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A Nonlocal Problem for Degenerate Hyperbolic Equation

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Abstract—We consider a nonlocal problem for a degenerate equation in a domain bounded by characteristics of this equation. The boundary-value conditions of the problem include linear combination of operators of fractional integro-differentiation in the Riemann–Liouville sense. The uniqueness of solution of the problem under consideration is proved by means of the modified Tricomi method, and existence is reduced to solvability of either singular integral equation with the Cauchy kernel or Fredholm integral equation of second kind.

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1. Statement of the problem. We consider the equation

$$|y|^m u_{yy} - u_{xx} + \alpha \operatorname{sign} y |y|^{m-1} u_y = 0, \tag{1}$$

where $\alpha = \text{const}$, $0 < m < 1$, in a finite domain Ω bounded by characteristics

$$AC: x - \frac{2}{m-2} y^{\frac{2-m}{2}} = 0, \quad BC: x + \frac{2}{m-2} y^{\frac{2-m}{2}} = 1,$$

$$AD: x - \frac{2}{m-2} (-y)^{\frac{2-m}{2}} = 0, \quad BD: x + \frac{2}{m-2} (-y)^{\frac{2-m}{2}} = 1$$

of Eq. (1). Let $\Omega_1 = \Omega \cap (y > 0)$, $\Omega_2 = \Omega \cap (y < 0)$, I be an interval $0 < x < 1$ of axis $y = 0$.

Problem. Find the solution

$$u(x,y) = \begin{cases} u_1(x,y), & (x,y) \in \Omega_1; \\ u_2(x,y), & (x,y) \in \Omega_2, \end{cases}$$

to Eq. (1) in the class $C(\overline{\Omega}) \cap C^1(\Omega_1 \cup I) \cap C^1(\Omega_2 \cup I) \cap C^2(\Omega_1 \cup \Omega_2)$ satisfying conditions

$$a_i(x) D_{0x}^{\beta_i} \delta_i(x) u_i[\Theta_0^i(x)] + b_i(x) D_{x1}^{\beta_i} w_i(x) u_i[\Theta_1^i(x)] + c_i(x) u_i(x,0) + d_i(x) u_{i\,y}(x,0) = f_i(x), \quad i = 1,2, \tag{2}$$

and conjugation condition

$$\lim_{y \to +0} y^\alpha u_y(x,y) = \lim_{y \to -0} (-y)^\alpha u_y(x,y), \tag{3}$$

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where $\Theta_0^i(x)$, $\Theta_1^i(x)$ are points of intersection of characteristics of Eq. (1) emerging from the point $(x,0) \in I$ with characteristics AC, AD, BC, and BD, respectively; $a_i(x)$, $b_i(x)$, $c_i(x)$, $d_i(x)$, $f_i(x)$, $\delta_i(x)$, and $w_i(x)$ are given continuous functions, and

$$a_i^2(x) + b_i^2(x) + c_i^2(x) + d_i^2(x) \neq 0, \tag{4}$$

where $a_i(x), b_i(x), c_i(x), d_i(x), f_i(x) \in C^1(\overline{I}) \cap C^3(I)$, $i = 1,2$, $D_{0x}^{\beta_i}$, and $D_{x1}^{\beta_i}$ are operators of fractional integro-differentiation in the sense of Riemann–Liouville ([1], pp. 42, 44).

The formulated above problem belongs to the class of boundary-value problems with displacements ([2], P. 29).

The boundary-value problems with displacements for degenerate hyperbolic and mixed-type equations including Riemann–Liouville and Saigo operators of fractional integro-differentiation are investigated by the authors in [3–5]. The present paper is continuation of these studies.

Uniqueness of solution to the problem. Let $\tau(x) = u(x,0)$, $\nu_1(x) = \lim\limits_{y\to -0}(-y)^\alpha u_y$, $\nu_2(x) = \lim\limits_{y\to +0} y^\alpha u_y$.

Using the known solutions to the Cauchy problem ([6], pp. 57–58) for Eq. (1) in domains Ω_1 and Ω_2 and satisfying boundary-value conditions (2) we obtain

$$\begin{aligned}\big[\mu a_1(x)(1-x)^\beta + \mu b_1(x)x^\beta + x^\beta(1-x)^\beta d_1(x)\big]\nu_1(x) = x^\beta(1-x)^\beta f_1(x)\\ - \frac{\Gamma(2\beta)}{\Gamma(\beta)}(1-x)^\beta a_1(x) D_{0x}^{1-2\beta}\tau(x) - \frac{\Gamma(2\beta)}{\Gamma(\beta)} x^\beta b_1(x) D_{x1}^{1-2\beta}\tau(x) - c_1(x)x^\beta(1-x)^\beta\tau(x),\end{aligned} \tag{5}$$

where

$$\mu = \frac{\Gamma(2-2\beta)}{(1-\alpha)\Gamma(1-\beta)}\left(\frac{2-m}{4}\right)^{1-2\beta}.$$

