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On Complete Sublattices of Formations of Finite Groups

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Abstract—We prove that the lattice of all τ-closed saturated formations of finite groups is a complete sublattice of the lattice of all τ-closed solubly saturated formations of finite groups.

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INTRODUCTION

All groups in question are finite. It is known that the set of all formations $\mathcal{F}$ is a complete lattice with respect to the inclusion $\subseteq$. Recall that a nonempty set of formations Θ is called a *complete lattice of formations* ([1], P. 151) if the intersection of any family of formations from Θ belongs to Θ and there is a formation $\mathfrak{F}$ in Θ such that $\mathfrak{H} \subseteq \mathfrak{F}$ for any formation $\mathfrak{H}$ from Θ. Various families of formations can form complete lattices, in particular, the family of all saturated formations $\mathcal{L}$ and the family of all solubly saturated formations $\mathcal{C}$ ([1], P. 151; [2], P. 97).

It is well-known that a sublattice of a complete lattice $\mathcal{P}$ may be a complete lattice and be not a complete sublattice of $\mathcal{P}$ ([3], Chap. V, P. 195). A *sublattice* $\mathcal{H}$ of a complete lattice $\mathcal{P}$ is called *complete* if $\sup_{\mathcal{P}} \mathcal{X} \in \mathcal{H}$ and $\inf_{\mathcal{P}} \mathcal{X} \in \mathcal{H}$ for any nonempty subset $\mathcal{X} \subseteq \mathcal{H}$.

This being the case, we have $\sup_{\mathcal{H}} \mathcal{X} = \sup_{\mathcal{P}} \mathcal{X}$ and $\inf_{\mathcal{H}} \mathcal{X} = \inf_{\mathcal{P}} \mathcal{X}$.

The property of completeness for sublattices of formations was studied in [1], [4–6], [7] (P. 273). Note that the fact that sublattices of saturated and solubly saturated formations are complete was established due to functor methods in the study of formations (see A. N. Skiba's monograph [1]). A formation $\mathfrak{F}$ is called *saturated* if the condition $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. A formation $\mathfrak{F}$ is called *solubly saturated* if the condition $G/\Phi(R(G)) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. The symbol $R(G)$ denotes the greatest soluble normal subgroup of a group G. For a nonempty saturated formation $\mathfrak{F}$, it is accepted to write $\mathfrak{F} = LF(f)$ and say that $\mathfrak{F}$ is a saturated formation with local satellite f ([2], P. 20; [8], P. 356).

In [9], A. N. Skiba introduced multiply saturated and totally saturated formations. Every formation is considered to be 0-*tuply saturated*. For $n \geq 1$, a formation $\mathfrak{F}$ is called an n-*tuply saturated* if $\mathfrak{F} = LF(f)$, where all nonempty values of the local satellite f are $(n-1)$-tuply saturated formations. A formation is called *totally saturated* if it is n-tuply saturated for all natural numbers n.

Let $\tau(G)$ be a system of subgroups in a group G. It is said that τ is a *subgroup functor* (in the sense of A. N. Skiba, [1], P. 16) if the following conditions hold:

1) $G \in \tau(G)$;

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2) for any epimorphism $\varphi : A \mapsto B$ and any groups $H \in \tau(A)$, $T \in \tau(B)$, we have

$$H^{\varphi} \in \tau(B) \text{ and } T^{\varphi^{-1}} \in \tau(A).$$

If $\tau(G) = \{G\}$, then the functor τ is called *trivial*. We will consider only subgroup functors τ such that, for any group G, all subgroups from $\tau(G)$ are subnormal in G. A formation $\mathfrak{F}$ is called τ*-closed* ([1], P. 23) if $\tau(G) \subseteq \mathfrak{F}$ for any group G from $\mathfrak{F}$.

In [1] (P. 158) it is proved that the lattice of all τ-closed n-tuply saturated formations $\mathcal{L}_n^{\tau}$ is a complete sublattice of the lattice of all n-tuply saturated formations $\mathcal{L}_n$ and the following question was posed.

Question 1 ([1], question 4.1.15, P. 159)**.** Is it true that the lattice of all τ-closed totally saturated formations $\mathcal{L}_{\infty}^{\tau}$ is a complete sublattice of the lattice of all totally saturated formations $\mathcal{L}_{\infty}$?

