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Periodic Solutions to Nonlinear Nonautonomous System of Differential Equations

M. T. Teryokhin* **and O. V. Baeva**

Ryazan State University named after S. A. Esenin
ul. Svobody 46, Ryazan, 390000 Russia

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Abstract—We prove a theorem on the existence of nonzero periodic solution to a system of differential equations by the method of fixed point of nonlinear operator defined on a topological product of two compact sets.

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Consider a nonautonomous nonlinear system of differential equations of the form

$$\dot{x} = (A + B(t,\lambda) + X(t,\varphi,x,\varepsilon,\lambda))x, \tag{1'}$$

$$\dot{\varphi} = \mu(t,\varepsilon) + \Phi(t,\varphi,x,\varepsilon,\lambda), \tag{1''}$$

where x, φ are n-, p-dimensional vectors respectively, ε, λ are l-, q-dimensional vector-parameters, A, $B(t,\lambda)$, $X(t,\varphi,x,\varepsilon,\lambda)$ are $(n\times n)$-matrices, $\mu(t,\varepsilon)$, $\Phi(t,\varphi,x,\varepsilon,\lambda)$ are p-dimensional vector-functions.

We introduce the following notations: $|z|=\max\limits_i\{|z_i|\}$, $\|D(t)\|=\max\limits_{|z|\le 1}|D(t)z|$, $\|D\|=\sup\limits_{[0,T]}\|D(t)\|$, z is a vector, $D(t)$ is a matrix, $M(\delta_0)=[0,T]\times M_0(\delta_0)\times E_n\times E(\delta_0)\times\Lambda(\delta_0)$, $M_0(\delta_0)=\{\varphi\in E_p : |\varphi|\le\delta_0\}$, $E(\delta_0)=\{\varepsilon\in E_l : |\varepsilon|\le\delta_0\}$, $\Lambda(\delta_0)=\{\lambda\in E_q : |\lambda|\le\delta_0\}$, E_s is an s-dimensional vector space, $T>0$, $\delta_0>0$ are some numbers.

We will assume from now on that vector-functions $\mu(t,\varepsilon)$, $\Phi(t,\varphi,x,\varepsilon,\lambda)$ and matrices $B(t,\lambda)$, $X(t,\varphi,x,\varepsilon,\lambda)$ are defined and continuous on the set $M(\delta_0)$, and the following conditions hold true on $M(\delta_0)$:

$$\|\Phi(t,\varphi',x,\varepsilon',\lambda')-\Phi(t,\varphi'',x,\varepsilon'',\lambda'')\|\le c_1(\delta)|\varphi'-\varphi''|+c_2(\delta)|\varepsilon'-\varepsilon''|+c_3(\delta)|\lambda'-\lambda''|,$$

$$\|X(t,\varphi',x,\varepsilon',\lambda')-X(t,\varphi'',x,\varepsilon'',\lambda'')\|\le l_1(\delta)|\varphi'-\varphi''|+l_2(\delta)|\varepsilon'-\varepsilon''|+l_3(\delta)|\lambda'-\lambda''|,$$

$\|B(t,\lambda')-B(t,\lambda'')\|\le a|\lambda'-\lambda''|$, $0<\delta\le\delta_0$, $|\mu(t,\varepsilon')-\mu(t,\varepsilon'')|\le b|\varepsilon'-\varepsilon''|$, a, b are some positive numbers, for each $i=\overline{1,3}$ and $\delta\to 0$, $c_i(\delta)\to 0$, $l_i(\delta)\to 0$ uniformly with respect to t, x on the set $[0,T]\times R$, R is an arbitrary fixed closed and bounded subset of the space E_n; $X(t,\varphi,0,\varepsilon,\lambda)=0$, $\Phi(t,\varphi,0,\varepsilon,\lambda)=0$, $B(t,0)=0$, $\mu(t,0)=0$, $\lim\limits_{\lambda\to 0}B(t,\lambda)=0$, $\lim\limits_{x\to 0}X(t,\varphi,x,\varepsilon,\lambda)=0$, $\lim\limits_{x\to 0}\Phi(t,\varphi,x,\varepsilon,\lambda)=0$; $\int\limits_0^T\mu(t,\varepsilon)dt=G\varepsilon+\Theta(\varepsilon)$, G is a matrix, $|\Theta(\varepsilon')-\Theta(\varepsilon'')|\le(\tau(\delta))|\varepsilon'-\varepsilon''|$, $\tau(\delta)\to 0$ for $\delta\to 0$; vector-functions $\mu(t,\varepsilon)$, $\Phi(t,\varphi,x,\varepsilon,\lambda)$ and matrices $B(t,\lambda)$, $X(t,\varphi,x,\varepsilon,\lambda)$ are T-periodic in t in the definition domain.

Problem. For system (1), derive conditions of the existence of nonzero T-periodic solution.

The problem of existence of periodic solution to system (1) was considered in [1–3], where one of the basic requirements was the assumption that all eigenvalues of matrix A have nonzero real parts. In this

*E-mail: m.terehin@rsu.edu.ru.

paper, we suggest the way to construct the transformation of a periodic function to a periodic one, at that the matrix A may have zero or pure imaginary eigenvalues.

The problem of finding conditions for the system of differential equations to have nonzero periodic solutions, was considered in several papers, among them [4–6]. Investigation of existence conditions for nonzero periodic solutions were restricted there to the search of existence conditions for small solutions to branching equations. The basic equation of branching in the above mentioned papers was the Lyapunov–Schmidt equation constructed by means of representation of the space in the form of direct sum of two subspaces, one of which is finite-dimensional with the dimension equal the number of zeros of linear operator. In order to define the existence conditions for small solution of branching equation, they used mainly the theory of indices of zero point of a vector field and representation of the solution in the form of series.

In this paper, the existence of nonzero periodic solutions is established by the method of fixed point of nonlinear operator. It is worthy to note that the system of the form

$$\dot{y} = S(v)y + f(t, y, v),$$

where y, f are n-dimensional vectors, $S(v)$ is a matrix, v is a vector-parameter, can be reduced to system (1) by introduction of polar coordinates.

We denote by $C(d, k)$ the set of T-periodic vector-functions $F(t)$ satisfying conditions $|F(t)| \leq d$, $|F(t') - F(t'')| \leq k|t' - t''|$ for all $[t', t''] \in [0, T]$, d, k are some positive numbers.