Analogously,

$$\begin{aligned}\big[\mu a_2(x)(1-x)^\beta + \mu b_2(x)x^\beta - x^\beta(1-x)^\beta d_2(x)\big]\nu_2(x) = -x^\beta(1-x)^\beta f_2(x)\\ + \frac{\Gamma(2\beta)}{\Gamma(\beta)}(1-x)^\beta a_2(x) D_{0x}^{1-2\beta}\tau(x) + \frac{\Gamma(2\beta)}{\Gamma(\beta)} x^\beta b_2(x) D_{x1}^{1-2\beta}\tau(x) + c_2(x)x^\beta(1-x)^\beta\tau(x)\end{aligned} \tag{6}$$

for

$$\beta_i = 1-\beta, \quad \beta = \frac{2\alpha - m}{2(2-m)}, \quad \delta_i(x) = w_i(x) = 1, \quad i = 1,2. \tag{7}$$

But if

$$\beta_i = \beta, \quad \delta_i(x) = x^{2\beta-1}, \quad w_i(x) = (1-x)^{2\beta-1},$$

then in domains Ω_i, $i = 1,2$, we have

$$\begin{aligned}\left[\frac{\Gamma(2\beta)}{\Gamma(\beta)}(1-x)^{1-\beta}a_i(x) + \frac{\Gamma(2\beta)}{\Gamma(\beta)}x^{1-\beta}b_i(x) + c_i(x)x^{1-\beta}(1-x)^{1-\beta}\right]\tau(x)\\ = -\mu\big[(1-x)^{1-\beta}a_i(x)D_{0x}^{2\beta-1}\nu_i(x) + x^{1-\beta}b_i(x)D_{x1}^{2\beta-1}\nu_i(x)\big]\\ - (1-x)^{1-\beta}x^{1-\beta}d_i(x)\nu_i(x) + x^{1-\beta}(1-x)^{1-\beta}f_i(x).\end{aligned} \tag{8}$$

Let conditions (7) hold and

$$M_i(x) = \mu a_i(x)(1-x)^\beta + \mu b_i(x)x^\beta + (-1)^{i-1}x^\beta(1-x)^\beta d_i(x) \neq 0, \quad i = 1,2. \tag{9}$$

We rewrite (5) and (6) as

$$\nu_i(x) = A_i(x)D_{0x}^{1-2\beta}\tau(x) + B_i(x)D_{x1}^{1-2\beta} + C_i(x)\tau(x) + F_i(x), \quad i = 1,2, \tag{10}$$

where

$$A_i(x) = \mp\frac{\Gamma(2\beta)}{\Gamma(\beta)}\frac{(1-x)^\beta a_i(x)}{M_i(x)}, \quad B_i(x) = \mp\frac{\Gamma(2\beta)}{\Gamma(\beta)}\frac{x^\beta b_i(x)}{M_i(x)},$$

$$C_i(x) = \mp\frac{x^\beta(1-x)^\beta c_i(x)}{M_i(x)}, \quad F_i(x) = \pm\frac{x^\beta(1-x)^\beta f_i(x)}{M_i(x)}.$$

For $F_i(x)=0$ we consider the integral $I_i^* = \int\limits_0^1 \tau(x)\nu_i(x)dx$ and obtain

$$\Gamma(2\beta)I_i^* = \int_0^1 A_i(x)\tau(x)\left[\frac{d}{dx}\int_0^x \frac{\tau(t)dt}{(x-t)^{1-2\beta}}\right]dx - \int_0^1 B_i(x)\tau(x)\left[\frac{d}{dx}\int_x^1 \frac{\tau(t)dt}{(t-x)^{1-2\beta}}\right]dx + \Gamma(2\beta)\int_0^1 C_i(x)\tau^2(x)dx.$$