The affirmative answer to Question 1 was obtained by V. G. Safonov and L. A. Shemetkov [4]. The following analog of Question 1 is of interest.

Question 2. Is it true that the lattice of all τ-closed saturated formations $\mathcal{L}^{\tau}$ is a complete sublattice of the lattice of all τ-closed solubly saturated formations $\mathcal{C}^{\tau}$?

The answer to Question 2 is the main result of this paper. We prove the following

Theorem. *The lattice of all τ-closed saturated formations $\mathcal{L}^{\tau}$ is a complete sublattice of the lattice of all τ-closed solubly saturated formations $\mathcal{C}^{\tau}$.*

As a consequence of the theorem, in the case when τ is a trivial subgroup functor, we obtain the following

Corollary ([6], theorem 1.1)**.** The lattice of all saturated formations $\mathcal{L}$ is a complete sublattice of the lattice of all solubly saturated formations $\mathcal{C}$.

We will use the standard terminology adopted in [1], [2], [7], [8], [10]–[14].

1. PRELIMINARIES

Recall that $\pi(G)$ denotes the set of all prime divisors of the order of a group G. For an arbitrary totality of groups $\mathfrak{X}$, $\mathrm{Com}\,(\mathfrak{X})$ denotes the class of all simple abelian groups A such that $A \cong H/K$ for some composition factor H/K of a group $G \in \mathfrak{X}$.

$C^p(G)$ denotes the intersection of the centralizers of the principal factors of a group G whose composition factors have prime order p (if a group G has no such factors, it is assumed that $C^p(G) = G$).

The symbols $\mathfrak{G}$, $\mathfrak{N}_p$, $\mathfrak{G}_{p'}$, and $\mathfrak{S}$ denote, respectively, the class of all groups, the class of all p-groups, the class of all p'-groups, and the class of all soluble groups. For an arbitrary class of groups $\mathfrak{F} \supseteq (1)$, $G_{\mathfrak{F}}$ denotes the product of all normal $\mathfrak{F}$-subgroups of G. In particular, we write

$$O_p(G) = G_{\mathfrak{N}_p}, \;\; R(G) = G_{\mathfrak{S}}, \;\; F_p(G) = G_{\mathfrak{G}_{p'}\mathfrak{N}_p}.$$

Let $\mathbb{P}$ be the set of all prime numbers. Then, for every formation function of the form

$$f : \mathbb{P} \to \{\text{formations of groups}\}, \tag{1}$$

$LF(f)$ denotes the totality of all groups G such that either $G = 1$ or $G \neq 1$ and $G/F_p(G) \in f(p)$ for all $p \in \pi(G)$. If a formation $\mathfrak{F}$ is such that $\mathfrak{F} = LF(f)$ for some function f of the form (1), then $\mathfrak{F}$ is called a *saturated formation* with *local satellite* f ([2], P. 20; [8], P. 356). If $\mathfrak{F} = LF(f)$ and $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, then f is called an *inner local satellite* of $\mathfrak{F}$. The symbol $\mathfrak{G}_pF(p)$ denotes the set of all groups A such that $A^{F(p)}$ is a p-group. According to [8] (Chap. IV, proposition 3.8, P. 360), for any nonempty saturated formation $\mathfrak{F}$, there exists a unique formation function F such that $\mathfrak{F} = LF(F)$ and $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all prime p. The formation function F is called the *canonical local satellite* of $\mathfrak{F}$.

For any formation function

$$f : \mathbb{P} \cup \{0\} \to \{\text{formations of groups}\}, \qquad (2)$$

we let [14] $CF(f) = (G \mid G/R(G) \in f(0)$ and $G/C^p(G) \in f(p)$ for all $p \in \pi(\mathrm{Com}\,(G)))$. If a formation $\mathfrak{F}$ is such that $\mathfrak{F} = CF(f)$ for some function f of the form (2), then $\mathfrak{F}$ is called *solubly saturated formation* with *composition satellite* f. If $\mathfrak{F} = CF(f)$ and $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, then f is called an *inner composition satellite* of $\mathfrak{F}$. According to [14], for any nonempty solubly saturated formation $\mathfrak{F}$, there exists a unique formation function F of the form (2) such that $\mathfrak{F} = CF(F)$, $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all prime p, and $F(0) = \mathfrak{F}$. The formation function F is called the *canonical composition satellite* of $\mathfrak{F}$.