Definition. Let $F(t) \in C(d, k)$. By a solution of system $(1'')$ for $x = F(t)$, $\varepsilon \in E(\delta_0)$, $\lambda \in \Lambda(\delta_0)$ we call a vector-function $\varphi(t)$ defined and continuously differentiable on the segment $[a, b]$, satisfying system $(1'')$ for any $t \in [a, b]$.

We denote the solution $\varphi(t)$ to the system

$$\dot{\varphi} = \mu(t, \varepsilon) + \Phi(t, \varphi, F(t), \varepsilon, \lambda)$$

satisfying initial conditions $\varphi(0) = \varphi_0$ by $\varphi(t, \varphi_0, F, \varepsilon, \lambda)$, $\varphi_0 \in E_p$.

Let $\varphi^* \in E_p$ be an arbitrary fixed vector. Our assumptions imply that for $x = 0$ and $\varepsilon = 0$, the solution of system of Eqs. $(1'')$ is the vector-function $\varphi(t) \equiv \varphi^*$. Then due to the theorem about uniqueness of solution and its continuous dependence on initial values and parameter, there exist numbers d^* and $\delta^* \in (0, \delta_0]$ such that for all φ_0, $|\varphi_0 - \varphi^*| \leq \delta^*$, $F(t) \in C(d^*, k)$, $\varepsilon \in E(\delta^*)$, $\lambda \in \Lambda(\delta^*)$, system $(1'')$ has a solution $\varphi(t, \varphi^* + \varphi_0, F, \varepsilon, \lambda)$, $\varphi(0, \varphi^* + \varphi_0, F, \varepsilon, \lambda) = \varphi^* + \varphi_0$ defined on the segment $[0, T]$, continuous on the set $[0, T] \times \{x : |x| \leq d^*\} \times E(\delta^*) \times \Lambda(\delta^*)$ and satisfying the inequality $|\varphi(t, \varphi^* + \varphi_0, F, \varepsilon, \lambda) - \varphi^*| \leq \delta_0$ for all $t \in [0, T]$.

For simplicity, we let $\varphi^* = 0$, $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$.

Let us find the conditions of existence of T-periodic solution for system $(1'')$.

Lemma. *Solution $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ to system $(1'')$ satisfies Lipschitz condition in the variables ε, λ.*

Proof. Let $\varphi_t' = \varphi(t, \varphi_0, F, \varepsilon', \lambda')$, $\varphi_t'' = \varphi(t, \varphi_0, F, \varepsilon'', \lambda'')$. Since

$$\varphi_t = \varphi_0 + \int_0^t (\mu(\xi, \varepsilon) + \Phi(\xi, \varphi_\xi, F(\xi), \varepsilon, \lambda))d\xi,$$

we get that for all $\varepsilon', \varepsilon'' \in E(\delta)$, $\lambda', \lambda'' \in \Lambda(\delta)$, $F(t) \in C(d, k)$, $d \in (0, d^*]$, $t \in [0, T]$,

$$|\varphi_t' - \varphi_t''| \leq \int_0^t (c_1(\delta)|\varphi_\xi' - \varphi_\xi''| + c_2(\delta)|\varepsilon' - \varepsilon''| + c_3(\delta)|\lambda' - \lambda''| + b|\varepsilon' - \varepsilon''|)d\xi$$

$$\leq T[(c_2(\delta) + b|\varepsilon' - \varepsilon''| + c_3(\delta)|\lambda' - \lambda''|)] + \int_0^t c_1(\delta)|\varphi_\xi' - \varphi_\xi''|d\xi.$$

Due to Grönwall–Bellman inequality ([3], P. 108), we obtain

$$|\varphi_t' - \varphi_t''| \leq T[(c_2(\delta) + b|\varepsilon' - \varepsilon''| + c_3(\delta)|\lambda' - \lambda''|)] \exp c_1(\delta)T.$$

With $\gamma_1(\delta) = T(c_2(\delta) + b)\exp c_1(\delta)T$, $\gamma_2(\delta) = Tc_3(\delta)\exp c_1(\delta)T$, we obtain the inequality

$$|\varphi'_t - \varphi''_t| \le \gamma_1(\delta)|\varepsilon' - \varepsilon''| + \gamma_2(\delta)|\lambda' - \lambda''|. \qquad (2) \quad \square$$

It follows straightforwardly from the definition of periodic solution that $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ is T-periodic solution of system $(1'')$ if and only if the equality

$$G\varepsilon + \Theta(\varepsilon) + \int_0^T \Phi(t, \varphi_t, F(t), \varepsilon, \lambda)dt = 0 \qquad (3)$$

holds true.

Theorem 1. *Let* $\operatorname{rank} G = p$. *Then there exist numbers* $\delta_1 \in (0, \delta^*]$, $d \in (0, d^*]$ *such that for any vector-function* $F(t) \in C(d, k)$ *and for any fixed vector* $\lambda \in \Lambda(\delta^*)$, *there exists a value of parameter* $\varepsilon \in \mathrm{E}(\delta_1)$ *for which the solution* $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ *to system* $(1'')$ *is* T*-periodic.*

Proof. As it was mentioned above, for all $F(t) \in C(d, k)$, $\varepsilon \in \mathrm{E}(\delta^*)$, $\lambda \in \Lambda(\delta^*)$, system $(1'')$ has a solution $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ defined and bounded on the segment $[0, T]$.

Let us prove that there exist numbers $\delta_1 \in (0, \delta^*]$, $d \in (0, d^*]$ such that for any vector-function $F(t) \in C(d, k)$ and any vector $\lambda \in \Lambda(\delta^*)$, there exists a vector $\varepsilon \in \mathrm{E}(\delta_1)$ satisfying equality (3).

