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Second-Order Linear Differential Equations in a Banach Space and Splitting Operators

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Abstract—We consider a second-order linear differential equation whose coefficients are bounded operators acting in a complex Banach space. For this equation with a bounded right-hand side, we study the question on the existence of solutions which are bounded on the whole real axis. An asymptotic behavior of solutions is also explored. The research is conducted under condition that the corresponding "algebraic" operator equation has separated roots or provided that an operator placed in front of the first derivative in the equation has a small norm. In the latter case we apply the method of similar operators, i.e., the operator splitting theorem. To obtain the main results we make use of theorems on the similarity transformation of a second order operator matrix to a block-diagonal matrix.

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1. INTRODUCTION. MAIN RESULTS

Let $\mathcal{X}$ be a complex Banach space and let $\operatorname{End}\mathcal{X}$ be a Banach algebra of all bounded linear operators acting in $\mathcal{X}$. We denote by $\mathcal{X}^2 = \mathcal{X} \times \mathcal{X}$ the Banach space whose elements are all ordered pairs $x = (x_1, x_2)$, where $x_1, x_2 \in \mathcal{X}$, and the norm is given by the formula $\|(x_1, x_2)\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$. In what follows, the symbol $C_b(\mathbb{J}, \mathcal{X})$, where $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, stands for the Banach space of all bounded continuous functions $x : \mathbb{J} \to \mathcal{X}$ endowed with the norm $\|x\|_C = \sup_{t\in\mathbb{J}} \|x(t)\|$. We denote by $C_b^{(k)}(\mathbb{J}, \mathcal{X})$ the Banach space of k times continuously differentiable functions $x \in C_b(\mathbb{J}, \mathcal{X})$ with the kth derivative $x^{(k)}$ belonging to the space $C_b(\mathbb{J}, \mathcal{X})$ and the norm defined by the formula $\|x\|_{C^{(k)}} = \|x\|_C + \|x^{(k)}\|_C$.

Throughout, we will write $X \sim \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ to denote the correspondence (the relation) between an operator $X \in \operatorname{End}\mathcal{X}^2$ and its matrix, where $X_{ij} \in \operatorname{End}\mathcal{X}$ and $i, j = 1, 2$. This matrix is defined by the following equalities: $X(x_1, x_2) = (y_1, y_2)$, $X_{11}x_1 = y_1$, $X_{21}x_1 = y_2$, $X_{12}x_2 = y_1$, $X_{22}x_2 = y_2$, where (x_1, x_2), $(y_1, y_2) \in \mathcal{X}^2$. Sometimes, we identify an operator acting in the Cartesian product of Banach spaces with its matrix.

In the space $C_b = C_b(\mathbb{R}, \mathcal{X})$ we consider a second-order differential equation

$$\ddot{x}(t) + B_1\dot{x}(t) + B_2x(t) = g(t), \quad t \in \mathbb{R}, \tag{1}$$

where $B_1, B_2 \in \operatorname{End}\mathcal{X}$, $g \in C_b$. We rewrite this equation in the form $\mathrm{L}x = g$, where the second-order differential operator $\mathrm{L} : D(\mathrm{L}) \subset C_b \to C_b$ is defined by the formula

$$\mathrm{L}x = \ddot{x} + B_1\dot{x} + B_2x, \quad x \in D(\mathrm{L}),$$

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and its domain $D(\mathrm{L})$ coincides with the space $C_b^{(2)} = C^{(2)}(\mathbb{R}, \mathcal{X})$.

Along with Eq. (1), we write out a first order differential equation

$$\dot{y} + \mathbb{B}y = f, \;\; f = (f_1, f_2) \in C_b(\mathbb{R}, \mathcal{X}^2). \tag{2}$$

This equation is considered in the Banach space $C_b(\mathbb{R}, \mathcal{X}^2)$ which is isomorphic to the space $C_b(\mathbb{R}, \mathcal{X}) \times C_b(\mathbb{R}, \mathcal{X})$. Here, the operator $\mathbb{B} \in \operatorname{End} \mathcal{X}^2$ is defined by the matrix

$$\mathbb{B} \sim \begin{pmatrix} 0 & -I \\ B_2 & B_1 \end{pmatrix},$$

i.e., we have the equality $\mathbb{B}(y_1, y_2) = (-y_2, B_2 y_1 + B_1 y_2)$ for $(y_1, y_2) \in \mathcal{X}^2$.

We note that Eqs. (1) and (2) are equivalent, and one differential equation is transformed into another by setting $f = (0, g)$.

Naturally, the question arises about the simultaneous invertibility of the operators $\mathrm{L} : D(\mathrm{L}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ and $\mathbb{L} : D(\mathbb{L}) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$, where the latter is defined by the following rule:

$$\mathbb{L}y = \dot{y} + \mathbb{B}y, \;\; y = (y_1, y_2) \in D(\mathbb{L}) = C_b^{(1)}(\mathbb{R}, \mathcal{X}^2),$$

i.e.,

$$\mathbb{L}y = (Dy_1 - y_2, B_2 y_1 + (D + B_1) y_2).$$

Here, as usual, the differentiation operator $D : C_b^{(1)}(\mathbb{R}, \mathcal{X}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ is defined by the formula $Dx = \dot{x}$, where $x \in C_b^{(1)}(\mathbb{R}, \mathcal{X})$.

The answer to this question is given by the following statement.

Theorem 1. *The operator* $\mathrm{L} : D(\mathrm{L}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ *is invertible if and only if the operator* $\mathbb{L} : D(\mathbb{L}) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$ *is invertible.*

Moreover, if the operator $\mathrm{L} : D(\mathrm{L}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ *is invertible, then the inverse* $\mathbb{L}^{-1} \in \operatorname{End} \mathcal{X}^2$ *of the operator* $\mathbb{L} : D(\mathbb{L}) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$ *is given by the matrix*

$$\begin{pmatrix} (D - \lambda_0 I)^{-1} - \mathrm{L}^{-1}\left((B_2 + \lambda_0 B_1 + \lambda_0^2 I)(D - \lambda_0 I)^{-1} + \lambda_0 I\right) & \mathrm{L}^{-1} \\ \lambda_0 (D - \lambda_0 I)^{-1} - D\,\mathrm{L}^{-1}\left((B_2 + \lambda_0 B_1 + \lambda_0^2 I)(D - \lambda_0 I)^{-1} + \lambda_0 I\right) & D\,\mathrm{L}^{-1} \end{pmatrix}. \tag{3}$$

Here, λ_0 *is an arbitrary number in the set* $\mathbb{C} \setminus (i\mathbb{R})$.

