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Complex Spherical Semi-Designs

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Abstract—We prove a complex analog of Sidelnikov's integral inequality. In discrete case an inequality turns into equality on the complex spherical semi-designs and only on them.

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1. MAIN RESULT

In 1974 V. M. Sidelnikov [1] proved two real inequalities. The first is simple, the second (even for $t = 2k$) is difficult and meaningful. It seems important for comparison with the complex case to give the inequality in [1].

Let $\langle x, y\rangle$ be an ordinary inner product in $\mathbb{R}^n$, $n \geq 2$, and $\|x\|^2 = \sqrt{\langle x, x\rangle}$. Let k be a natural number, $U \subset \mathbb{R}^n$ and μ be a measure on U such that $\int\limits_U \|x\|^{2k}\, d\mu(x) < \infty$. Then the inequality

$$\int_U \int_U \langle x, y\rangle^{2k}\, d\mu(x)\, d\mu(y) \geq c_k \biggl(\int_U \|x\|^{2k}\, d\mu(x) \biggr)^2 \tag{1}$$

holds, here $c_k = (2k-1)!!/(n(n+2)\cdots(n+2k-2))$. It is a strong result though the proof of [1] is complicated. Paper [2] contains simpler proof and considers the case of equality.

Here we find an analog of inequality (1) in the complex case. For $k > 1$ we give the original proof of the new inequality. In the space $\mathbb{C}^n$ we consider the same notation $\langle z, w\rangle$ for the inner product and the norm $\|z\|^2 = \langle z, z\rangle$.

Theorem. *Assume that $U \subset \mathbb{C}^n$ and μ is the measure on U such that $\int\limits_U \|z\|^{2k}\, d\mu(z) < \infty$. Then the inequality*

$$\int_U \int_U |\langle z, w\rangle|^{2k}\, d\mu(z)\, d\mu(w) \geq \widetilde{c}_k \biggl(\int_U \|z\|^{2k}\, d\mu(z) \biggr)^2 \tag{2}$$

holds for

$$\widetilde{c}_k = \binom{n+k-1}{k}^{-1} = \frac{k!}{n(n+1)\cdots(n+k-1)}. \tag{3}$$

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2. PROOF OF THEOREM

First try to prove inequality (2) with the help of inequality (1). Consider $z = x + iy$, $w = u + iv$, here $x, y, u, v \in \mathbb{R}^n$, i is the imaginary unit. Then

$$\bigl|\langle z, w\rangle\bigr|^2 = \bigl(\langle x, u\rangle + \langle y, v\rangle\bigr)^2 + \bigl(\langle y, u\rangle - \langle x, v\rangle\bigr)^2.$$

In the case of $k = 1$ the integral of $\bigl|\langle z, w\rangle\bigr|^{2k}$ equals the sum of two integrals, each of which is subject to inequality (1). The result is (2) with $k = 1$.

In the case of $k > 1$ we need the new proof. Then the constant $\widetilde{c}_k$ in (2) is smaller than the constant c_k of (1) for $k > 1$ (Section 4). We give the cases when inequality (2) turns into equality (Sections 3 and 4).

In order to prove the main theorem we need certain auxiliary statements and constructions.

Let $\Omega_n = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ be a unit sphere in $\mathbb{C}^n$. Let ω be a measure on Ω_n, invariant under rotations, $\omega(\Omega_n) = 1$.

In [3] (P. 25) one can find the integral

$$I(\alpha, \beta) := \int_{\Omega_n} z^\alpha \overline{z}^\beta \, d\omega(z) = \begin{cases} 0, & \alpha \neq \beta; \\ \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}, & \alpha = \beta. \end{cases} \tag{4}$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\overline{z}^\beta = \overline{z}_1^{\beta_1} \cdots \overline{z}_n^{\beta_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$.