By means of the notation

$$\tau_1(x) = \frac{\sin(2\pi\beta)}{\pi}\frac{d}{dx}\int_0^x \frac{\tau(t)dt}{(x-t)^{1-2\beta}}, \quad \tau_2(x) = -\frac{\sin(2\pi\beta)}{\pi}\frac{d}{dx}\int_x^1 \frac{\tau(t)dt}{(t-x)^{1-2\beta}}$$

and inversion formula for the Abel integral equation we obtain

$$\Gamma(2\beta)I_i^* = \frac{\pi}{\sin(2\pi\beta)}\int_0^1 A_i(x)\tau_1(x)dx\int_0^x \frac{\tau_1(\xi)d\xi}{(x-\xi)^{2\beta}} + \frac{\pi}{\sin(2\pi\beta)}\int_0^1 B_i(x)\tau_2(x)dx\int_x^1 \frac{\tau_2(\xi)d\xi}{(\xi-x)^{2\beta}} + \Gamma(2\beta)\int_0^1 C_i(x)\tau^2(x)dx.$$

Then we apply the well-known formula for gamma-function

$$\int_0^\infty t^{\mu-1}\cos(kt)dt = \frac{\Gamma(\mu)}{k^\mu}\cos\left(\frac{\mu\pi}{2}\right), \quad k>0,\ 0<\mu<1.$$

We put $k=|x-\xi|$, $\mu=2\beta$, and obtain

$$\frac{1}{|x-\xi|^{2\beta}} = \frac{1}{\Gamma(2\beta)\cos(\pi\beta)}\int_0^\infty t^{2\beta-1}\cos(t|x-\xi|)dt.$$

Then

$$\frac{1}{\pi}\Gamma^2(2\beta)\sin(2\pi\beta)\cos(\pi\beta)I_i^* = \int_0^1 A_i(x)\tau_1(x)dx\int_0^x \tau_1(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(x-\xi)]dt + {} + \int_0^1 B_i(x)\tau_2(x)dx\int_x^1 \tau_2(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(\xi-x)]dt + \frac{1}{\pi}\Gamma^2(2\beta)\sin(2\pi\beta)\cos(\pi\beta)\int_0^1 C_i(x)\tau^2(x)dx.$$

We replace the order of integration, integrate by parts, and by means of equalities $B_i(0)=0$, $A_i(1)=0$ conclude that

$$\frac{1}{\pi}\Gamma^2(2\beta)\sin(2\pi\beta)\cos(\pi\beta)I_i^* = -\frac{1}{2}\int_0^\infty t^{2\beta-1}dt\int_0^1 A_i'(x)\left[\left(\int_0^x \tau_1(\xi)\cos(t\xi)d\xi\right)^2 + \left(\int_0^x \tau_1(\xi)\sin(t\xi)d\xi\right)^2\right]dx$$

$$+\frac{1}{2}\int_0^\infty t^{2\beta-1}dt\int_0^1 B_i'(x)\left[\left(\int_x^1 \tau_2(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_x^1 \tau_2(\xi)\sin(t\xi)d\xi\right)^2\right]dx$$

$$+\frac{1}{\pi}\Gamma^2(2\beta)\sin(2\pi\beta)\cos(\pi\beta)\int_0^1 C_i(x)\tau^2(x)dx.$$

The conditions $A_1'(x)\le 0$, $B_1'(x)\ge 0$, $C_1(x)\ge 0$ ensure fulfillment of $I_1^*\ge 0$, and conditions $A_2'(x)\ge 0$, $B_2'(x)\le 0$, $C_2(x)\le 0$ imply $I_2^*\le 0$. Since condition (3) is valid, it follows

$$I_i^*=\int_0^1 \tau(x)\nu_i(x)dx=0,\quad i=1,2.$$

Consequently,

$$\int_0^\infty t^{2\beta-1}dt\left(\int_0^x \tau_1(\xi)\cos(t\xi)d\xi\right)^2=0,\quad \int_0^\infty t^{2\beta-1}dt\left(\int_0^x \tau_1(\xi)\sin(t\xi)d\xi\right)^2=0.$$

Since $t^{2\beta-1}>0$, we have

$$\int_0^x \tau_1(\xi)\cos(t\xi)d\xi=0,\quad \int_0^x \tau_1(\xi)\sin(t\xi)d\xi=0$$

for any $t\in[0,\infty)$, particularly, for $t=2\pi k$, $k=0,1,2,\dots$ For these values of t functions $\sin(t\xi)$, $\cos(t)\xi$ form complete orthogonal system of functions on L^2. Hence, $\tau_1(\xi)=0$ almost everywhere. But by assumption $\tau_1(\xi)$ is continuous. Then $\tau_1(\xi)=0$ everywhere, and, consequently, $\tau(x)=0$. Relation (10) implies that $\nu_i(x)=0$ and $u_i(x,y)\equiv 0$ $(i=1,2)$ as the solution to the Cauchy problem with null data.