Theorem 1 allows us to use the results of [1–7].

The study of differential equations (1) and (2) is carried out using some properties of roots of the “algebraic” operator equation

$$X^2 + B_1 X + B_2 = 0 \tag{4}$$

in the Banach algebra $\operatorname{End} \mathcal{X}$. This method of studying a second-order differential equation was proposed in [8] and showed its effectiveness.

In general, the set of roots for Eq. (4) may be infinite.

Let Λ_1 and Λ_2 be two roots of Eq. (4). They are said to be *separated roots* provided that the operator $\Lambda_1 - \Lambda_2$ is an invertible element of the algebra $\operatorname{End} \mathcal{X}$. Conditions for the existence of separated roots are given, for example, in [1] (Chap. II, § 4). It is worth noting here that the fractional powers of operators are defined in [2] (Chap. I, § 5). In particular, we recall the sufficient conditions for Eq. (4) with $B_1 = 0$. It possesses two separated roots $\pm\sqrt{-B_2}$, provided that the following two conditions are fulfilled. The operator B_2 is invertible as well as the origin $z = 0$ and the point ∞ in the extended plane belong to the same connected component of the resolvent set $\varrho(-B_2)$ of the operator $-B_2$.

For linear operators $A_1, A_2 \in \operatorname{End}\mathcal{X}$, we denote by $A_1 \oplus A_2$ the direct sum of the operators defined by a block-diagonal matrix: $A_1 \oplus A_2 \sim \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.

One of the main result of the present paper is the following statement.

Theorem 2. *Assume that Eq.* (4) *possesses two separated roots* $\Lambda_1, \Lambda_2 \in \operatorname{End}\mathcal{X}$. *Then the operator* $\mathbb{B} \in \operatorname{End}\mathcal{X}^2$ *is similar to the block-diagonal operator* $\Lambda \in \operatorname{End}\mathcal{X}^2$ *defined by the matrix*

$$\Lambda \sim \begin{pmatrix} -\Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix}.$$

Moreover, the equalities hold true:

$$\mathbb{B} = U^{-1}\Lambda U, \quad e^{-\mathbb{B}t} = U^{-1}(e^{\Lambda_1 t} \oplus e^{\Lambda_2 t})U. \tag{5}$$

Here the operators $U, U^{-1} \in \operatorname{End}\mathcal{X}^2$ *are defined by matrices in the following way:*

$$U \sim \mathcal{U} = \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix}, \quad U^{-1} \sim \mathcal{U}^{-1} = \begin{pmatrix} -(\Lambda_1 - \Lambda_2)^{-1}\Lambda_2 & (\Lambda_1 - \Lambda_2)^{-1} \\ (\Lambda_1 - \Lambda_2)^{-1}\Lambda_1 & -(\Lambda_1 - \Lambda_2)^{-1} \end{pmatrix}. \tag{6}$$

As the example constructed by A. S. Markus [9] showed, the continuous spectrum is not always contained in the spectrum of the operator $\mathbb{B}$. However, the similarity of the operators $\mathbb{B}$ and Λ together with equalities (5) imply the the following result.

Theorem 3. *Let* Λ_1 *and* Λ_2 *be separated roots of Eq.* (4). *Then the operators* $\mathrm{L} : D(\mathrm{L}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ *and* $\mathbb{L} : D(\mathbb{L}) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$ *are invertible if and only if the following condition is satisfied*:

$$(\sigma(\Lambda_1) \cup \sigma(\Lambda_2)) \cap (i\mathbb{R}) = \varnothing, \tag{7}$$

where $\sigma(\Lambda_k)$ *is the spectrum of the operator* Λ_k *for* $k = 1, 2$.

In the hypotheses of the following theorem we consider a pair of separated roots Λ_1 and Λ_2 of Eq. (4) such that condition (7) holds. Since the spectrum of the operator Λ_k and the imaginary axis $i\mathbb{R}$ are mutually disjoint for each $k = 1, 2$, we have the decompositions $\sigma(\Lambda_k) = \sigma_k^- \cup \sigma_k^+$, $k = 1, 2$, where the spectral sets σ_k^- and σ_k^+ lie in the left and the right half-planes, respectively, and one of these sets may be empty, $k = 1, 2$.

Further, let us denote by $P_k^{\mp}$ the Riesz spectral projections corresponding to the spectral sets $\sigma_k^{\mp}$, $k = 1, 2$. We consider the convolution integral operators

$$(G_k * g)(t) = \int_{\mathbb{R}} G_k(t - s)g(s)\,ds, \;\; t \in \mathbb{R}, \;\; g \in C_b, \;\; k = 1, 2,$$

where

$$G_k(t) = \begin{cases} -e^{\Lambda_k t}P_k^+, & t \le 0; \\ e^{\Lambda_k t}P_k^-, & t > 0, \;\; k = 1, 2, \end{cases} \tag{8}$$

is the Green function ([1], Chap. II, § 4) constructed for the corresponding differential equation

$$\dot{x} = \Lambda_k x, \;\; k = 1, 2.$$

Theorem 4. *Assume that condition* (7) *holds. The inverse to the operators* L *and* $\mathbb{L}$ *have the following form*:

$$\mathrm{L}^{-1}g = (\Lambda_1 - \Lambda_2)^{-1}\Lambda_2(G_2 - G_1) * g, \;\; g \in C_b(\mathbb{R}, \mathcal{X}), \tag{9}$$

$$\mathbb{L}^{-1}f = \varphi = (\varphi_1, \varphi_2) \in C_b(\mathbb{R}, \mathcal{X}^2), \;\; f = (f_1, f_2) \in C_b(\mathbb{R}, \mathcal{X}^2),$$

where

$$\varphi_1 = (\Lambda_1 - \Lambda_2)^{-1}(-\Lambda_2 G_1 * (f_1 + f_2) + G_2 * (\Lambda_1 f_1 + \Lambda_2 f_2)),$$
$$\varphi_2 = (\Lambda_1 - \Lambda_2)^{-1}(\Lambda_1 G_1 * (f_1 + f_2) - G_2 * (\Lambda_1 f_1 + \Lambda_2 f_2)).$$

For the homogeneous differential equation

$$\ddot{x}(t) + B_1\dot{x}(t) + B_2 x(t) = 0, \quad t \geq 0, \tag{10}$$

decomposition (5) of the operator exponential in Theorem 2 and the results of [10] allow us to obtain an asymptotic representation of its bounded solutions on the real semiaxis $\mathbb{R}_+ = [0, \infty)$.