For simplicity, assume that nonzero minor of order p is placed on the first p columns of matrix G. Then system (3) can be rewritten as

$$\varepsilon_1 = -G_1^{-1}\left(G_2\varepsilon_2 + \int_0^T \Phi(t, \varphi_t, F(t), \varepsilon, \lambda)dt + \Theta(\varepsilon)\right). \qquad (4)$$

Define an operator Γ by the right-hand side of equality (4). Let $\nu \in (0, 1)$ be some number. Taking into consideration the properties of quantities $c_1(\delta)$, $\gamma_1(\delta)$, $c_2(\delta)$, $\tau(\delta)$, we choose a number $\delta_1 \in (0, \delta^*]$ such that for any $\delta \in (0, \delta_1]$, inequality $\|G_1^{-1}\|T(c_1(\delta)\gamma_1(\delta) + c_2(\delta) + \tau(\delta)) < \nu$ holds true. Then for all $\varepsilon'_1, \varepsilon''_1$ such that $(\varepsilon'_1, \varepsilon_2) \in \mathrm{E}(\delta_1)$, $(\varepsilon''_1, \varepsilon_2) \in \mathrm{E}(\delta_1)$, we have

$$\begin{aligned}|\Gamma\varepsilon'_1 - \Gamma\varepsilon''_1| &\le \|G_1^{-1}\|(T(c_1(\delta)|\varphi'_t - \varphi''_t| + c_2(\delta)|\varepsilon'_1 - \varepsilon''_1|) + \tau(\delta)|\varepsilon'_1 - \varepsilon''_1|)\\ &\le \|G_1^{-1}\|(T(c_1(\delta)\gamma_1(\delta) + c_2(\delta)) + \tau(\delta_1))||\varepsilon'_1 - \varepsilon''_1|| \le \nu|\varepsilon'_1 - \varepsilon''_1|,\end{aligned}$$

i.e., the operator Γ on the set $\mathrm{E}^*(\delta_1) = \{\varepsilon_1 : |\varepsilon_1| \le \delta_1\}$ is a contraction. Moreover, we choose numbers $d, \overline{\delta} \in (0, \delta_1]$ such that for all $F(t) \in C(d, k)$, any vector ε_2 ($|\varepsilon_2| \le \overline{\delta}$), and any vector $\lambda \in \Lambda(\delta^*)$, $\varepsilon_1 = 0$, the inequality

$$|\Gamma_0| \le \|G_1^{-1}\|\left(|G_2\varepsilon_2| + \int_0^T |\Phi(t, \varphi_t, F(t), \varepsilon, \lambda)|dt + |\Theta(\varepsilon)|\right) \le (1 - \nu)\delta_1$$

holds true. Thus, $|\Gamma\varepsilon_1| \le |\Gamma\varepsilon_1 - \Gamma_0| + |\Gamma_0| < \delta_1$.

Hence, due to the Banach theorem, for all fixed vectors $|\varepsilon_2| \le \overline{\delta}$, $\lambda \in \Lambda(\delta^*)$, and vector-function $F(t) \in C(d, k)$, operator Γ on the set $\{\varepsilon_1 : |\varepsilon_1| \le \delta_1\}$ has a unique fixed point. Hence, for fixed $|\varepsilon_2^*| \le \overline{\delta}$ we get that $\varepsilon = (\varepsilon_1, \varepsilon_2^*) \in \mathrm{E}(\delta_1)$ is the unique solution to system (3). This means that for all $F(t) \in C(d, k)$, $\lambda \in \Lambda(\delta^*)$, there exists a unique vector $\varepsilon = (\varepsilon_1, \varepsilon_2^*) \in \mathrm{E}(\delta_1)$ such that system $(1'')$ has a solution φ_t satisfying equality (4), and this solution is T-periodic. $\square$

We assume further that numbers d and δ_1 are chosen according to Theorem 1. Proof of Theorem about existence of nonzero periodic solution of system (1) is based on the following

Theorem 2 ([7]). *Let*

1) K *and* Λ *be closed compact subsets of linear normed spaces,* K *be a convex set*;

2) *on the subset of the set* $K \times \Lambda$, *there exists an operator* T_λ *such that for all* $x \in K$, *there exists a unique* $\lambda \in \Lambda$ *satisfying inclusion* $T_\lambda x \in K$;

3) *limits* $x_n \to x_0$, $\lambda_n \to \lambda_0$, $y_n = T_{\lambda_n}x_n$, $y_n \to y_0$ *imply* $y_0 = T_{\lambda_0}x_0$.

Then there exist $x^* \in K$, $\lambda^* \in \Lambda$ *such that* $x^* = T_{\lambda^*}x^*$.

Notice that contraction mapping principle and Bohl–Brouwer theorem about fixed point are generalized in M. A. Krasnoselskii's fixed point principle. The latter can be used to prove local theorems on existence of solutions to systems of differential equations in more general statements [8].

Let $F(t) \in C(d,k)$, $\lambda \in \Lambda(\delta^*)$. Then due to Theorem 1, there exists a unique vector $\varepsilon = (\varepsilon_1, \varepsilon_2^*) \in$ E(δ_1) such that $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ is a T-periodic solution of system $(1'')$.

Together with $(1')$, consider the system

$$\dot{y} = [A + B(t,\lambda) + X(t, \varphi_t, F(t), \varepsilon, \lambda)]y. \tag{5}$$

Let $Y(t, \varphi_t, F, \varepsilon, \lambda)$ be fundamental matrix of system (5), $Y(0, \varphi_t, F, \varepsilon, \lambda) = E$, E be identity matrix. Then a solution $y(t, \varphi_t, F, \varepsilon, \lambda)$, $y(0, \varphi_t, F, \varepsilon, \lambda) = \alpha$ to system (5) is defined by the equality

$$y(t, \varphi_t, F, \varepsilon, \lambda) = Y(t, \varphi_t, F, \varepsilon, \lambda)\alpha, \tag{6}$$

where α is some constant vector. A solution $y(t, \varphi_t, F, \varepsilon, \lambda)$ is T-periodic if and only if

$$(Y(T, \varphi_t, F, \varepsilon, \lambda) - E)\alpha = 0.$$

Notice that equality (6) for any fixed $\lambda \in \Lambda(\delta^*)$, defines an operator $L(\lambda) : F(t) \to y(t, \varphi_t, F, \varepsilon, \lambda)$ on the set $C(d,k)$ which maps any vector-function $F(t) \in C(d,k)$ to a vector-function $y(t, \varphi_t, F, \varepsilon, \lambda)$.

Theorem 3 ([7]). *Fixed points of operator $L(\lambda)$ are T-periodic solutions to system* $(1')$.

