions that are non-zero on L_k, the components of which belong to the class $H(L_k)$. The functions $A(t)$, $C(t)$, $F(t)$ are representable in the complex form:

$$A(t) = A_0(t)e_0 + A_1(t)e_{12} + A_2(t)e_{13} + A_3(t)e_{23}, \;\; C(t) = C_0(t)e_0 + C_1(t)e_{12} + C_2(t)e_{13} + C_3(t)e_{23}, \tag{23}$$

$$F(t) = F_0(t)e_0 + F_1(t)e_{12} + F_2(t)e_{13} + F_3(t)e_{23}.$$

Since e_{123} commutes with all elements of the Pauli algebra, we rewrite boundary condition (22) in the matrix form

$$R(t)\Phi^+(t) + G(t)\Phi^-(t) = F(t), \;\; t \in L_k, \tag{24}$$

$$\Phi^-(\infty) = 0,$$

$$R(t) = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix}, \;\; G(t) = \begin{pmatrix} C_0 & -C_1 & -C_2 & -C_3 \\ C_1 & C_0 & -C_3 & C_2 \\ C_2 & C_3 & C_0 & -C_1 \\ C_3 & -C_2 & C_1 & C_0 \end{pmatrix},$$

$$\Phi^+(t) = \begin{pmatrix} \Phi_0^+(t) \\ \Phi_1^+(t) \\ \Phi_2^+(t) \\ \Phi_3^+(t) \end{pmatrix}, \;\; \Phi^-(t) = \begin{pmatrix} \Phi_0^-(t) \\ \Phi_1^-(t) \\ \Phi_2^-(t) \\ \Phi_3^-(t) \end{pmatrix}, \;\; F(t) = \begin{pmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ F_3(t) \end{pmatrix}.$$

We obtain a matrix problem for a vector-valued function. We note that the matrices $R(t)$ and $G(t)$ are orthogonal, $\det R(t) = (A_0^2 + A_1^2 + A_2^2 + A_3^2)^2$, $\det G(t) = (C_0^2 + C_1^2 + C_2^2 + C_3^2)^2$. Further on we assume that $\det R(t) \neq 0$, $\det G(t) \neq 0$. Multiplying (24) by the inverse matrix R^{-1}(t) we obtain an inhomogeneous Riemann boundary-value problem for the vector-function $\Phi(z)$:

$$\Phi^+(t) - G_1(t)\Phi^-(t) = F^*(t), \;\; t \in L_k, \tag{25}$$

$$\Phi^-(\infty) = 0,$$

here the matrix $G_1(t) = -R^{-1}(t)G(t)$ is of class H and $\det G_1(t) = \frac{\det G(t)}{\det R(t)} \neq 0$ on L_k, $F^*(t) = R^{-1}(t)F(t)$. Denote by $\chi = \operatorname{Ind}\det G_1(t)$. Let $\mathrm{X}^\beta(z_k) = (\mathrm{X}_0^\beta(z_k),\; \mathrm{X}_1^\beta(z_k),\; \mathrm{X}_2^\beta(z_k)\mathrm{X}_3^\beta(z_k))$, $\beta = 0, 1, 2, 3$ be the canonical system of solutions to the homogeneous problem, and $\mathrm{X}(z_k)$ be the corresponding canonical matrix ([1], pp. 427–430) of the form

$$\mathrm{X}(z_k) = \|\mathrm{X}_\alpha^\beta(z_k)\| = \begin{pmatrix} \mathrm{X}_0^0 & \mathrm{X}_0^1 & \mathrm{X}_0^2 & \mathrm{X}_0^3 \\ \mathrm{X}_1^0 & \mathrm{X}_1^1 & \mathrm{X}_1^2 & \mathrm{X}_1^3 \\ \mathrm{X}_2^0 & \mathrm{X}_2^1 & \mathrm{X}_2^2 & \mathrm{X}_2^3 \\ \mathrm{X}_3^0 & \mathrm{X}_3^1 & \mathrm{X}_3^2 & \mathrm{X}_3^3 \end{pmatrix}.$$

The canonical system has the following properties:

1) the canonical matrix is normal, i.e., its determinant $\Delta(z_k) = \det \mathrm{X}(z_k)$ does not vanish anywhere in the finite part of the plane;

2) let $(-\chi_\beta)$ be the order of the solution $\mathrm{X}^\beta(z_k)$ at infinity. Then the determinant $\Delta^0(z_k) = \det \left\| z_k^{\chi_\beta} \mathrm{X}_\alpha^\beta(z_k) \right\| = z_k^{\chi_0+\chi_1+\chi_2+\chi_3} \Delta(z_k)$ is non-zero for $z_k = \infty$;

3) $\mathrm{X}^+(t) = G_1(t)\mathrm{X}^-(t)$.

Put G_1(t) into (25) and obtain

$$[\mathrm{X}^+(t)]^{-1}\Phi^+(t) = [\mathrm{X}^-(t)]^{-1}\Phi^-(t) + [\mathrm{X}^+(t)]^{-1}F^*(t).$$

Hence all the solutions to the problem under consideration are given by the formula

$$\Phi(z_k) = \frac{\mathrm{X}(z_k)}{2\pi e_{123}} \int_{L_k} \frac{[\mathrm{X}^+(t)]^{-1} F^*(t)dt}{t - z_k} + \mathrm{X}(z_k)P(z_k),$$

here $P(z_k) = (P_0(z_k), P_1(z_k), P_2(z_k), P_3(z_k))$ is a vector with arbitrary polynomial coefficients.

If the index $\chi = \chi_0 + \chi_1 + \chi_2 + \chi_3$ is nonnegative, then ([1], pp. 440–445) the solution vanishes at infinity if and only if $P(z_k)$ is a vector whose components are polynomials of degree at most $\chi - 1$, and if $\chi \leq 0$ then $P(z_k) \equiv 0$. If $\chi < 0$, then the solution vanishes at infinity if we have the solvability conditions

$$\int_{L_k} Q(t)[\mathrm{X}^+(t)]^{-1} F^*(t)dt = 0,$$

$Q(z_k) = (Q_{-\chi_0-1}, Q_{-\chi_1-1}, Q_{-\chi_2-1}, Q_{-\chi_3-1})$, here $Q_\alpha = Q_\alpha(z_k)$ are arbitrary polynomials of degree at most α, $Q_\alpha(z_k) = 0$ for $\alpha < 0$.

4.2. Now consider the case $\det R(t) = 0$, $\det G(t) = 0$. Then [20] the coefficients $A(t)$, $C(t)$ of the boundary-value problem (22) are zero divisors in the algebra $R_{3,0}$. We consider a special case of problem (22), where the coefficients $A(t)$ and $C(t)$ are spinors. Boundary-value problem (22) has a solution if the function $F(t)$ is a spinor. The basis of the spinors ([21], P. 60) in $R_{3,0}$ is formed by the elements $f_0 = \frac{e_0+e_3}{2}$, $f_1 = \frac{e_{23}+e_2}{2}$, $f_2 = \frac{-e_{13}-e_1}{2}$, $f_1 = \frac{e_{12}+e_{123}}{2}$. In the spinor space, the coefficients $A(t)$, $C(t)$, $F(t)$ can be represented as (23), where the coefficients $A_k(t)$, $C_k(t)$, $F_k(t)$ are real functions. Boundary problem (22) can be written in the form of (24), where the matrices $R(t)$, $G(t)$ depend on the real variables, $\det R(t) \neq 0$, $\det G(t) \neq 0$. Multiplying (24) by the inverse matrix R^{-l} (t), we obtain the inhomogeneous Riemann boundary-value problem (25) for the vector-valued function $\Phi(z)$. The index $\chi = \operatorname{Ind} \det G_1(t) = 0$. The solution to the problem is given by the formula

$$\Phi(z_k) = \frac{\mathrm{X}(z_k)}{2\pi e_{123}} \int_{L_k} \frac{[\mathrm{X}^+(t)]^{-1} F^*(t)dt}{t - z_k}.$$

4.3. We consider the case of the function $f(w)$ dependent on two complex variables. For definiteness, we take as variables the variables z_0 and z_1.