We have the following theorem.

Theorem 5. *Let Λ_1 and Λ_2 be separated roots of Eq.* (4). *Assume that all solutions to the differential equation* (10) *are bounded on $\mathbb{R}_+$, and the set*

$$(\sigma(-\Lambda_1) \cup \sigma(-\Lambda_2)) \cap (i\mathbb{R}) = \{i\lambda_1, \dots, i\lambda_m\} \tag{11}$$

is finite. Then there exist projection-valued uniformly continuous functions

$$\mathbb{P}_k : \mathbb{R}_+ \to \operatorname{End}\mathcal{X}^2,\ \ \mathbb{P}_k \in C_b(\mathbb{R}_+, \operatorname{End}\mathcal{X}^2),\ \ 1 \leq k \leq m,$$

such that the representation

$$(x(t), \dot{x}(t)) = \sum_{k=1}^{m} e^{i\lambda_k t}\mathbb{P}_k(t)(x(0), \dot{x}(0)),\ \ t \geq 0$$

is valid for every solution $x : \mathbb{R}_+ \to \mathcal{X}$ to the homogeneous differential equation (10).

The functions $\mathbb{P}_k$, $1 \leq k \leq m$, possess the following properties:

1) the operators $\mathbb{P}_k(t)$, $t \geq 0$, $1 \leq k \leq m$, belong to the smallest closed subalgebra generated by both the operator $\mathbb{B}$ and the identity operator $\mathbb{I} \in \operatorname{End}\mathcal{X}^2$;

2) the functions $\mathbb{P}_k(t)$, $t \geq 0$, $1 \leq k \leq m$ are extendable to $\mathbb{C}$ up to entire functions of the exponential type whose derivatives satisfy the condition

$$\lim_{t\to\infty} \|\dot{\mathbb{P}}_k(t)\| = 0,\ \ 1 \leq k \leq m;$$

3) $\lim\limits_{t\to\infty} \|\mathbb{B}\mathbb{P}_k(t) - i\lambda_k\mathbb{P}_k(t)\| = 0$, $1 \leq k \leq m$;

4) $\lim\limits_{t\to\infty} \|\mathbb{P}_k(t)\mathbb{P}_j(t)\| = 0$, $k \neq j$, $1 \leq k, j \leq m$;

5) $\lim\limits_{t\to\infty} \|\mathbb{P}_k^2(t) - \mathbb{P}_k(t)\| = 0$, $1 \leq k \leq m$;

6) $\lim\limits_{t\to\infty} \|\sum\limits_{k=1}^{m} \mathbb{P}_k(t) - \mathbb{I}\| = 0$.

The family of the functions $\mathbb{P}_1, \dots, \mathbb{P}_m$ is called *the resolution of the identity at infinity*. By property 2) these functions are slowly varying at infinity (see [11, 12]).

Theorem 6. *Under condition* (11), *each bounded solution $x_0 : \mathbb{R}_+ \to \mathcal{X}$ to the homogeneous differential equation* (10) *is representable in the form*

$$x_0(t) = \sum_{k=1}^{m} a_k(t)e^{i\lambda_k t},\ \ t \geq 0. \tag{12}$$

Here the functions $a_k : \mathbb{R}_+ \to \mathcal{X}$, $1 \leq k \leq m$, belonging to $C_b^{(l)}(\mathbb{R}_+, \mathcal{X})$ for every $l \in \mathbb{N}$, are extendable to $\mathbb{C}$ up to functions of exponential type. In addition, the following condition holds:

$$\lim_{t\to\infty} \dot{a}_k(t) = 0,\ \ 1 \leq k \leq m.$$

It is worth noting that according to [11] and [12], any function represented in the form (12) is almost periodic at infinity.

Also we notice that under the hypotheses of Theorems 5 and 6, the complex numbers $i\lambda_1, \dots, i\lambda_m$ may belong to a single connected component of the set $(\sigma(-\Lambda_1) \cup \sigma(-\Lambda_2)) \cap (i\mathbb{R})$.

In mathematical models which describe real world processes, the operator B_1 in Eq. (1) and (2) reflects the presence of a friction in mechanical systems, a resistance in electrical networks etc. It is natural to assume that the influence of B_1 is small enough. In Section 3 of the present paper our study is carried out under the assumption

$$0 \notin \sigma(B_2), \tag{13}$$

as well as we suppose that the norm of the operator B_1 is sufficiently small. The main result is contained in Theorem 7 about splitting operators. This theorem allows us to obtain the analogs of Theorems 2–6.

2. PROOFS OF THEOREMS

Proof of Theorem 1. Assume that the operator L is invertible.

At first, we prove that the operator $\mathbb{L}$ is injective, i.e., the equality $\operatorname{Ker}\mathbb{L} = \{0\}$ holds, where $\operatorname{Ker}\mathbb{L} = \{y = (y_1, y_2) \in D(\mathbb{L}) \mid \mathbb{L}y = 0\}$. To this end, we take an element $y = (y_1, y_2) \in \operatorname{Ker}\mathbb{L}$. Then we have $\dot{y}_1 = y_2$, $\dot{y}_2 = -B_2y_1 - B_1y_2$, whence, $y_1 \in D(\mathrm{L})$ and $\mathrm{L}y_1 = 0$. In other words, we have $y_1 \in \operatorname{Ker}\mathrm{L} = \{0\}$. Therefore, we get $y_2 = 0$ and $y = (y_1, y_2) = 0$. Thus, the condition $Ker\mathbb{L} = \{0\}$ is fulfilled, as desired.