Let Θ be a complete lattice of formations. A formation function f of the form either (1) or (2) is called Θ-*valued* if all its values belong to Θ. The symbol Θ^l denotes the set of all formations possessing a local Θ-valued satellite ([2], P. 78). The symbol Θ^c denotes the set of all formations possessing a composition Θ-valued satellite.

We denote by $\mathcal{L}^\tau$ the set of all τ-closed saturated formations and by $\mathcal{C}^\tau$ the set of all τ-closed solubly saturated formations. The sets $\mathcal{L}^\tau$ and $\mathcal{C}^\tau$ are complete lattices with respect to the inclusion $\subseteq$ ([1], P. 151). In a lattice $\mathcal{L}^\tau$ ($\mathcal{C}^\tau$), for an arbitrary nonempty totality $\Sigma = \{\mathcal{H}_i \mid i \in \Lambda\}$ of its elements, $\bigcap_{i\in\Lambda} \mathcal{H}_i$ is the greatest lower bound for Σ in the lattice $\mathcal{L}^\tau$ (in the lattice $\mathcal{C}^\tau$, respectively); $l^\tau\,\mathrm{form}\left(\bigcup_{i\in\Lambda} \mathcal{H}_i\right)$ is the least upper bound for Σ in the lattice $\mathcal{L}^\tau$ $\left(c^\tau\,\mathrm{form}\left(\bigcup_{i\in\Lambda} \mathcal{H}_i\text{, respectively}\right)\text{ is the least upper bound for } \Sigma \text{ in the lattice } \mathcal{C}^\tau\right)$. The symbol $l^\tau\,\mathrm{form}(\mathfrak{X})$ ($c^\tau\,\mathrm{form}(\mathfrak{X})$, respectively) denotes the intersection of all τ-closed saturated (τ-closed solubly saturated) formations containing a totality of groups $\mathfrak{X}$.

In the proof of the theorem we will use the following results.

Lemma 1 ([1], theorem 1.3.7, P. 29). *Let $\mathfrak{F}$ be a saturated formation. Then the following assertions hold*:

1) *if $\mathfrak{F}$ has an inner τ-valued local satellite, then $\mathfrak{F}$ is a τ-closed formation*;

2) *if $\mathfrak{F}$ is a τ-closed formation, then its canonical local satellite is τ-valued.*

Lemma 2 ([2], lemma 18.3, P. 168). *Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations, $\mathfrak{H}$ be saturated, and let G be a group of the minimal order from $\mathfrak{F} \setminus \mathfrak{H}$. Then G is monolithic, its monolith coincides with $G^{\mathfrak{H}}$ and if $G^{\mathfrak{H}}$ is a p-group, then $G^{\mathfrak{H}} = C_G(G^{\mathfrak{H}}) = F_p(G)$.*

Lemma 3 ([8], Chap. IV, proposition 1.5, P. 335). *Let $\mathfrak{F}$ be a formation and R/S a normal section of a $\mathfrak{F}$-group G. Let K be a normal subgroup of G contained in $C_G(R/S)$. Let $H = (R/S) \rtimes (G/K)$ be the semidirect product with respect to the following action of G/K on R/S*:

$$(rS)^{gK} = g^{-1}rgS, \;\; r \in R, \;\; g \in G.$$

Then $H \in \mathfrak{F}$.

Let Θ be a complete lattice of formations. For any totality of groups $\mathfrak{X}$, denote by $\Theta\,\mathrm{form}(\mathfrak{X})$ the intersection of all formations from Θ which contain all groups from $\mathfrak{X}$. In the case when $\Theta = \mathcal{F}^\tau$ is the lattice of all τ-closed formations, we write $\tau\,\mathrm{form}(\mathfrak{X})$ instead of $\Theta\,\mathrm{form}(\mathfrak{X})$.

For any totality of formations $\{\mathfrak{F}_i \mid i \in I\}$ from Θ, we let

$$\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\,\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right).$$

In the case when $\Theta = \mathcal{F}^\tau$, we write $\vee^\tau(\mathfrak{F}_i \mid i \in I)$ instead of $\vee_\Theta(\mathfrak{F}_i \mid i \in I)$.