Statement of the Riemann problem. Let $L^2 = L_0 \times L_1$ be the common frame of four domains $D^{\pm\pm} = D_1^\pm \times D_2^\pm$. Find a strongly regular function $f(z_0, z_1) = \Phi_0(z_0, z_1)e_0 + \Phi_1(z_0, z_1)e_{12} + \Phi_2(z_0, z_1)e_{13} + \Phi_3(z_0, z_1)e_{23}$ by the conditions

$$f^{++}(t_0, t_1)A(t_0, t_1) + f^{-+}(t_0, t_1)B(t_0, t_1) + f^{+-}(t_0, t_1)C(t_0, t_1) + f^{--}(t_0, t_1)D(t_0, t_1) = F(t_0, t_1),$$

$$f^{\pm=}(z_0, \infty) = 0, \;\; z_0 \in D_0^\pm, \;\; f^{=\pm}(\infty, z_1) = 0, \;\; z_1 \in D_1^\pm, \tag{26}$$

here the variables $(t, \omega) \in L^2 = L_1 \times L_2$, $L_1 = \partial D_1$, $L_2 = \partial D_2$ and the coefficients $A(t_0, t_1)$, $B(t_0, t_1)$, $C(t_0, t_1)$, $D(t_0, t_1)$ are Clifford-valued functions that are not zero on L^2, with components of the class $H(L^2)$, $F(t_0, t_1) \in H(L^2)$.

We consider the degenerate case of $A(t_0, t_1) = B(t_0, t_1)$, $C(t_0, t_1) = D(t_0, t_1)$. Introduce the notation $f^{++}(t_0, t_1) + f^{-+}(t_0, t_1) = \phi_{t_0}^+(t_1)$, $f^{++}(t_0, t_1) + f^{-+}(t_0, t_1) = \phi_{t_0}^+(t_1)$, here $t_0 \in L_0$ is the parameter, and t_1 is the variable. Boudary conditions (26) take the form

$$\phi_{t_0}^+(t_1)A(t_0, t_1) + \phi_{t_0}^-(t_1)C(t_0, t_1) = F(t_0, t_1),$$

$$\phi_\infty(z_1) = \phi_{z_0}(\infty) = 0.$$

Represent the functions $A(t_0, t_1)$, $C(t_0, t_1)$, $F(t_0, t_1)$, $\phi_{t_0}^+(t_1)$, $\phi_{t_0}^-(t_1)$ as (23), and write the boundary conditions in the matrix form

$$R(t_0, t_1)\phi_{t_0}^+(t_1) + G(t_0, t_1)\phi_{t_0}^-(t_1) = F(t_0, t_1), \;\; t_1 \in L_1, \tag{27}$$

$$\phi_\infty(z_1) = \phi_{z_0}(\infty) = 0,$$

$$R(t_0,t_1)=\begin{pmatrix}A_0 & -A_1 & -A_2 & -A_3\\ A_1 & A_0 & -A_3 & A_2\\ A_2 & A_3 & A_0 & -A_1\\ A_3 & -A_2 & A_1 & A_0\end{pmatrix},\ G(t_0,t_1)=\begin{pmatrix}C_0 & -C_1 & -C_2 & -C_3\\ C_1 & C_0 & -C_3 & C_2\\ C_2 & C_3 & C_0 & -C_1\\ C_3 & -C_2 & C_1 & C_0\end{pmatrix},$$

$$\phi^+_{t_0}(t)=\begin{pmatrix}\phi^+_{0t_0}(t_1)\\ \phi^+_{1t_0}(t_1)\\ \phi^+_{2t_0}(t_1)\\ \phi^+_{3t_0}(t_1)\end{pmatrix},\ \phi^-_{t_0}(t)=\begin{pmatrix}\phi^-_{0t_0}(t_1)\\ \phi^-_{1t_0}(t_1)\\ \phi^-_{2t_0}(t_1)\\ \phi^-_{3t_0}(t_1)\end{pmatrix},\ F(t_0,t_1)=\begin{pmatrix}F_0(t_0,t_1)\\ F_1(t_0,t_1)\\ F_2(t_0,t_1)\\ F_3(t_0,t_1)\end{pmatrix}.$$