Next, we check that the operator $\mathbb{L}$ is surjective. To do this, we consider the equation $\mathbb{L}y = f$, where $f = (f_1, f_2)$ is an arbitrary function from the space $C_b(\mathbb{R}, \mathcal{X}^2) \simeq C_b \times C_b$. Further, we take any complex number λ_0 such that $\operatorname{Re}\lambda_0 \neq 0$. Then the operator $D - \lambda_0 I$ is invertible in C_b ([1, 3, 5]), and the operator L is representable in the form

$$\mathrm{L} = (D - \lambda_0 I)^2 + (B_1 + 2\lambda_0 I)(D - \lambda_0 I) + B_2 + \lambda_0 B_1 + \lambda_0^2 I.$$

It is straightforward to see that the equation $\mathbb{L}y = f$ is solvable, and the coordinates of its solution $y = (y_1, y_2) \in C_b \times C_b$ have the following forms:

$$y_1 = ((D - \lambda_0 I)^{-1} - \mathrm{L}^{-1}(B_2 + \lambda_0 B_1 + \lambda_0^2 I)(D - \lambda_0 I)^{-1} + \lambda_0 I)f_1 + \mathrm{L}^{-1}f_2,$$
$$y_2 = (\lambda_0(D - \lambda_0 I)^{-1} - D\mathrm{L}^{-1}((B_2 + \lambda_0 B_1 + \lambda_0^2 I)(D - \lambda_0 I)^{-1} + \lambda_0 I))f_1 + D\mathrm{L}^{-1}f_2.$$

Obviously, we have $y_k \in D(\mathbb{L})$ for $k = 1, 2$.

It follows from the above representation of the solution that the inverse of the operator $\mathbb{L}$ is given by matrix (3).

Now, we assume that the operator $\mathbb{L}$ is invertible.

We claim that the operator L is injective. We take an arbitrary element $x \in \operatorname{Ker}\mathrm{L}$. Let us show that $x = 0$. Indeed, we note that $(x, \dot{x}) \in D(\mathbb{L}) = C_b^{(1)} \times C_b^{(1)}$ and $\mathbb{L}(x, \dot{x}) = (Dx - Dx, B_2x + (D + B_1)Dx) = (0, \mathrm{L}x) = (0, 0)$. The injectivity of the operator $\mathbb{L}$ immediately implies the equality $x = 0$, as desired.

Finally, we prove that the operator L is surjective. To this end, we consider the equation $\mathrm{L}x = g$, where g is an arbitrary function from the space C_b. The invertibility of the operator $\mathbb{L}$ implies the existence of a solution $y = (x_1, x_2) \in C_b^{(1)} \times C_b^{(1)}$ to the equation $\mathbb{L}y = (0, g)$. Therefore, we have the equalities

$$\dot{x}_1 - x_2 = 0, \ \ \dot{x}_2 + B_2x_1 + B_1x_2 = g.$$

Hence, we get $x_1 \in C_b^{(2)} = D(\mathrm{L})$ and $\mathrm{L}x_1 = g$. Thus, the surjectivity of the operator L is proved. □

Proof of Theorem 2. Since Λ_1 and Λ_2 are separated roots, the operator $\Lambda_1 - \Lambda_2$ is invertible. It is straightforward to check that the inverse of the operator U has the matrix $\mathcal{U}^{-1}$ in (6). Also one can easily see that the first equality in (5) holds. The second equality in (5) for the operator exponential follows from the similarity of the operators $\mathbb{B}$ and Λ. □

Since the spectra of similar operators coincide, we have the equality $\sigma(\mathbb{B}) = \sigma(\Lambda)$. This equality together with the decomposition $\sigma(\Lambda) = -\sigma(\Lambda_1) \cup (-\sigma(\Lambda_2))$ as well as condition (7) and the corresponding result in [1] (Chap. II, § 4) yield the proof of *Theorem* 3.

Proof of Theorem 4. Let us consider the operators $\mathrm{L}_k : C_b^{(1)}(\mathbb{R}, \mathcal{X}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ defined by equalities $\mathrm{L}_k x = \dot{x} - \Lambda_k x$, where $x \in C_b^{(1)}(\mathbb{R}, \mathcal{X})$ and $k = 1, 2$. According to [1] (Chap. II, § 4), under condition (7), each differential operator L_k, $k = 1, 2$, is invertible. Their inverses are the convolution operators given by the formulas

$$\mathrm{L}_k^{-1} y = G_k * y, \;\; y \in C_b, \;\; k = 1, 2,$$

where G_k, $k = 1, 2$, is defined by (8). It follows from Theorem 2 that the differential operator $\mathbb{L}$ is similar to the direct sum $\mathrm{L}_1 \oplus \mathrm{L}_2 : C_b^{(1)}(\mathbb{R}, \mathcal{X}^2) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$. Here the operator $\widetilde{U} : C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$ of multiplication by the operator U in (6) serves as the operator of transforming $\mathbb{L}$ into the operator $\mathrm{L}_1 \oplus \mathrm{L}_2$. Thus, we have the equalities

$$\mathbb{L} = U^{-1}(\mathrm{L}_1 \oplus \mathrm{L}_2)U, \quad \mathbb{L}^{-1} = U^{-1}(\mathrm{L}_1^{-1} \oplus \mathrm{L}_2^{-1})U.$$

These observations together with the above representations for L_k^{-1}, $k = 1, 2$, establish the representation for $\mathbb{L}^{-1}$ mentioned in the assertion of the theorem which, in turn, yields formula (9) for the operator L^{-1}. Indeed, one should put $f_1 = 0$, $f_2 = g$ and take into account the permutability of the operator Λ_2 and the Green function G_2. □

Proof of Theorem 5. The boundedness of all solutions to the homogeneous differential equation (10) on the semiaxis $\mathbb{R}_+$ implies the boundedness of all solutions to the homogeneous differential equation

$$\dot{y} + \mathbb{B} y = 0, \;\; t \geq 0. \tag{14}$$

Since, by Theorem 2, the spectrum $\sigma(\mathbb{B})$ of the operator $\mathbb{B}$ coincides with the set $\sigma(-\Lambda_1) \cup \sigma(-\Lambda_1)$, condition (11) means that

$$\sigma(\mathbb{B}) \cap (i\mathbb{R}) = \{i\lambda_1, \dots, i\lambda_m\}. \tag{15}$$