A fundamental matrix of solutions to system (5) is representable by the equality

$$Y(t, \varphi_t, F, \varepsilon, \lambda) = \overline{X}(t) + H(t, \varphi_t, F, \varepsilon, \lambda), \tag{7}$$

where $\overline{X}(t)$ is a fundamental matrix of solutions to system $\dot{x} = Ax$, $\overline{X}(0) = E$, matrix $H(t, \varphi_t, F, \varepsilon, \lambda)$, $H(0, \varphi_t, F, \varepsilon, \lambda) = 0$, is a solution to matrix differential equation

$$\dot{D} = [A + B(t,\lambda) + X(t, \varphi_t, F(t), \varepsilon, \lambda)]D + [B(t,\lambda) + X(t, \varphi_t, F(t), \varepsilon, \lambda)]\overline{X}(t)$$

and, hence, it is defined by the equality

$$H(t, \varphi_t, F, \varepsilon, \lambda) = Y(t, \varphi_t, F, \varepsilon, \lambda) \cdot \int_0^t Y^{-1}(\tau, \varphi_\tau, F(\tau), \varepsilon, \lambda)[B(\tau, \lambda) + X(\tau, \varphi_\tau, F(\tau), \varepsilon, \lambda)]\overline{X}(\tau)d\tau, \tag{8}$$

$H(t, \varphi_t, F, \varepsilon, \lambda) = O(|\lambda|) + O(d)$, because matrices $Y(t, \varphi_t, F\varepsilon, \lambda)$, $Y^{-1}(t, \varphi_t, F, \varepsilon, \lambda)$ are continuous in their definition domain. Let us check that a solution to system (5) satisfies the Lipschitz condition in the variables ε, λ. Since for all $t \in [0,T]$,

$$\dot{y}(t, \varphi_t, F(t), \varepsilon, \lambda) = (A + B(t,\lambda) + X(t, \varphi_t, F(t), \varepsilon, \lambda))y(t, \varphi_t, F(t), \varepsilon, \lambda),$$

we get

$$y(t, \varphi_t, F(t), \varepsilon, \lambda) = \alpha + \int_0^t (A + B(\xi, \lambda) + X(\xi, \varphi_\xi, F(\xi), \varepsilon, \lambda))y(\xi, \varphi_\xi, F, \varepsilon, \lambda)d\xi.$$

Hence,

$$|y(t, \varphi_t, F, \varepsilon, \lambda)| \le |\alpha| + \int_0^t (\|A\| + \|B(\xi,\lambda)\| + \|X(\xi, \varphi_\xi, F(\xi), \varepsilon, \lambda)\|)|y(\xi, \varphi_\xi, F, \varepsilon, \lambda)|d\xi.$$

The lemma by Grönwall–Bellman implies

$$|y(t, \varphi_t, F, \varepsilon, \lambda)| \le |\alpha| \exp(\|A\| + R^* + L)T,$$

where $\|B(t,\lambda)\| \le R^*$, $\|X(t, \varphi_t, F(t), \varepsilon, \lambda)\| \le L$ on the set $M_0(\delta^*)$ for any vector-function $F(t) \in C(d_0, k)$, R^*, L are some numbers. Then on the set $[0,T] \times E(\delta^*) \times \Lambda(\delta^*)$,

$$|y(t,\varphi_t',F,\varepsilon',\lambda')-y(t,\varphi_t'',F,\varepsilon'',\lambda'')| \leq \int_0^t |(A+B(\xi,\lambda')+X(\xi,\varphi_t',F(t),\varepsilon',\lambda'))(y(\xi,\varphi_t',F,\varepsilon',\lambda')-y(\xi,\varphi_t'',F,\varepsilon'',\lambda''))+[(B(\xi,\lambda')+X(\xi,\varphi_t',F(t),\varepsilon',\lambda')-(B(\xi,\lambda'')+X(\xi,\varphi_t'',F(t),\varepsilon'',\lambda'')))]y(\xi,\varphi_t'',F,\varepsilon'',\lambda'')|d\xi \leq (\|A\|+R^*+L)\int_0^t |y(\xi,\varphi_t',F,\varepsilon',\lambda')-y(\xi,\varphi_t'',F,\varepsilon'',\lambda'')|d\xi$$
$$+|\alpha|\exp(\|A\|+R^*+L)T(a|\lambda'-\lambda''|+l_1(\delta)|\varphi_t'-\varphi_t''|+l_2(\delta)|\varepsilon'-\varepsilon''|+l_3(\delta)|\lambda'-\lambda''|).$$

Due to Grönwall—Bellman Lemma and inequality (2), we get $|y(t,\varphi_t',F,\varepsilon',\lambda')-y(t,\varphi_t'',F,\varepsilon'',\lambda'')| \leq T(N_1(\delta)|\varepsilon'-\varepsilon''|+N_2(\delta)|\lambda'-\lambda''|)\exp(\|A\|+R^*+L)T$.

This means that on the set $E(\delta)\times\Lambda(\delta)$, $\delta\in(0,\delta^*]$, a solution to system (5), and, hence, elements of matrix $Y(t,\varphi_t,F,\varepsilon,\lambda)$ and matrix $H(t,\varphi_t,F,\varepsilon,\lambda)$ (due to equality (7)) satisfy the Lipschitz condition in the variables ε, λ.

Let us verify the representation

$$H(t,\varphi_t,F,\varepsilon,\lambda)=\overline{X}(t)\int_0^t X^{-1}(\xi)B(\xi,\lambda)\widetilde{X}(\xi)d\xi+o(|\lambda|)+O(d). \tag{9}$$

Straightforwardly, equality $\widetilde{X}(t)=Y(t,\varphi_t,F,\varepsilon,\lambda)-H(t,\varphi_t,F,\varepsilon,\lambda)$ implies $Y^{-1}(t,\varphi_t,F,\varepsilon,\lambda)=\widetilde{X}^{-1}(t)-\widetilde{X}^{-1}(t)H(t,\varphi_t,F,\varepsilon,\lambda)Y^{-1}(t,\varphi_t,F,\varepsilon,\lambda)$. Then due to (8), after appropriate transformations,

$$H(t,\varphi_t,F,\varepsilon,\lambda)=\widetilde{X}(t)\int_0^t X^{-1}(\xi)B(\xi,\lambda)\widetilde{X}(\xi)d\xi+\Psi_1(t,\varphi_t,F,\varepsilon,\lambda)+\Psi_2(t,\varphi_t,F,\varepsilon,\lambda),$$