Let $\{f_i \mid i \in I\}$ be an arbitrary totality of Θ-valued satellites. Then $\vee_\Theta(f_i \mid i \in I)$ denotes a satellite f such that $f(p) = \Theta\,\mathrm{form}\left(\bigcup_{i\in I} f_i(p)\right)$ for all $p \in \mathbb{P}$.

A complete lattice of formations Θ^l is called *inductive* ([1], P. 151) if, for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ from Θ^l and every such a collection $\{f_i \mid i \in I\}$ of inner Θ-valued local satellites f_i of formations $\mathfrak{F}_i$, we have

$$\vee_{\Theta^l}(\mathfrak{F}_i \mid i \in I) = LF\big(\vee_\Theta\,(f_i \mid i \in I)\big).$$

Lemma 4 ([1], theorem 4.1.1, P. 152). *The lattice of all τ-closed saturated formations $\mathcal{L}^\tau$ is inductive.*

Similarly, a complete lattice of formations Θ^c is called *inductive* ([1], P. 151; [12], P. 220) if, for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ from Θ^c and every such a collection $\{f_i \mid i \in I\}$ of inner Θ-valued composition satellites f_i of formations $\mathfrak{F}_i$, we have

$$\vee_{\Theta^c}(\mathfrak{F}_i \mid i \in I) = CF\big(\vee_\Theta\,(f_i \mid i \in I)\big).$$

Lemma 5 ([5], theorem 2.1). *The lattice $\mathcal{C}^\tau$ of all τ-closed solubly saturated formations is inductive.*

2. PROOF OF THE THEOREM

Let $\{\mathfrak{F}_i \mid i \in I\}$ be an arbitrary collection of τ-closed saturated formations, and let F_i be the canonical local satellite of $\mathfrak{F}_i$. Then, by Lemma 1, the satellite F_i is τ-valued. Let

$$\mathfrak{F} = \vee_{\mathcal{L}^\tau}(\mathfrak{F}_i \mid i \in I) = l^\tau\,\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right) \text{ and } \mathfrak{H} = \vee_{\mathcal{C}^\tau}(\mathfrak{F}_i \mid i \in I) = c^\tau\,\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right).$$

It is clear that $\bigcap_{i\in I}\mathfrak{F}_i$ is the τ-closed saturated formation which is the greatest lower bound for $\{\mathfrak{F}_i \mid i \in I\}$ in the lattice $\mathcal{L}^\tau$. On the other hand, it is clear that $\mathfrak{F}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in the lattice $\mathcal{L}^\tau$ and $\mathfrak{H}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in the lattice $\mathcal{C}^\tau$. Let us prove that $\mathfrak{F} = \mathfrak{H}$. The inclusion $\mathfrak{H} \subseteq \mathfrak{F}$ is obvious. Therefore, we only need to prove that $\mathfrak{F} \subseteq \mathfrak{H}$.

Let $\mathfrak{H}_i = CF(H_i)$, where the composition satellite H_i is such that

$$H_i(a) = \begin{cases} \mathfrak{F}_i, & \text{if } a = 0;\\ F_i(a), & \text{if } a = p \in \mathbb{P}. \end{cases}$$

Let us first show that $\mathfrak{F}_i = \mathfrak{H}_i$ for all i.

Assume that $\mathfrak{H}_i \not\subseteq \mathfrak{F}_i$. Let G be a group of the minimal order from $\mathfrak{H}_i \setminus \mathfrak{F}_i$. Then G is a monolithic group with monolith $R = G^{\mathfrak{F}_i}$.

If R is a nonabelian group, then $R(G) = 1$. Therefore, $G \cong G/1 = G/R(G) \in H_i(0) = \mathfrak{F}_i$, a contradiction. Consequently, R is an abelian p-group, where $p \in \pi(\mathrm{Com}\,(R))$. By Lemma 2, $R = C_G(R) = F_p(G)$. Therefore, $R = O_p(G) = C^p(G)$. Consequently,

$$G/F_p(G) = G/C^p(G) \in H_i(p) = F_i(p).$$

Therefore, $G \in \mathfrak{F}_i$, a contradiction. Thus, $\mathfrak{H}_i \subseteq \mathfrak{F}_i$.