Each bounded solution $x \in C_b(\mathbb{R}_+, \mathcal{X})$ to Eq. (14) can be represented on $\mathbb{R}_+$ in the form $x(t) = (x_1(t), x_2(t)) = e^{\mathbb{B}t}(x(0), \dot{x}(0))$. The operator function $t \mapsto e^{\mathbb{B}t} : \mathbb{R}_+ \to \operatorname{End} \mathcal{X}^2$ is bounded and is extendable to $\mathbb{C}$ up to entire function of exponential type [11, 12]. The fulfilment of condition (15) allows us to make use of the result from [10] which guarantees the existence of a family of operator-valued functions $\mathbb{P}_k : \mathbb{R}_+ \to \operatorname{End} \mathcal{X}^2$ possessing the properties stated in Theorem 5. □

Theorem 6 is an immediate consequence of Theorem 5. The properties of the functions a_k, $1 \leq k \leq m$, in representation (12) follow from property 2) of the functions $\mathbb{P}_k$, $1 \leq k \leq m$.

3. ON SPLITTING A DIFFERENTIAL OPERATOR BY THE METHOD OF SIMILAR OPERATORS

In this Section we deal with the differential equations (1), (2) as well as the corresponding operators $\mathrm{L} : D(\mathrm{L}) \subset C_b(\mathbb{R}, \mathcal{X}) \to C_b(\mathbb{R}, \mathcal{X})$ and $\mathbb{L} : D(\mathbb{L}) \subset C_b(\mathbb{R}, \mathcal{X}^2) \to C_b(\mathbb{R}, \mathcal{X}^2)$ provided that the operator coefficient B_2 is representable in the form $B_2 = C^2$ for some invertible operator $C \in \operatorname{End} \mathcal{X}$. We denote by B the operator $(-B_1)$. Thus, we assign to the operator $\mathbb{B} \in \operatorname{End} \mathcal{X}^2$ the matrix

$$\mathbb{B} \sim \begin{pmatrix} 0 & -I \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ C^2 & -B \end{pmatrix}.$$

Further, let us consider the representation

$$\mathbb{B} = \mathcal{A}_0 - \mathcal{B}_0,$$

where the operators $\mathcal{A}_0, \mathcal{B}_0$ belong to $\operatorname{End}\mathcal{X}^2$ and their matrices are written as follows:

$$\mathcal{A}_0 \sim \begin{pmatrix} 0 & -I \\ C^2 & 0 \end{pmatrix}, \quad \mathcal{B}_0 \sim \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Condition (13) means that $0 \notin \sigma(C)$ and guarantees the existence of the separated roots $\Lambda_1 = -iC$ and $\Lambda_2 = iC$ for the operator equation

$$X^2 + C^2 = 0.$$

By Theorem 2, the operator $\mathcal{A}_0$ is similar to the operator $\mathcal{A} \in \operatorname{End}\mathcal{X}^2$ defined by the matrix

$$\mathcal{A} \sim \begin{pmatrix} -\Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix} = \begin{pmatrix} iC & 0 \\ 0 & -iC \end{pmatrix}.$$

The operator U and its inverse U^{-1} have, respectively, the following matrices:

$$U \sim \mathcal{U} = \begin{pmatrix} I & I \\ -iC & iC \end{pmatrix}, \quad U^{-1} \sim \mathcal{U}^{-1} = \frac{1}{2}\begin{pmatrix} I & iC^{-1} \\ I & -iC^{-1} \end{pmatrix}.$$

One can easily see that the equalities

$$U^{-1}\mathbb{B}U = U^{-1}\mathcal{A}_0 U - U^{-1}\mathcal{B}_0 U = \mathcal{A} - \mathcal{B}$$

are valid. This means that the operator $\mathbb{B}$ is similar to the operator $\mathcal{A} - \mathcal{B}$. The matrix of the operator $\mathcal{B}$ is given by the formula

$$\mathcal{U}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\mathcal{U} = \frac{1}{2}\begin{pmatrix} C^{-1}BC & -C^{-1}BC \\ -C^{-1}BC & C^{-1}BC \end{pmatrix},$$

whence the following estimate is valid for the norm on the space $\mathcal{X}^2$:

$$\|\mathcal{B}(x_1, x_2)\|^2 = \frac{1}{4}(\|C^{-1}BCx_1 - C^{-1}BCx_2\|^2 + \|-C^{-1}BCx_1 + C^{-1}BCx_2\|^2)$$
$$\leq \frac{1}{4}\|C^{-1}BC\|^2 4(\|x_1\|^2 + \|x_2\|^2) = \|C^{-1}BC\|^2\|(x_1, x_2)\|^2, \ (x_1, x_2) \in \mathcal{X}^2.$$

As a consequence, we have the estimate

$$\|\mathcal{B}\| \leq \|C\|\,\|C^{-1}\|\,\|B\|. \tag{16}$$

Thus, we have proved

Lemma 1. *The operator $\mathbb{B}$ is similar to the operator $\mathcal{A} - \mathcal{B}$, and estimate* (16) *holds.*

We will apply the method of similar operators to the operators $\mathcal{A} - \mathcal{B}$ and $\mathbb{B}$. This method was developed in [13–17].

The main result of this Section is Theorem 7 about the similarity of the operator $\mathcal{A} - \mathcal{B}$ and hence, by Lemma 1, about the similarity of the operator $\mathbb{B}$, to the operator whose matrix has the following block-diagonal form: $\mathcal{A} - \mathcal{B} \sim \begin{pmatrix} \widetilde{\Lambda}_1 & 0 \\ 0 & \widetilde{\Lambda}_2 \end{pmatrix}$, where $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2 \in \operatorname{End}\mathcal{X}$.