for all $i\in\{1,2\}$, matrix $\Psi_i(t,\varphi_t,F,\varepsilon,\lambda)$ on the set $E(\delta^*)\times\Lambda(\delta^*)$ satisfies the Lipschitz condition in the variables ε, λ with constant $\gamma_{ij}(\delta)\to 0$ for $\delta\to 0$, $j\in\{1,2\}$. The matrix $\Psi_1(t,\varphi_t,F,\varepsilon,\lambda)$ is a sum of summands each containing a product of matrices $H(t,\varphi_t,F,\varepsilon,\lambda)$, $B(t,\lambda)$ as multiplier; $\Psi_2(t,\varphi_t,F,\varepsilon,\lambda)$ is a sum of summands each containing a matrix $X(t,\varphi_t,F,\varepsilon,\lambda)$ as multiplier. Due to our assertion about matrices $B(t,\lambda)$, $X(t,\varphi_t,x,\varepsilon,\lambda)$, we arrive at conclusion that $\Psi_1(t,\varphi_t,F,\varepsilon,\lambda)=o(|\lambda|)+O(d)$, $\Psi_2(t,\varphi_t,F,\varepsilon,\lambda)=O(d)$ uniformly on the sets $[0,T]\times\{\varphi:|\varphi|\leq\delta_0\}\times\{d:0\leq d\leq d^*\}\times E(\delta^*)$, $[0,T]\times\{\varphi:|\varphi|\leq\delta_0\}\times E(\delta^*)\times\Lambda(\delta^*)$, i.e., equality (9) holds true.

Thus, solution $y(t,\varphi_t,F,\varepsilon,\lambda)$ to system (5) can be presented in the form

$$y(t,\varphi_t,F,\varepsilon,\lambda)=\left(\overline{X}(t)+\overline{X}(t)\int_0^t \overline{X}^{-1}(\xi)B(\xi,\lambda)\overline{X}(\xi)d\xi+o(|\lambda|)+O(d)\right)\alpha.$$

The system of equations for the existence conditions of T-periodic solution to system (5) obtains the form

$$\overline{X}(T)-E+B^*(\lambda)+o(|\lambda|)+O(d)\alpha=0, \tag{10}$$

where

$$B^*(\lambda)=\overline{X}(t)\int_0^T \overline{X}^{-1}(T)B(t,\lambda)\overline{X}(t)dt.$$

It is possible to verify that for $\det(\overline{X}(T)-E)\neq 0$, there exist δ, d such that for any vector-function $F(t)\in C(d,k)$, and any $\varepsilon\in E(\delta)$, $\lambda\in\Lambda(\delta)$, equality (10) does not hold true. Thus, we will assume further that $\det(\overline{X}(T)-E)=0$.

Let $r=\operatorname{rank}(\overline{X}(T)-E)$, $0\leq r<n$. The change of variables $\alpha=S\beta$, where $S=(s_1,s_2,\ldots,s_{n-r})$, $s_1,s_2,\ldots,s_{n-r}$ are linearly independent solutions to the system $(X(T)-E)s=0$, reduces system (10) to the system

$$M(\varphi_t,F,\varepsilon,\lambda)\beta=0, \tag{11}$$

matrix $M(\varphi_t,F,\varepsilon,\lambda)=\overline{B}(\lambda)+o(|\lambda|)+O(d)$, $\overline{B}(\lambda)=B^*(\lambda)S$, $\overline{B}(\lambda)=(\overline{b}_{kj}(\lambda))_{1^n}^{n-r_1}$.

We write the last column of the matrix $M(\varphi_t, F, \varepsilon, \lambda)$ in the form $m_{n-r}(\varphi_t, F, \varepsilon, \lambda) = \overline{b}_{n-r}(\lambda) + o(|\lambda|) + O(d)$, $\overline{b}_{n-r}(\lambda)$ is the last column of the matrix $\overline{B}(\lambda)$.

Let us define the conditions of existence of numbers δ, d such that for all $\lambda \in \Lambda(\delta)$, $F(t) \in C(d, k)$, there exists a vector $\varepsilon \in E(\delta)$ satisfying the equality

$$m_{n-r}(\varphi_t, F, \varepsilon, \lambda) = 0. \tag{12}$$

Suppose that

$$\overline{b}_{n-r}(\lambda) = w_s(\lambda) + o(|\lambda|^s), \tag{13}$$

where $w_s(\lambda)$ is a vector-form of degree s with respect to λ. Due to (13), system (12) obtains the form

$$w_s(\lambda) + o(|\lambda|^s) + O(d) = 0. \tag{14}$$

The change of variables $\lambda = \rho e$, $\rho > 0$, reduces system (14) to the system

$$w_s(e) + O(\rho|e|) + \frac{1}{\rho^s}O(d) = 0. \tag{15}$$

Let $Q = \{e : |e| = 1\}$. If $w_s(e) \neq 0$ for all $e \in Q$, there exist numbers $\rho^* > 0$, $d > 0$ such that for all $\rho \in (0, \rho^*)$, $\varepsilon \in E(\delta^*)$, $F(t) \in C(d, k)$, $e \in Q$ equality (15) does not hold true. Thus, we assume further that there exists a vector $e^* \in Q$ satisfying the equality $w_s(e^*) = 0$. Using Taylor's formula in the neighborhood of point e^* for vector-form $w_s(e)$, we rewrite system (15) in the form

$$D(e^*)\tau + \sum_{k=2}^{s} P_k(e^*, \tau) + O(\rho|e|) + \frac{1}{\rho^s}O(d) = 0, \tag{16}$$

where $D(e^*)$ is a value of Jacobi matrix of vector-form $w_s(e)$ at point e^*, $P_k(e^*, \tau)$ is a vector-form of degree k with respect to τ for all $k = \overline{2, s}$; $e = e^* + \tau$, $|\tau| \leq \Delta$, $\Delta \in (0, 1)$ is some number.

Theorem 4. *Let* 1) $n \leq q$, $\operatorname{rank} D(e^*) = n$; 2) $p \leq l$, $\operatorname{rank} G = p$. *Then system* $(1')$, $(1'')$ *has a nonzero* T*-periodic solution.*

Proof. Due to Theorem 1, there exist numbers $d > 0$, $\delta_1 > 0$, $\overline{\delta} \in (0, \delta_1]$ such that for all $F(t) \in C(d, k)$, $\lambda \in \Lambda(\delta)$ and any fixed vector $|\varepsilon_2^*| \leq \overline{\delta}$, there exists a unique vector $\varepsilon = (\varepsilon_1, \varepsilon_2^*) \in E(\delta_1)$ such that $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ is a T-periodic solution to system $(1'')$. Hence, on the set $\Lambda(\delta^*)$ Theorem 1 defines a function $\varepsilon(\lambda) = (\varepsilon_1(\lambda), \varepsilon_2(\lambda) = \varepsilon_2^*)$ such that $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon(\lambda), \lambda)$ is a T-periodic solution to system $(1'')$. Let us prove that vector-function $\varepsilon(\lambda)$ satisfies the Lipschitz condition on the set $\Lambda(\delta^*)$.