The same conclusions are valid for

$$\int_0^\infty t^{2\beta-1}dt\left(\int_x^1 \tau_2(\xi)\cos(t\xi)d\xi\right)^2=0,\quad \int_0^\infty t^{2\beta-1}dt\left(\int_x^1 \tau_2(\xi)\sin(t\xi)d\xi\right)^2=0.$$

Remark. The equality $\int_0^1 C_i(x)\tau^2(x)dx=0$ means that $\tau(x)=0$.

Now let the following conditions be valid

$$\beta_i=\beta,\quad \delta_i(x)=x^{2\beta-1},\quad w_i(x)=(1-x)^{2\beta-1},$$

$$N_i(x)=\frac{\Gamma(2\beta)}{\Gamma(\beta)}[(1-x)^{1-\beta}a_i(x)+x^{1-\beta}b_i(x)]+x^{1-\beta}(1-x)^{1-\beta}c_i(x)\ne 0,\quad i=1,2. \tag{11}$$

We rewrite (8) as

$$\tau(x)=\overline{A}_i(x)D_{0x}^{2\beta-1}\nu_i(x)+\overline{B}_i(x)D_{x1}^{2\beta-1}\nu_i(x)+\overline{C}_i(x)\nu_i(x)+\overline{F}_i(x),\quad i=1,2, \tag{12}$$

where

$$\overline{A}_i(x)=\mp\frac{\mu(1-x)^{1-\beta}a_i(x)}{N_i(x)},\quad \overline{B}_i(x)=\mp\frac{\mu x^{1-\beta}b_i(x)}{N_i(x)},$$

$$\overline{C}_i(x)=-\frac{(1-x)^{1-\beta}x^{1-\beta}d_i(x)}{N_i(x)},\quad \overline{F}_i(x)=\frac{x^{1-\beta}(1-x)^{1-\beta}f_i(x)}{N_i(x)}.$$

As in the first case, under conditions $\overline{A}_1'(x)\le 0$, $\overline{B}_1'(x)\ge 0$, $\overline{C}_1(x)\ge 0$; $\overline{A}_2'(x)\ge 0$, $\overline{B}_2'(x)\le 0$, $\overline{C}_2(x)\le 0$ we can conclude that $\nu(x)=0$ and $\tau(x)=0$. Consequently, the solution is unique.

There is valid

Theorem. *Problem* (1)–(4) *cannot have more than one solution in the domain* Ω, *if either*

$$\beta_i = 1-\beta, \quad \delta_i(x) = w_i(x) = 1, \quad i = 1,2,$$

and conditions (9) *and inequalities*

$$\left[\frac{(1-x)^{\beta}a_i(x)}{M_i(x)}\right]' \geq 0, \quad \left[\frac{x^{\beta}b_i(x)}{M_i(x)}\right]' \leq 0, \quad \frac{x^{\beta}(1-x)^{\beta}c_i(x)}{M_i(x)} \leq 0, \quad i = 1,2,$$

are fulfilled, or

$$\beta_i = \beta, \quad \delta_i(x) = x^{2\beta-1}, \quad w_i(x) = (1-x)^{2\beta-1}, \quad i = 1,2,$$

and conditions (11) *and inequalities*

$$\left[\frac{(1-x)^{1-\beta}a_i(x)}{N_i(x)}\right]' \geq 0, \quad \left[\frac{x^{1-\beta}b_i(x)}{N_i(x)}\right]' \leq 0,$$

$$\frac{x^{1-\beta}(1-x)^{1-\beta}d_1(x)}{N_1(x)} \leq 0, \quad \frac{x^{1-\beta}(1-x)^{1-\beta}d_2(x)}{N_2(x)} \geq 0$$

are fulfilled.

3. Existence of solution. Let $\beta_i = 1-\beta$, $\delta_i(x) = w_i(x) \equiv 1$. The equality $\nu_1(x) = \nu_2(x)$ enables us to deduce from relation (10) the following equation:

$$[A_1(x) - A_2(x)]D_{0x}^{1-2\beta}\tau(x) + [B_1(x) - B_2(x)]D_{x1}^{1-2\beta}\tau(x) + [C_1(x) - C_2(x)]\tau(x) = F_2(x) - F_1(x).$$

Let $A_1(x) - A_2(x) \neq 0$. Then we rewrite the latter equation in the form

$$D_{0x}^{1-2\beta}\tau(x) + P(x)D_{x1}^{1-2\beta}\tau(x) + Q(x)\tau(x) = F_2^*(x), \tag{13}$$

where

$$P(x) = \frac{B_1(x) - B_2(x)}{A_1(x) - A_2(x)}, \quad Q(x) = \frac{C_1(x) - C_2(x)}{A_1(x) - A_2(x)}, \quad F_2^*(x) = \frac{F_2(x) - F_1(x)}{A_1(x) - A_2(x)}.$$