Further, we will consider two so-called transformers $J : \operatorname{End}\mathcal{X}^2 \to \operatorname{End}\mathcal{X}^2$, $\Gamma : \operatorname{End}\mathcal{X}^2 \to \operatorname{End}\mathcal{X}^2$. We recall that, according to M. G. Krein's terminology, an operator acting in the space of operators is called a transformer. To this end, we introduce the canonical projections $P_1, P_2 \in \operatorname{End}\mathcal{X}^2$ defined by the formulas

$$P_1 x = (x_1, 0), \quad P_2 x = (0, x_2), \quad x = (x_1, x_2) \in \mathcal{X}^2.$$

Of course, we have the following properties: $P_1 + P_2 = \mathbb{I}$, $\|P_1\| = \|P_1\| = 1$.

The transformer $J : \operatorname{End}\mathcal{X}^2 \to \operatorname{End}\mathcal{X}^2$ is defined by the formula

$$JX = P_1XP_1 + P_2XP_2, \quad X \in \operatorname{End}\mathcal{X}^2.$$

Thus, if $X \sim (X_{ij})$, $i, j = 1, 2$, then the matrix of the operator JX has the block-diagonal form

$$\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}.$$

The image of an operator $X \in \operatorname{End}\mathcal{X}^2$ under the transformer $\Gamma : \operatorname{End}\mathcal{X}^2 \to \operatorname{End}\mathcal{X}^2$ is defined as the solution $Y^0 \in \operatorname{End}\mathcal{X}^2$ to the linear equation

$$\mathcal{A}Y - Y\mathcal{A} = X - JX$$

satisfying the additional condition $JY^0 = 0$. In that case, the matrix $\left(\begin{smallmatrix} 0 & Y_{12}^0 \\ Y_{21}^0 & 0 \end{smallmatrix}\right)$ of the operator Y^0 is a solution to the matrix equation

$$\begin{pmatrix} iC & 0 \\ 0 & -iC \end{pmatrix}\begin{pmatrix} 0 & Y_{12} \\ Y_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & Y_{12} \\ Y_{21} & 0 \end{pmatrix}\begin{pmatrix} iC & 0 \\ 0 & -iC \end{pmatrix} = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix}.$$

Therefore, the operators $Y_{12}^0, Y_{21}^0 \in \operatorname{End}\mathcal{X}$ are solutions to the equations

$$iCY_{12} + iY_{12}C = X_{12}, \quad -iCY_{21} - iY_{21}C = X_{21}. \tag{17}$$

In what follows, we assume that the following condition holds:

$$0 \notin \sigma(C) + \sigma(C) = \{\lambda + \mu \mid \lambda \in \sigma(C),\ \mu \in \sigma(C)\}.$$

This condition guarantees (see [1], Chap. I, § 3) the solvability as well as the uniqueness of solution to both of Eqs. (17). For these solutions we have the integral representations

$$Y_{12}^0 = -\frac{1}{4\pi^2}\oint_{\partial(iC)}\oint_{\partial(-iC)} \frac{(iC - \lambda I)^{-1}X_{12}(-iC - \mu I)^{-1}}{\lambda - \mu}\, d\lambda\, d\mu, \tag{18}$$

$$Y_{21}^0 = -\frac{1}{4\pi^2}\oint_{\partial(-iC)}\oint_{\partial(iC)} \frac{(-iC - \lambda I)^{-1}X_{21}(iC - \mu I)^{-1}}{\lambda - \mu}\, d\lambda\, d\mu, \tag{19}$$

where $\partial(-iC)$ and ∂iC are closed curves surrounding the spectra of the operators $-iC$ and iC, respectively (see [1], Chap. I, § 3).

We define the transformers $\Gamma_{12} : \operatorname{End}\mathcal{X} \to \operatorname{End}\mathcal{X}$, $\Gamma_{21} : \operatorname{End}\mathcal{X} \to \operatorname{End}\mathcal{X}$ by means of the formulas $\Gamma_{12}X_{12} = Y_{12}^0$ and $\Gamma_{21}X_{21} = Y_{21}^0$. Consequently, we have the following equalities:

$$iC(\Gamma_{12}X_{12}) + i(\Gamma_{12}X_{12})C = X_{12}, \quad -iC(\Gamma_{21}X_{21}) - i(\Gamma_{21}X_{21})C = X_{21}. \tag{20}$$

It follows from representations (18)–(20) that the transformers Γ_{12} and Γ_{21} are bounded. Further, we put

$$\gamma = \max\{\|\Gamma_{12}\|, \|\Gamma_{21}\|\}. \tag{21}$$

Under the transformer $\Gamma : \operatorname{End}\mathcal{X}^2 \to \operatorname{End}\mathcal{X}^2$, to each operator $X \in \operatorname{End}\mathcal{X}^2$ with a matrix (X_{ij}), $i, j = 1, 2$, there corresponds the operator with the matrix

$$\Gamma X \sim \begin{pmatrix} 0 & \Gamma_{12}X_{12} \\ \Gamma_{21}X_{21} & 0 \end{pmatrix}.$$

In other words, we have the formula

$$(\Gamma X)x = ((\Gamma_{12}X_{12})x_2, (\Gamma_{21}X_{21})x_1),\ \ x = (x_1, x_2) \in \mathcal{X}^2.$$

It is worth noting that for an arbitrary operator $X \in \operatorname{End} \mathcal{X}^2$ and for every element $x = (x_1, x_2) \in \mathcal{X}^2$ we have the estimate

$$\|(\Gamma X)x\|^2 = \|(\Gamma_{12}X_{12})x_2\|^2 + \|(\Gamma_{21}X_{21})x_1\|^2 \leq \max\{\|\Gamma_{12}X_{12}\|^2, \|\Gamma_{21}X_{21}\|^2\}\|x\|^2. \tag{22}$$

In the case when we deal with a Hilbert space and a self-adjoint operator C, the number γ can be effectively calculated. More precisely, we have the following statement.

Lemma 2. *Let X be a Hilbert space and $C = C^*$ a self-adjoint operator. Assume that C is either uniformly positive or uniformly negative. Then the following equalities hold*:

$$2\gamma = \|C^{-1}\| = r(C^{-1}) = 1/\min_{\lambda\in\sigma(C)} |\lambda|, \tag{23}$$

where $r(C^{-1})$ denotes the spectral radius of the operator C^{-1}.