For any fixed vector-function $F(t) \in C(d, k)$, and any vectors $\lambda', \lambda'' \in \Lambda(\delta^*)$ due to (4) and (2), we get $|\varepsilon(\lambda') - \varepsilon(\lambda'')| = |\varepsilon_1(\lambda') - \varepsilon_1(\lambda'')| \leq \eta_1(\delta)|\varepsilon(\lambda') - \varepsilon(\lambda'')| + \eta_2(\delta)|\lambda' - \lambda''|$, $\eta_i(\delta) \to 0$ for $\delta \to 0$, $i \in \{1, 2\}$. We choose a number δ_1 such that for any $\delta \in (0, \delta_1]$, the inequality

$$|\varepsilon(\lambda') - \varepsilon(\lambda'')| \leq \frac{\eta_2(\delta)}{1 - \eta_1(\delta)}|\lambda' - \lambda''| \tag{17}$$

holds true, i.e., vector-function $\varepsilon(\lambda)$ on the set $\Lambda(\delta^*)$ satisfies the Lipschitz condition.

Substituting solution $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon(\lambda), \lambda)$ and vector-function $F(t) \in C(d, k)$, $\varepsilon = \varepsilon(\lambda)$ into system $(1')$, we get system (6), whose solution is vector-function $y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)$ defined by equality (6). We reduce the problem to the proof of existence of numbers d and δ_1 such that for all vector-function $F(t) \in C(d, k)$, there exists a vector $\lambda \in \Lambda(\delta_1)$ satisfying the equality $m_{n-r}(\varphi_t, F, \varepsilon(\lambda), \lambda) = 0$. For simplicity, we assume that a nonzero minor of order n of the matrix $D(e^*)$ of system (16) is placed at the first n columns of the matrix $D(e^*)$. Then system (16) obtains the form

$$\tau_1 = -D_1^{-1}\Big(D_2\tau_2 + \sum_{k=2}^{s} P_k(e^*, \tau) + O(\rho|e^* + \tau|) + \frac{1}{\rho^s}O(d)\Big), \tag{18}$$

where $\det D_1 \neq 0$, $D = [D_1, D_2]$, τ_1 is an n-dimensional vector, τ_2 is a $(q-n)$-dimensional vector, $\tau = (\tau_1, \tau_2)$.

Define an operator Γ_1 by the right-hand side of equality (18). Then due to inequalities (2), (17), we can choose a number σ_1 such that for all $(\tau_1', \tau_1'') \in \{\tau_1 : |\tau_1| \leq \sigma_1\}$, inequality $|\Gamma_1\tau_1' - \Gamma_1\tau_1''| \leq \nu|\tau_1' - \tau_1''|$ holds true, $\nu \in (0,1)$ is a number. Moreover, we choose numbers ρ^*, $d > 0$, $\sigma_2 \in (0, \sigma_1]$ such that for any vector-function $F(t) \in C(d,k)$, and any vector $|\tau_2| \leq \sigma_2$, $\tau_1 = 0$, $\rho = \rho^*$, inequality $|\Gamma 0| < (1-\nu)\cdot\sigma_1$ holds true. Hence, $|\Gamma_1\tau_1| \leq |\Gamma_1\tau_1 - \Gamma_1 0| + |\Gamma_1 0| < \sigma_1$. This means that for any fixed $|\tau_2| \leq \sigma_2$, operator Γ_1 has a unique fixed point on the set $\{\tau_1 : |\tau_1| \leq \sigma_2\}$.

Fix $|\tau_2^*| \leq \sigma_2$. Then letting $\delta_1 = \rho^*(1+\sigma_2)$, we get that for all vector-function $F(t) \in C(d,k)$ there exists a unique vector

$$\lambda \in P(\delta_1) = \{\lambda = \rho^*(e^* + \tau) : \tau = (\tau_1, \tau_2^*),\ |\tau_1| \leq \sigma_2\} \subset \Lambda(\delta_1)$$

satisfying system (12). This means that the last column of matrix $M(\varphi_t, F, \varepsilon(\lambda), \lambda)$ of system (11) $m_{n-r}(\varphi_t, F, \varepsilon(\lambda), \lambda) = 0$. Hence solution to system (11) is vector β, whose first $n-r-1$ coordinates equal zero, a coordinate at position $n-r$ is nonzero, solution $y(t, \varphi_t, F, \varepsilon(\lambda), \lambda) = Y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)$ $S\beta$ to system (5) is nonzero T-periodic. Let us prove that we can choose a vector β such that solution $y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)$ belongs to the set $C(d,k)$.

Since matrix $Y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)$ is bounded, with the choice of vector β we get

$$|y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)| \leq \|Y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)\|\, \|S\|\beta \leq d,$$

$$|\dot{y}(t, \varphi_t, F, \varepsilon(\lambda), \lambda)| \leq (\|A\| + \|B(t,\lambda)\| + \|X(t, \varphi_t, F, \varepsilon(\lambda), \lambda)\|)|y(t, \varphi_t, F, \varepsilon(\lambda), \lambda)| \leq R\beta \leq k,$$

$R > 0$ is some number. Due to Lagrange's mean value theorem,

$$|y(t', \varphi_t, F, \varepsilon(\lambda), \lambda) - y(t'', \varphi_t, F, \varepsilon(\lambda), \lambda)| \leq \max_{[0,T]} |\dot{y}(t, \varphi_t, F, \varepsilon(\lambda), \lambda)| \leq k|t' - t''|.$$

This implies $y(t, \varphi_t, F, \varepsilon(\lambda), \lambda) \in C(d,k)$. Since matrix $X(t, \varphi, F, \varepsilon(\lambda), \lambda)$ is uniformly continuous on the set $C(d,k) \times \Lambda(\delta_1)$ with respect to $t \in [0,T]$, $|\varphi| \leq \delta^*$, and matrix $B(t,\lambda)$ is continuous on the set $[0,T] \times \Lambda(\delta_1)$, operator $\Gamma(\lambda)$ defined by equality (6) is continuous on the set $C(d,k) \times \Lambda(\delta_1)$.