We then have a matrix problem for the vector function. Note that the matrices $R(t_0,\, t_1)$ and $G(t_0,\, t_1)$ are orthogonal, $\det R(t)=(A_0^2+A_1^2+A_2^2+A_3^2)^2$, $\det G(t)=(C_0^2+C_1^2+C_2^2+C_3^2)^2$. Assume further on that $\det R(t)\neq 0$, $\det G(t)\neq 0$. Multiplying (27) by the inverse matrix $R^{-1}(t_0,\, t_1)$, we obtain an inhomogeneous Riemann boundary-value problem for the vector function $\phi_{t_0}(z_1)$:

$$\phi^+_{t_0}(t_1)-G_1(t_0,t_1)\phi^-_{t_0}(t_1)=F^*(t_0,t_1),\ \ t_1\in L_1, \tag{28}$$

$$\phi_\infty(z_1)=\phi_{z_0}(\infty)=0,$$

here the matrix $G_1(t_0,t_1)=-R^{-1}(t_0,t_1)G(t_0,t_1)$ is of the class H and $\det G_1(t_0,t_1)=\frac{\det G(t_0,t_1)}{\det R(t_0,t_1)}\neq 0$ on L_1, $F^*(t_0,t_1)=R^{-1}(t_0,t_1)F(t_0,t_1)$. Denote by $\chi=\operatorname{Ind}\det G_1(t_0,t_1)$. Let $\mathrm{X}^\beta_{t_0}(z_1)=(\mathrm{X}^\beta_{0t_0}(z_1), \mathrm{X}^\beta_{1t_0}(z_1), \mathrm{X}^\beta_{2t_0}(z_1)\mathrm{X}^\beta_{3t_0}(z_1))$, $\beta=0,1,2,3,$ be the canonical system of solutions to the homogeneous problem, and $\mathrm{X}_{t_0}(z_1)$ be the corresponding canonical matrix of the form

$$\mathrm{X}_{t_0}(z_1)=\|\mathrm{X}^\beta_{\alpha t_0}(z_1)\|=\begin{pmatrix}\mathrm{X}^0_{0t_0} & \mathrm{X}^1_{0t} & \mathrm{X}^2_{0t_0} & \mathrm{X}^3_{0t_0}\\ \mathrm{X}^0_{1t_0} & \mathrm{X}^1_{1t_0} & \mathrm{X}^2_{1t_0} & \mathrm{X}^3_{1t_0}\\ \mathrm{X}^0_{2t_0} & \mathrm{X}^1_{2t_0} & \mathrm{X}^2_{2t_0} & \mathrm{X}^3_{2t_0}\\ \mathrm{X}^0_{3t_0} & \mathrm{X}^1_{3t_0} & \mathrm{X}^2_{3t_0} & \mathrm{X}^3_{3t_0}\end{pmatrix}.$$

Put $G_1(t_0,t_1)$ into (28) and achieve

$$[\mathrm{X}^+_{t_0}(t_1)]^{-1}\phi^+_{t_0}(t_1)=[\mathrm{X}^-_{t_0}(t_1)]^{-1}\phi^-_{t_0}(t_1)+[\mathrm{X}^+_{t_0}(t_1)]^{-1}F^*(t_0,t_1).$$

Hence all the solutions of the problem under consideration are given by the formula

$$\phi_{t_0}(z_1)=\frac{\mathrm{X}_{t_0}(z_1)}{2\pi e_{123}}\int_{L_1}\frac{[\mathrm{X}^+_{t_0}(t_1)]^{-1}F^*(t_0,t_1)dt_1}{t_1-z_1}+\mathrm{X}_{t_0}(z_1)P_{t_0}(z_1),$$

here $P_{t_0}(z_1)=(P_{0t_0}(z_1),P_{1t_0}(z_1),P_{2t_0}(z_1),P_{3t_0}(z_1))$ is a vector with arbitrary polynomial coefficients depending on the variable t_0. The solution to problem (26) we obtain from formulas (15).