Proof. For definiteness, we suppose that the operator C is uniformly negative. In (17) we consider the first equation which defines the transformer Γ_{12}. It follows from the results of [12] and [17] that the following integral representation is valid:

$$\Gamma_{12}X = i\int_0^\infty e^{Ct}Xe^{Ct}\,dt, \quad X \in \operatorname{End}\mathcal{X}. \tag{24}$$

As an immediate consequence of (24), we obtain (23). Indeed, it follows from the inequality

$$\|\Gamma_{12}X\| \leq \int_0^\infty e^{-2t\min\limits_{\lambda\in\sigma(C)}|\lambda|}\|X\|\,dt = \frac{1}{2\min_{\lambda\in\sigma(C)}|\lambda|}\|X\|, \ X \in \operatorname{End}\mathcal{X}. \qquad \square$$

Further, we make use of the method of similar operators for determining conditions for the existence of an operator $X \in \operatorname{End}\mathcal{X}$ satisfying the equality

$$(\mathcal{A} - \mathcal{B})(I + \Gamma X) = (I + \Gamma X)(\mathcal{A} - JX), \tag{25}$$

where the operator $I + \Gamma X$ is invertible in the algebra $\operatorname{End}\mathcal{X}$.

Using the matrix representations of the operators $\mathcal{A}$, $\mathcal{B}$, X, ΓX, JX and formulas (20), in view of (25), we get the following system of operator equations:

$$\begin{aligned}
2X_{11} &= -C^{-1}BC(\Gamma_{21}X_{21}) + C^{-1}BC,\\
2X_{12} &= C^{-1}BC(\Gamma_{12}X_{12}) - (\Gamma_{12}X_{12})C^{-1}BC + (\Gamma_{12}X_{12})C^{-1}BC(\Gamma_{12}X_{12}) - C^{-1}BC,\\
2X_{21} &= C^{-1}BC(\Gamma_{21}X_{21}) - (\Gamma_{21}X_{21})C^{-1}BC + (\Gamma_{21}X_{21})C^{-1}BC(\Gamma_{21}X_{21}) - C^{-1}BC,\\
2X_{22} &= -C^{-1}BC(\Gamma_{12}X_{12}) + C^{-1}BC.
\end{aligned} \tag{26}$$

We note that representations (18)–(20) for the transformers Γ_{ij}, $i, j = 1, 2$, $i \neq j$, yield the equalities

$$C(\Gamma_{ij}X_{ij})C^{-1} = \Gamma_{ij}(CX_{ij}C^{-1}), \ \ i, j = 1, 2, \ \ i \neq j. \tag{27}$$

The next step is the change of variables $Y_{ij} = CX_{ij}C^{-1}$, $i, j = 1, 2$, in system (26). As a result, we get the following system of equations in $\operatorname{End}\mathcal{X}$:

$$\begin{aligned}
2Y_{11} &= -B(\Gamma_{21}Y_{21}) + B,\\
2Y_{12} &= B(\Gamma_{12}Y_{12}) - (\Gamma_{12}Y_{12})B + (\Gamma_{12}Y_{12})B(\Gamma_{12}Y_{12}) - B,\\
2Y_{21} &= B(\Gamma_{21}Y_{21}) - (\Gamma_{21}Y_{21})B + (\Gamma_{21}Y_{21})B(\Gamma_{21}Y_{21}) - B,\\
2Y_{22} &= -B(\Gamma_{12}Y_{12}) + B.
\end{aligned} \tag{28}$$

Before solving this system we make a few observations about its properties. It is enough to find solutions $\widetilde{Y}_{12}$ and $\widetilde{Y}_{21}$ to the second and third equations of the system for determining the operators $\widetilde{Y}_{11} = -\frac{1}{2}B(\Gamma_{21}\widetilde{Y}_{21}) + \frac{1}{2}B$ and $\widetilde{Y}_{22} = -\frac{1}{2}B(\Gamma_{12}\widetilde{Y}_{12}) + \frac{1}{2}B$ in the first and fourth equations, and hence, for solving the whole system (28). We note that the second and third equations in this system are identical. Therefore, it suffices to solve only one of these equations.

Let us consider the second equation in system (28). We define the transformer $\Phi : \operatorname{End}\mathcal{X} \to \operatorname{End}\mathcal{X}$ by the equality

$$2\Phi(Y) = B(\Gamma_{12}Y) - (\Gamma_{12}Y)B + (\Gamma_{12}Y)B(\Gamma_{12}Y) - B,\ \ Y \in \operatorname{End}\mathcal{X}.$$

We claim that there exists a real number $r > 0$ such that the inequality $\|Y\| \le \|B\|r$ implies the estimate $\|\Phi(Y)\| \le \|B\|r$. In other words, the operator Φ transforms the ball $\overline{B}(0,\beta r) \subset \operatorname{End}\mathcal{X}$ of radius βr, where $\beta = \|B\|$, into itself. Indeed, the estimate

$$\|\Phi(Y)\| \le \gamma\|B\|^2 r + \frac{1}{2}\gamma^2\|B\|^3 r^2 + \frac{1}{2}\|B\| \le \|B\|r$$

implies that r must satisfy the inequality

$$\gamma^2\|B\|^2 r^2 + 2(\gamma\|B\| - 1)r + 1 \le 0.$$

We assume that the condition

$$2\gamma\|B\| < 1 \tag{29}$$

holds, where γ is defined in (21). Then one can easily see that the required r exists. Indeed, it is given by the formula

$$0 < r = \frac{1 - \gamma\|B\| - \sqrt{1 - 2\gamma\|B\|}}{\gamma^2\|B\|^2} = (1 - \gamma\|B\| + \sqrt{1 - 2\gamma\|B\|})^{-1}. \tag{30}$$

Further, we will show that the operator $\Phi : \operatorname{End}\mathcal{X} \to \operatorname{End}\mathcal{X}$ is contractive on the ball $\overline{B}(0,\beta r)$. To this end, let us fix two points $Z_1, Z_2 \in \overline{B}(0,\beta r)$. Obviously, we have the inequality

$$\|\Phi(Z_1) - \Phi(Z_2)\| \le (\gamma\|B\| + r\gamma^2\|B\|^2)\|Z_1 - Z_2\| = q\|Z_1 - Z_2\|,$$

where $q = \gamma\|B\|(1 + \gamma\|B\|r)$. Using the first expression for r in (30) and assumption (29), one can easily see that the inequality $q \le 2\gamma\|B\| < 1$ holds. Thus, the operator $\Phi : \overline{B}(0,\beta r) \subset \operatorname{End}\mathcal{X} \to \overline{B}(0,\beta r)$ is a contraction mapping with the Lipschitz constant $q < 1$. Therefore, there exists a unique solution to the second equation of system (28) in the ball $\overline{B}(0,\beta r)$. As is well-known, this solution can be found by the successive approximations method.