Thus, we arrive at conclusion that there exist numbers $d > 0$, $\delta_1 > 0$ such that on the set $C(d,k) \times P(\delta_1)$ operator $\Gamma(\lambda)$ is continuous, and for any vector-function $F(t) \in C(d,k)$, there exists a unique vector $\lambda \in P(\delta_1)$ satisfying relation $\Gamma(\lambda)F(t) = y(t, \varphi_t, F, \varepsilon(\lambda), \lambda) \in C(d,k)$. Due to Theorem 2, there exist $F^*(t) \in C(d,k)$ and $\lambda^* \in P(\delta_1)$ such that $F^*(t)$ is a fixed point of operator $\Gamma(\lambda^*)$, i.e., $F^*(t) = Y(t, \varphi_t, F^*, \varepsilon(\lambda^*), \lambda^*)\alpha$, $\alpha = S\beta$. Then due to Theorem 3, $F^*(t)$ is nonzero T-periodic solution of the system $(1')$ when $\varepsilon^* = \varepsilon(\lambda^*)$, λ^*.

As a result, there exist values ε^*, λ^*, $\varepsilon^* = \varepsilon(\lambda^*)$ of parameters ε, λ for which $F^*(t) = Y(t, \varphi_t, F^*, \varepsilon(\lambda^*), \lambda^*)\alpha$, $\varphi_t = \varphi(t, \varphi_0, F^*, \varepsilon^*, \lambda^*)$ is nonzero T-periodic solution to system $(1')$, $(1'')$. $\square$

Suppose that $\Theta(\varepsilon) = f_m(\varepsilon) + o(|\varepsilon|^m)$, where $f_m(\varepsilon)$ is a vector-form of degree m with respect to ε, $\lim\limits_{\varepsilon\to 0} \frac{o(|\varepsilon|^m)}{|\varepsilon|^m} = 0$. From the properties of vector-function $\Theta(\varepsilon)$ it follows that $o(|\varepsilon|^m)$ satisfies the Lipschitz condition with a constant tending to zero as $\delta \to 0$. Then system (3) can be rewritten in the form

$$G\varepsilon + f_m(\varepsilon) + \int_0^T \Phi(t, \varphi_t, F(t), \varepsilon, \lambda)dt + o(|\varepsilon|^m) = 0. \tag{19}$$

Let $\operatorname{rank} G = \omega$, $0 \leq \omega < p$. With the help of elementary transformations and the change of variable $\varepsilon = \sigma\upsilon$, where $\sigma > 0$, $|\upsilon| \leq \eta$, $\eta > 1$ is some number, system (19) can be transformed into the system

$$N(\upsilon) + O(\sigma|\upsilon|) + h(\sigma, \varphi_t, F, \varepsilon, \lambda) = 0, \tag{20}$$

where $N(\upsilon) = \operatorname{colon}(G^*\upsilon, f_m^*(\upsilon))$, $f_m^*(\upsilon)$ is a vector-form of degree m with respect to υ, G^* is a matrix, $\operatorname{rank} G^* = \omega$, $\lim\limits_{\sigma\to 0} O(\sigma|\upsilon|) = 0$, $\lim\limits_{x\to 0} h(\sigma, \varphi_t, F, \varepsilon, \lambda) = 0$ uniformly with respect to $|\upsilon| \leq \eta$, $|\ \varphi_t| \leq \delta^*$, $\varepsilon \in \mathrm{E}(\delta^*)$, $\lambda \in \Lambda(\delta^*)$.

Theorem 5. *Let $N(v) \neq 0$ for all $|v| = 1$. Then there exists a number $d > 0$ such that for any vector-function $F(t) \in C(d, k)$, any fixed $\lambda \in \Lambda(\delta^*)$, in any neighborhood of the point $\varepsilon = 0$, there exists a set such that for any of its point ε, solution $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ to system $(1'')$ is not T-periodic.*

Proof is based on the fact that there exists a number $b > 0$ satisfying inequality $|N(v)| \geq b$ on the set $\{v : |v| = 1\}$. Then numbers $d > 0$, σ can be chosen in such a way that for all $F(t) \in C(d, k)$, $\lambda \in \Lambda(\delta^*)$, $|v| = 1$ and, consequently, any $\varepsilon = \sigma v$, we have $|N(v) + O(\sigma|v|) + h(\sigma, \varphi_t, F, \varepsilon, \lambda)| > 0$. This means that the solution $\varphi_t = \varphi(t, \varphi_0, F, \varepsilon, \lambda)$ is not T-periodic. □

Thus, we assume that there exists a vector $v^*(|v^*| = 1)$ such that $N(v^*) = 0$. Using Taylor's formula in the neighbourhood of point v^* for vector-form $f_m^*(v)$, and the change of variables $v - v^* = z$, $|z| \leq \Delta$, $\Delta \in (0, 1)$, we rewrite system (20) in the form

$$S^* z + \sum_{k=2}^{m} P_k^*(v^*, z) + O(\sigma|v|) + h(\sigma, \varphi_t, F, \varepsilon, \lambda) = 0,$$

where $S^* = \operatorname{colon}(G^*, G^{**}(v^*))$ is a matrix, $G^{**}(v^*)$ is a value of Jacobi matrix for vector-form $f_m^*(v)$ at point v^*, $P_k^*(v^*, z)$ is a vector-form of degree k with respect to z.

Theorem 6. *Let $\operatorname{rank} D(e^*) = n$, $\operatorname{rank} S^* = p$. Then system $(1')$, $(1'')$ has nonzero T-periodic solution.*

Proof is similar to that of Theorem 4.

Suppose that the rank of matrix $D(e^*)$ in system (16) is r_1, $0 \leq r_1 < n$. Then with the help of elementary transformations and the change of variables $\tau = \rho_1 e$, $\rho_1 > 0$, $|e| \leq \eta_1$, $\eta_1 > 1$ is some number, we reduce system (16) to the system whose main term is defined by equality $D^*(e) = \operatorname{colon}(D^{(1)}(e^*)e, P_{k^*}^*(e^*, e))$, $P_{k^*}^*(e^*, e)$ is a vector-form of degree k^* with respect to vector e provided that $P_k^*(e^*, e) \equiv 0$ for $0 \leq k < k^*$, $P_{k^*}^*(e^*, e)$ is not identically zero on the set $\{e : |e| = 1\}$, $\operatorname{rank} D^{(1)}(e^*) = r_1$.