Remark. If $\operatorname{Ind}\det G(t_0,t_1)=\operatorname{Ind}\det R(t_0,t_1)$ then $\chi=\operatorname{Ind}\det G(t_0,t_1)$=0 and degenerate problem (26) meets the Noether theory.

5. An arbitrary Clifford algebra $R_{p,q}$. Let $R_{p,q}$ be an arbitrary Clifford algebra. The commutation coefficients $a_{\alpha\beta}$ are determined from the equality $e_\alpha e_\beta=a_{\alpha\beta}e_\beta e_\alpha$. Note that $a_{\alpha\beta}=1$ if the elements e_α, e_β commute with each other, and $a_{\alpha\beta}=-1$ if the elements e_α, e_β anticommute. In order to split the basis of the algebra $R_{p,q}$ into disjoint B-sets of two elements, it is necessary to construct the set $B_0=\{e_0,e_k\}$, where $e_k^2=-1$. If $q\neq 0$, then we can take $e_n, e_n^2=-1$, $n=p+q$ as the generating element; if $q=0$, $e_\tau^2=e_{12\dots n}^2=-1$, then the generating element is e_τ; If $q=0$, $e_\tau^2=1$, then the generating element can be taken as $e_{n-1,n}$, $e_{n-1,n}^2=-1$.

In what follows we consider the Clifford algebra $R_{p,q}$, $q\neq 0$. The basis of the algebra is decomposed into disjoint B-sets of two elements $\{e_\beta\}_{\beta\in\Gamma_n}=\bigsqcup_{\alpha\in\Gamma_{n-1}} B_0^\alpha$, where $B_0=\{e_0,e_n\}$. An arbitrary element, a differential operator, and a function with values in the Clifford algebra can be represented in the form

$$w=\sum_{\alpha\in\Gamma_{n-1}} e_\alpha w_{B_0^\alpha},\ \ f(w)=\sum_{\alpha\in\Gamma_{n-1}} e_\alpha g_{B_0^\alpha}=\sum_{\alpha\in\Gamma_{n-1}}\widetilde{g}_{B_0^\alpha}e_\alpha,\ \ \overline{D}=\frac{1}{2^n}\sum_{\alpha\in\Gamma_{n-1}} e_\alpha\overline{D}_{B_0^\alpha},$$

$$w_{B_0^\alpha}=x_\alpha e_\alpha+x_{\alpha n}e_n,\ \ g_{B_0^\alpha}=f_\alpha e_0+f_{\alpha n}e_n,\ \ \widetilde{g}_{B_0^\alpha}=f_\alpha e_0+a_{\alpha n}f_{\alpha n}e_n,\overline{D_{B_0^\alpha}}=e_0\frac{\partial}{\partial x_\alpha}+e_n\frac{\partial}{\partial x_{\alpha_n}}.$$

Let L_k be a simple smooth closed contour bounding in the complex variable plane the domain $D_k^+\subset\overline{\mathbb{C}}$, D_k^- completes $D_k^+\bigcup L_k$ to the extended complex plane $\overline{\mathbb{C}}$.

Statement of the Riemann problem: Find a regular function $f(w)=\sum\limits_{\alpha\in\Gamma_{n-1}}\Phi_{B_0^\alpha}(z_k)e_\alpha$ by the conditions

$$f^+(t)A(t)+f^-(t)C(t)=F(t),\ \ t\in L_k, \tag{29}$$

$$f^-(\infty)=0,$$

here $A(t)$, $C(t)$, $F(t)$ are Clifford-valued functions that are non-zero on L_k, the components of which belong to the class $H(L_k)$. Functions $A(t)$, $C(t)$, $F(t)$ have the form

$$A(t)=\sum_{\alpha\in\Gamma_{n-1}}A_{B_0^\alpha}(t)e_\alpha,\ \ C(t)=\sum_{\alpha\in\Gamma_{n-1}}C_{B_0^\alpha}(t)e_\alpha,\ \ F(t)=\sum_{\alpha\in\Gamma_{n-1}}F_{B_0^\alpha}(t)e_\alpha.$$