We apply operator $D_{0x}^{2\beta-1}$ to both sides of (13), and obtain

$$\tau(x) + D_{0x}^{2\beta-1}P(x)D_{x1}^{1-2\beta}\tau(x) + D_{0x}^{2\beta-1}Q(x)\tau(x) = D_{0x}^{2\beta-1}F_2^*(x). \tag{14}$$

Then we use results of the work [7] (pp. 97–103), and write Eq. (14) in the form

$$A^*(x)\tau(x) + \int_0^1 \frac{K(x,\xi)\tau(\xi)d\xi}{\xi - x} = \overline{F}_2^*(x), \tag{15}$$

where $A^*(x) = 1 + \pi\cot(2\pi\beta)P(x)$,

$$K(x,\xi) = \begin{cases} \left[\frac{\sin(2\pi\beta)}{\pi}K_3(x,\xi) - \frac{\sin(2\pi\beta)}{\pi}K_1(x,\xi) - K_5(x,\xi)\right](\xi - x) & \text{for} \quad \xi \leq x; \\ \frac{\sin(2\pi\beta)}{\pi}[K_4(x,\xi) - K_2(x,\xi)](x-\xi) & \text{for} \quad \xi \geq x, \end{cases}$$

$$K_1(x,\xi) = \int_0^{\xi} P_1(x,t)dt, \quad K_2(x,\xi) = \int_0^{x} P_1(x,t)dt,$$

$$K_3(x,\xi) = \frac{d}{dx}\int_0^{\xi} P_2(x,t)dt, \quad K_4(x,\xi) = \frac{d}{dx}\int_0^{x} P_2(x,t)dt,$$

$$P_1(x,t) = \frac{P'(t)}{(x-t)^{2\beta}(\xi-t)^{1-2\beta}}, \quad P_2(x,t) = \frac{P(t)}{(x-t)^{2\beta}(\xi-t)^{1-2\beta}},$$

$$K_5(x,\xi)=\frac{1}{\Gamma(1-2\beta)}\frac{Q(\xi)}{(x-\xi)^{1-2\beta}},\quad \overline{F}_2^*(x)=\frac{1}{\Gamma(1-2\beta)}\int_0^x\frac{F_2^*(\xi)d\xi}{(x-\xi)^{1-2\beta}}.$$

Then we deduce from properties of kernels $K_i(x,\xi)$, $i=\overline{1,5}$, that the kernel $K(x,\xi)$ is twice continuously differentiable in square $0<\xi, x<1$ for $\xi\neq x$; $\overline{F}_2^*(x)\in C(\overline{I})\cap C^2(I)$.

Thus, Eq. (15) is singular integral equation of second kind for $A^*(x)\neq 0$. The condition $[A^*(x)]^2+K^2(x,x)\neq 0$ guarantees the existence of regularizer that reduces Eq. (15) to a Fredholm integral equation of second kind. Therefore, the uniqueness of desired solution implies existence of solution to problem (1)–(4).

If $\beta_i=\beta$, $\delta_i(x)=x^{2\beta-1}$, $w_i(x)=(1-x)^{2\beta-1}$, then we obtain from equalities $\nu_1(x)=\nu_2(x)=\nu(x)$ and (12) the relation

$$\nu(x)+\int_0^1\frac{K(x,t)\nu(t)dt}{|t-x|^{2\beta}}=F_1^*(x), \tag{16}$$

where

$$K(x,t)=\begin{cases}\frac{1}{\Gamma(1-2\beta)}\frac{\overline{A}_1(x)-\overline{A}_2(x)}{\overline{C}_1(x)-\overline{C}_2(x)}, & x\geq t;\\ \frac{1}{\Gamma(1-2\beta)}\frac{\overline{B}_1(x)-\overline{B}_2(x)}{\overline{C}_1(x)-\overline{C}_2(x)}, & x\leq t,\end{cases}$$

$$F_1^*(x)=\frac{\overline{F}_2(x)-\overline{F}_1(x)}{\overline{C}_1(x)-\overline{C}_2(x)},\quad \overline{C}_1(x)-\overline{C}_2(x)\neq 0.$$

Thus, the question on existence of solution to the problem is equivalently reduced to a Fredholm equation of second kind (16), and its unconditional solvability in the required class of functions follows from the uniqueness of solution. The obtained $\nu(x)$ determines $\tau(x)$ and solution to problem (1)–(4) in domains Ω_1 and Ω_2 as solutions to the Cauchy problems.

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