Let there exist a vector $\overline{e}$, $|\overline{e}| = 1$ satisfying equality $D^*(\overline{e}) = 0$. Using Taylor's formula in the neighborhood of point $\overline{e}$ for $P_{k^*}^*(e^*, e)$, we get that matrix of linear (with respect to $e - \overline{e} = \Delta e$) summand of transformed system (16) has the form $Q = \operatorname{colon}(D^{(1)}(e^*), P^*(\overline{e}))$ where $P^*(\overline{e})$ is a value of Jacobi matrix for vector-form $P_{k^*}^*(e^*, e)$ at point $\overline{e}$.

Theorem 7. *Let $\operatorname{rank} Q = n$, $\operatorname{rank} G = p$. Then system $(1')$, $(1'')$ has nonzero T-periodic solution.*

Theorem 8. *Let $\operatorname{rank} Q = n$, $\operatorname{rank} S^* = p$. Then system $(1')$, $(1'')$ has nonzero T-periodic solution.*

Proofs of Theorems 7 and 8 are similar to that of Theorem 4.

Remark. 1) In the above theory concerning the search of existence conditions for nonzero periodic solution to system $(1')$, $(1'')$, we solve the problem of vanishing of $n{-}r$th column of matrix $M(\varphi_t, F, \varepsilon, \lambda)$ of system (11). The existence conditions for nonzero periodic solution to system (1) can also be obtained when solving the problem of vanishing of any column of matrix $M(\varphi_t, F, \varepsilon, \lambda)$, and even several columns simultaneously.

2) If at least one of the inequalities $\operatorname{rank} Q < n$, $\operatorname{rank} S^* < p$ is fulfilled, one can use the procedure of obtaining the existence (non-existence) conditions for periodic solutions to system $(1')$, $(1'')$ even further on.

Example. Consider a system of the form

$$\dot{x} = (B(t,\lambda) + X(t,\varphi,x,\varepsilon,\lambda))x, \quad \dot{\varphi} = \mu(t,\varepsilon) + \Phi(t,\varphi,x,\varepsilon,\lambda), \tag{21}$$

where $t \in [0, 2\pi]$, $\varphi \in E_2$, $x \in E_3$, $\varepsilon \in E_2$, $\lambda \in E_4$,

$$B(t,\lambda) = \begin{pmatrix} \lambda_1^2 \cos^2 t & \lambda_1\lambda_3(1-\sin 2t) & \lambda_1\lambda_2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_4 - 4\lambda_4^2 + \lambda_4^3 \cos t, \\ \lambda_1\lambda_4 \sin^2 t & \lambda_2\lambda_3(1-\cos t) & 2\lambda_1^2 + \lambda_2\lambda_4 + 2\lambda_1\lambda_3 - 5\lambda_1\lambda_4 + \lambda_1\lambda_2 \sin t \\ \lambda_2\lambda_3 \sin^2 4t & \lambda_1\lambda_4(2-\cos 2t) & \lambda_2^2 + \lambda_1\lambda_2 + 2\lambda_2\lambda_4 - 4\lambda_3\lambda_4 + \lambda_1\lambda_3 \cos 2t \end{pmatrix},$$

$$X(t,\varphi,x,\varepsilon,\lambda) = \begin{pmatrix} x_1x_2 \cos t & x_1x_3\varepsilon_2\lambda_3 & x_2x_3\lambda_1 \sin t \\ x_2x_3 \sin t \cos\varphi_1 & x_1x_3 \cos 2t & x_1x_2 \sin^2\varphi_1 \\ x_1x_2\varepsilon_1\lambda_1 \cos\varphi_2 & x_1^2x_2\varepsilon_1\lambda_4 \sin t \cos\varphi_2 & x_3\varepsilon_2\lambda_4 \sin 4t \end{pmatrix},$$

$$\mu(t,\varepsilon) = \begin{pmatrix} 2\varepsilon_1 \cos^2 t + 3\varepsilon_1^2(1-\sin 3t) + 4\varepsilon_2^2(3+\cos t) \\ \varepsilon_2 \sin^2 t + 4\varepsilon_1^2 \cos^2 t + 2\varepsilon_2^2 \sin^2 t \end{pmatrix},$$

$$\Phi(t,\varphi,x,\varepsilon,\lambda) = \begin{pmatrix} x_1\lambda_1 \cos\varphi_1 + x_1x_3\varepsilon_1 \sin t \\ x_1x_2 \sin t + x_2x_3\lambda_3 \cos\varphi_2 \end{pmatrix}.$$

By direct calculations, we find

$$\int_0^{2\pi} \mu(t,\varepsilon)dt = \begin{pmatrix} 2\pi\varepsilon_1 + 3\varepsilon_1^2 + 12\varepsilon_2^2 \\ \pi\varepsilon_2 + 4\pi\varepsilon_1^2 + 2\pi\varepsilon_2^2 \end{pmatrix}, \quad G = \begin{pmatrix} 2\pi & 0 \\ 0 & \pi \end{pmatrix}, \quad w_2(\lambda) = \begin{pmatrix} \lambda_1\lambda_2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_4 - 4\lambda_4^2 \\ 2\lambda_1^2 + \lambda_2\lambda_4 + 2\lambda_1\lambda_3 - 5\lambda_1\lambda_4 \\ \lambda_2^2 + \lambda_1\lambda_2 + 2\lambda_2\lambda_4 - 4\lambda_3\lambda_4 \end{pmatrix}.$$

Letting $\lambda = \rho e$, $\rho > 0$, we get

$$w_2(e) = \begin{pmatrix} e_1e_2 + 2e_1e_3 + e_2e_4 - 4e_4^2 \\ 2e_1^2 + e_2e_4 + 2e_1e_3 - 5e_1e_4 \\ e_2^2 + e_1e_2 + 2e_2e_4 - 4e_3e_4 \end{pmatrix}, \quad e = (e_1, e_2, e_3, e_4),$$

$w_2(e^*) = 0$ for $e^* = (1,1,1,1)$.

Using Taylor's formula for $w_2(\varepsilon)$ in the neighborhood of point ε^*, we get $w_2(\varepsilon) = D(\varepsilon^*)\tau + o(|\tau|)$ where $\tau = \varepsilon - \varepsilon^*$,

$$D(\varepsilon^*) = \begin{pmatrix} 3 & 2 & 2 & -7 \\ 1 & 1 & 2 & -4 \\ 1 & 5 & 4 & -2 \end{pmatrix}.$$

Since conditions of Theorem 4 are fulfilled ($\operatorname{rank} D(\varepsilon^*) = 3$, $\operatorname{rank} G = 2$), we arrive at conclusion that system (21) has a nonzero 2π-periodic solution.

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