Write boundary condition (29) in the matrix form

$$R(t)\Phi^+(t)+G(t)\Phi^-(t)=F(t),\ \ t\in L_k,$$

$$\Phi^-(\infty)=0,$$

here the matrices $R(t)$, $G(t)$ are determined from the equalities

$$f^+(t)A(t)=\sum_{\alpha\in\Gamma_{n-1}}\Phi^+_{B_0^\alpha}(t)e_\alpha\cdot\sum_{\alpha\in\Gamma_{n-1}}A_{B_0^\beta}(t)e_\beta=\sum_{\alpha\in\Gamma_{n-1}}\Phi^+_{B_0^\alpha}(t)\sum_{\alpha\in\Gamma_{n-1}}\widetilde{A}_{B_0^\beta}(t)e_\alpha e_\beta,$$

$$f^-(t)C(t)=\sum_{\alpha\in\Gamma_{n-1}}\Phi^-_{B_0^\alpha}(t)e_\alpha\cdot\sum_{\alpha\in\Gamma_{n-1}}C_{B_0^\beta}(t)e_\beta=\sum_{\alpha\in\Gamma_{n-1}}\Phi^-_{B_0^\alpha}(t)\sum_{\alpha\in\Gamma_{n-1}}\widetilde{C}_{B_0^\beta}(t)e_\alpha e_\beta,$$

where $\widetilde{A}_{B_0^\beta}(t)=A_\beta e_0+a_{\beta n}A_{\beta n}e_n$, $\widetilde{C}_{B_0^\beta}(t)=C_\beta e_0+a_{\beta n}C_{\beta n}e_n$,

$$\Phi^+(t)=\begin{pmatrix}\Phi_1^+(t)\\ \vdots\\ \Phi^+_{2^n-1}(t)\end{pmatrix},\ \ \Phi^-(t)=\begin{pmatrix}\Phi_1^-(t)\\ \vdots\\ \Phi^-_{2^n-1}(t)\end{pmatrix},\ \ F(t)=\begin{pmatrix}F_1(t)\\ \vdots\\ F_{2^n-1}(t)\end{pmatrix}.$$

If the determinants of the matrices $\det R(t)$ and $\det G(t)$ are non-zero, then repeating verbatim the arguments of Item 1.4 we find the solution to the problem (29) by the formula

$$\Phi(z_k)=\frac{\mathrm{X}(z_k)}{2\pi e_k}\int_{L_k}\frac{[\mathrm{X}^+(t)]^{-1}F^*(t)dt}{t-z_k}+\mathrm{X}(z_k)P(z_k),$$

here $X(z_k)$ is the canonical matrix, $F^*(t) = R^{-1}(t)F(t)$, $P(z_k) = (P_0(z_k), \ldots, P_{2^n-1}(z_k))$ is a vector with arbitrary polynomial coefficients.

If the index $\chi = \sum_{\alpha\in\Gamma_{n-1}} \chi_\alpha$ is nonnegative, then by [1] (pp. 440-445) the solution vanishes at infinity if and only if $P(z_k)$ is a vector whose components are polynomials of degree at most $\chi - 1$, and if $\chi \leq 0$, then $P(z_k) \equiv 0$. If $\chi < 0$, then the solution vanishes at infinity if it meets the solvability conditions

$$\int_{L_k} Q(t)[\mathrm{X}^+(t)]^{-1}F^*(t)dt = 0,$$

$Q(z_k) = (Q_{-\chi_0-1}, \ldots, Q_{-\chi_{2^n-1}-1})$, here $Q_\alpha = Q_\alpha(z_k)$ are arbitrary polynomials of degree at most α, $Q_\alpha(z_k) = 0$ for $\alpha < 0$.