Under the same hypotheses, the third equation of system (28) is also solvable.

The inclusions $\widetilde{Y}_{ij} \subset \overline{B}(0,\beta r)$, $i \ne j$, $i,j = 1,2$, and the representations of $\widetilde{Y}_{ii}$, $i = 1,2$, together with the first and last equations of system (28) yield the following estimates:

$$\|\widetilde{Y}_{ij}\| \le \begin{cases} \|B\|(1 - \gamma\|B\| + \sqrt{1 - 2\gamma\|B\|})^{-1}, & i \ne j; \\ \frac{1}{2}\|B\|(1 + \gamma\|B\|(1 - \gamma\|B\| + \sqrt{1 - 2\gamma\|B\|})^{-1}), & i = j. \end{cases} \tag{31}$$

Next we will show that the operator $I + \Gamma\widetilde{X}$ is invertible in $\operatorname{End}\mathcal{X}$. Here $\widetilde{X}$ is the operator given by the matrix $(\widetilde{X}_{ij})$, $i,j = 1,2$. To do this, we consider the matrix representation of the operator $I + \Gamma\widetilde{X}$ and make use of property (27) of the transformers $\Gamma_{ij} : \operatorname{End}\mathcal{X} \to \operatorname{End}\mathcal{X}$, $i \ne j$, $i,j = 1,2$. We have the following formulas:

$$I + \Gamma\widetilde{X} \sim \begin{pmatrix} I & (\Gamma_{12}\widetilde{X}_{12}) \\ (\Gamma_{21}\widetilde{X}_{21}) & I \end{pmatrix} = \begin{pmatrix} I & C^{-1}(\Gamma_{12}\widetilde{Y}_{12})C \\ C^{-1}(\Gamma_{21}\widetilde{Y}_{21})C & I \end{pmatrix}$$
$$= \begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} I & (\Gamma_{12}\widetilde{Y}_{12}) \\ (\Gamma_{21}\widetilde{Y}_{21}) & I \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Note that the estimates $\|\Gamma\widetilde{Y}_{12}\| \le \gamma\|B\|r = \gamma\|B\|(1 - \gamma\|B\| + \sqrt{1 - 2\gamma\|B\|})^{-1} < 2\gamma\|B\| < 1$ are valid. Analogously, we have the inequality $\|\Gamma\widetilde{Y}_{21}\| < 2\gamma\|B\| < 1$. Therefore, estimate (22) implies the inequality $\|\Gamma\widetilde{Y}\| < 1$ which guarantees that the operator $I + \Gamma\widetilde{Y}$ is invertible. Hence, the operator $I + \Gamma\widetilde{X}$ is also invertible.

Thus, equality (25) is proved. It implies the following representation:

$$\mathcal{A} - \mathcal{B} = (I + \Gamma\widetilde{X})(\mathcal{A} - J\widetilde{X})(I + \Gamma\widetilde{X})^{-1}.$$

Moreover, by Lemma 1, we have the decomposition $\mathcal{A} - \mathcal{B} = U^{-1}\mathbb{B}U$. Equating these two expressions for the operator $\mathcal{A} - \mathcal{B}$, we get

$$U^{-1}\mathbb{B}U = (I + \Gamma\widetilde{X})(\mathcal{A} - J\widetilde{X})(I + \Gamma\widetilde{X})^{-1}$$

and, hence,

$$\mathbb{B}U(I + \Gamma\widetilde{X}) = U(I + \Gamma\widetilde{X})(\mathcal{A} - J\widetilde{X}).$$

This means the similarity of the operators $\mathbb{B}$ and $\mathcal{A} - J\widetilde{X}$.

Therefore, we have proved the following statement.

Theorem 7. *Assume that the operators $C \in \operatorname{End}\mathcal{X}$ and $B \in \operatorname{End}\mathcal{X}$ satisfy the conditions $0 \notin \sigma(C) + \sigma(C)$ and* (29)*, respectively. Here the number γ is defined by equality* (21) *and, moreover, $\gamma = \|C^{-1}\|/2 = (2\min_{\lambda\in\sigma(C)}|\lambda|)^{-1}$ provided that C is a self-adjoint uniformly definite operator.*

Then there exist operators $\widetilde{X}_{ij} \in \operatorname{End}\mathcal{X}$, $i, j = 1, 2$, such that the operator $\mathbb{B}$ is similar to the operator $\widetilde{\mathbb{B}}$ possessing the block-diagonal matrix $\widetilde{\mathbb{B}} \sim (iC - \widetilde{X}_{11}) \oplus (-iC - \widetilde{X}_{22})$, and the differential operator $\mathbb{L}$ is similar to the operator $(D + iC - \widetilde{X}_{11}) \oplus (D - iC - \widetilde{X}_{22})$.

The operators $\widetilde{X}_{ij} \in \operatorname{End}\mathcal{X}$, $i, j = 1, 2$, are representable in the form $\widetilde{X}_{ij} = C^{-1}\widetilde{Y}_{ij}C$, where the operators $\widetilde{Y}_{ij} \in \operatorname{End}\mathcal{X}$, $i, j = 1, 2$, are the solutions to the system of operator equations (28) *and can be found by the the successive approximations method. Furthermore, the estimates $\|\widetilde{X}_{ij}\| \le \|C\|\,\|C^{-1}\|\,\|\widetilde{Y}_{ij}\|$, $i, j = 1, 2$, hold, and for the operators $\widetilde{Y}_{ij}$, $i, j = 1, 2$, inequalities* (31) *are valid.*

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