We show now that $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Assume the contrary. Let G be a group of the minimal order from $\mathfrak{F}_i \setminus \mathfrak{H}_i$. Then G is a monolithic group with monolith $R = G^{\mathfrak{H}_i}$. Let $p \in \pi(R)$.

If R is a nonabelian group, then $F_p(G) = 1$. Therefore, $G \cong G/1 = G/F_p(G) \in F_i(p) = H_i(p) \subseteq \mathfrak{H}_i$, a contradiction.

Thus, R is an abelian p-group. Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}_i$, by Lemma 3, we have $T \in \mathfrak{F}_i$.

If $|T| < |G|$, then $T \in \mathfrak{H}_i$ by the choice of G. Consequently,

$$G/C_G(R) \cong T/R = T/C_T(R) = T/C^p(T) \in H_i(p).$$

Therefore, $G \in \mathfrak{H}_i$, a contradiction.

Thus, $|T| = |G|$. Consequently, $R = C_G(R)$, which implies $R = C_G(R) = O_p(G) = C^p(G) = F_p(G)$. Thus, $G/C^p(G) = G/F_p(G) = G/O_p(G) \in F_i(p) = H_i(p)$. Therefore, $G \in \mathfrak{N}_p H_i(p) = H_i(p) \subseteq \mathfrak{H}_i$. Consequently, $G \in \mathfrak{H}_i$, a contradiction. Therefore, $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Thus, $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$.

Since, by Lemma 4, the lattice $\mathcal{L}^\tau$ is inductive, we have

$$\mathfrak{F} = \vee_{\mathcal{L}^\tau}(\mathfrak{F}_i \mid i \in I) = LF\big(\vee^\tau \, (F_i \mid i \in I)\big).$$

By Lemma 5, the lattice $\mathcal{C}^\tau$ is inductive, then

$$\mathfrak{H} = \vee_{\mathcal{C}^\tau}(\mathfrak{F}_i \mid i \in I) = CF\big(\vee^\tau \, (H_i \mid i \in I)\big).$$

Now we proceed to the proof of the equality $\mathfrak{F} = \mathfrak{H}$. It is easy to see that $\mathfrak{H} \subseteq \mathfrak{F}$. Assume that $\mathfrak{F} \not\subseteq \mathfrak{H}$. Let G be a group of the minimal order from $\mathfrak{F} \setminus \mathfrak{H}$. Then G is a monolithic group with monolith $R = G^{\mathfrak{H}}$. Let $p \in \pi(R)$.

If R is a nonabelian group, then $F_p(G) = 1$. Since the canonical local satellite F_i is inner, we have

$$\begin{aligned} G \cong G/1 = G/F_p(G) \in \big(\vee^\tau \, (F_i \mid i \in I)\big)(p) = \vee^\tau(F_i(p) \mid i \in I) \\ \subseteq \vee^\tau(\mathfrak{F}_i \mid i \in I) \subseteq \vee_{\mathcal{C}^\tau}(\mathfrak{F}_i \mid i \in I) = \mathfrak{H}, \end{aligned}$$

a contradiction. Therefore, R is an abelian p-group. Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}$, by Lemma 3, we have $T \in \mathfrak{F}$.

If $|T| < |G|$, then $T \in \mathfrak{H}$ by the choice of G. Consequently,

$$G/C_G(R) \cong T/R = T/C_G(R) = T/C_T(R) = T/C^p(T) \in \big(\vee^\tau \, (H_i \mid i \in I)\big)(p).$$

Therefore, $G \in \mathfrak{H}$, a contradiction. Therefore, $|T| = |G|$. Consequently, by Lemma 2, we have $R = C_G(R) = O_p(G) = C^p(G) = F_p(G)$. Since $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$, we have

$$\begin{aligned} G/C^p(G) = G/F_p(G) \in \big(\vee^\tau \, (F_i \mid i \in I)\big)(p) = \vee^\tau(F_i(p) \mid i \in I) \\ = \vee^\tau(H_i(p) \mid i \in I) = \big(\vee^\tau \, (H_i \mid i \in I)\big)(p). \end{aligned}$$

Consequently, $G \in \mathfrak{H}$. Therefore, $\mathfrak{F} \subseteq \mathfrak{H}$. Thus, $\mathfrak{F} = \mathfrak{H}$. The theorem has been proved.

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