REFERENCES

1. Muskhelishvili, N. I. *Singular Integral Equations* (Springer, 1958; Nauka, Moscow, 1968).
2. Vekua, N. P. *Systems of Singular Integral Equations* (Gordon & Breach Science Publ., 1967; Nauka, Moscow, 1970).
3. Kakichev, V. A. *Methods for Solving Some Boundary-Value Problems for Analytic Functions of Two Complex Variables* (Tyumen University Press, Tyumen, 1978) [in Russian].
4. Simonenko, I. B. “On the Question of the Solvability of Bisingular and Polysingular Equations”, Functional Analysis and Its Applications **5**, No. 1, 81–83 (1971).
5. Pilidi, V. S. “On Multidimensional Bisingular Eperators”, Sov. Phys. Dokl. **201**, No. 4, 787–789 (1971) [in Russian].
6. Bernstein, S. “On the Left Linear Riemann Problem in Clifford Analysis”, Bull. Belg. Math. Soc. **3**, No. 5, 557–576 (1996).
7. Abreu-Blaya, R., Bory-Reyes, J., and Peña Peña, D. “Jump Problem and Removable Singularities for Monogenic Functions”, J. Geom. Anal. **17**, No. 1, 1–14 (2007).
8. Kats, B. “On Solvability of the Jump Problem”, J. Math. Anal. Appl. **356**, No. 2, 577-581 (2009).
9. Abreu-Blaya, R. and Bory-Reyes, J. “Criteria for Monogenicity of Clifford Algebra-Valued Functions on Fractal Domains”, Arch. Math. (Basel) **95**, No. 1, 45–51 (2010).
10. Abreu-Blaya, R., Bory-Reyes, J., and Kats, B. A. “Approximate Dimension Applied to Criteria for Monogenicity on Fractal Domains”, Bull. Braz. Math. Soc., New Ser. **43**, No. 4, 529–544 (2012).
11. Abreu-Blaya, R., Bory-Reyes, J., and Kats, B. “On the Solvability of the Jump Problem in Clifford Analysis”, J. Math. Sci. (N. Y.) **189**, No. 1, 1–9 (2013).
12. Kuznetsov, S. P., Mochalov, V. V., Chuev, V. P. “On the Riemann Boundary-Value Problem for Quaternion-Valued Functions”, in *Mathematical Models and Their Applications*, No. 13, 16–24 (Chuvash University, Cheboksary, 2011) [in Russian].
13. Kuznetsov, S. P., Mochalov, V. V., Chuev, V. P. “On the Riemann Boundary-Value Problem for Clifford-Calued Functions in the Algebra $R_{2,0}$”, in *Mathematical Models and Their Applications*, No. 14, 28–33 (Chuvash University, Cheboksary, 2012) [in Russian].
14. Kuznetsov, S. P., Mochalov, V. V., Chuev, V. P. “On the Riemann Boundary-Value Problem for Clifford-Valued Functions in the Pauli Algebra”, in *Mathematical Models and Their Applications*, No. 15, 46–52 (Chuvash University, Cheboksary, 2013) [in Russian].
15. Marchuk, N. G. *Introduction to the Theory of Clifford Algebras* (Fazis, Moscow, 2012) [in Russian].
16. Kuznetsov, S. P. “B-Sets in Clifford Algebras”, in *Studies on Boundary-Value Problems and Their Applications*, 91–96 (Chuvash University, Cheboksary, 1992) [in Russian].
17. Kuznetsov, S. P., Mochalov, V. V. “Automorphisms of Clifford Algebra and Strong Regular Functions”, Russian Mathematics **36** (10), 81–84 (1992).
18. Kuznetsov, S. P., Mochalov, V. V. “Representation of the Laplace Operator and Strongly Regular Functions in Clifford Algebras”, in *Relevant Problems of Mathematics and Mechanics*, 56–70 (Chuvash University, Cheboksary, 1995) [in Russian)].
19. Kuznetsov, S. P., Mochalov, V. V. “The Structure of Certain Classes of Regular Functions With Values in the Clifford Algebra”, Izv. National Academy of Sciences and Arts of the Chuvash Republic, No. 3, 19–29 (2003) [in Russian)].
20. Kuznetsov, S. P., Mochalov, V. V., Chuev, V. P. “Clifford Groups and Zero Divisors in Clifford Algebras”, Vestn. Chuvash. Univ, Natural and Technical Science, No, 3, 164-171 (2015) [in Russian)].
21. Lounesto P. *Clifford Algebras and Spinors* (Cambridge Univ. Press., Cambridge, 2011).

Translated by P. N. Ivan'shin