Let us introduce the space Hom (k), consisting of generalized polynomials, for a natural k:

$$Q(z) = \sum_{|\alpha|=|\beta|=k} q(\alpha,\, \beta) z^\alpha \overline{z}^\beta, \quad z \in \mathbb{C}^n,$$

here $q(\alpha, \beta)$ are arbitrary complex coefficients. Let

$$F(z) = \sum_{|\alpha|=|\beta|=k} f(\alpha, \beta) z^\alpha \overline{z}^\beta \tag{5}$$

be a polynomial of Hom (k). We introduce the scalar product

$$[Q, F] = \sum_{|\alpha|=|\beta|=k} \frac{\overline{q(\alpha, \beta)} f(\alpha, \beta)}{c(\alpha)c(\beta)}, \ \text{ where } \ c(\alpha) = \frac{k!}{\alpha!}.$$

The complex conjugation over the left factor $q(\alpha, \beta)$ ensures important condition (6). In the real case the similar scalar product can be found in [4]. We introduce the polynomial

$$\rho_w(z) = |\langle w, z\rangle|^{2k} = \langle w, z\rangle^k \langle z, w\rangle^k = \sum_{|\alpha|=|\beta|=k} r(\alpha, \beta) z^\alpha \overline{z}^\beta,$$

for a fixed $w \in \mathbb{C}^n$, here $r(\alpha, \beta) = c(\alpha)c(\beta)\overline{w}^\alpha w^\beta$. The condition

$$[\rho_w, F] = \sum_{|\alpha|=|\beta|=k} \frac{\overline{r(\alpha, \beta)} f(\alpha, \beta)}{c(\alpha)c(\beta)} = F(w) \tag{6}$$

holds.

Consider the polynomial of Hom (k)

$$\omega_k(z) = \|z\|^{2k} = \sum_{|\alpha|=k} c(\alpha) z^\alpha \overline{z}^\alpha. \tag{7}$$

Lemma. *The equality $[\omega_k, \omega_k] = 1/\widetilde{c}_k$ holds for the polynomial $\omega_k(z)$, here $\widetilde{c}_k$ is defined by formula* (3).

Proof. For polynomial (7) the coefficients $q(\alpha, \beta)$ vanish for $\alpha \neq \beta$ and $q(\alpha, \alpha) = c(\alpha)$. Hence

$$[\omega_k, \omega_k] = \sum_{|\alpha|=k} \frac{c(\alpha)c(\alpha)}{c(\alpha)c(\alpha)} = \sum_{|\alpha|=k} 1.$$

The number of multi-indices α such that $|\alpha| = k$, equals

$$\binom{n+k-1}{k} = \frac{1}{\widetilde{c}_k}.$$

□

Proof of Theorem. Inequality (2) is nothing else but the Cauchy–Schwartz equation $[A, A][\omega_t, \omega_t] \geq [A, \omega_t]^2$, applied to the polynomials

$$A(z) = \int_U \rho_w(z)\, d\mu(w), \quad \omega_k(z) = \|z\|^{2k}.$$

Really, one can write down

$$A(z) = \sum_{|\alpha|=|\beta|=k} c(\alpha)c(\beta)M(\alpha, \beta)z^\alpha \overline{z}^\beta, \text{ here } M(\alpha, \beta) = \int_U \overline{w}^\alpha w^\beta\, d\mu(w).$$

We scalarly multiply A by an arbitrary polynomial F of the type (5). Then

$$[A, F] = \sum_{|\alpha|=|\beta|=k} \overline{M(\alpha, \beta)} f(\alpha, \beta) = \int_U F(w)\, d\mu(w).$$

Hence

$$[A, A] = \int_U A(w)\, d\mu(w) = \int_U \int_U |\langle z, w\rangle|^{2k}\, d\mu(z)\, d\mu(w),$$
$$[A, \omega_k] = \int_U \omega_k(w)\, d\mu(w) = \int_U \|w\|^{2k}\, d\mu(w).$$

Moreover, $[\omega_k, \omega_k] = 1/\widetilde{c}_k$ by Lemma 1. Put the inner products into Cauchy–Schwartz inequality and multiply it by $\widetilde{c}_k$. We obtain inequality (2). □

Proof of Theorem yields the condition under which inequality (2) turns into equality.

Proposition 1. *Assume that* $I_k = \int_U \|w\|^{2k}\, d\mu(w)$. (2) *turns into equality if and only if the following relation holds*:

$$\int_U |\langle z, w\rangle|^{2k}\, d\mu(w) = \widetilde{c}_k I_k \|z\|^{2k}, \quad z \in \mathbb{C}^n. \tag{8}$$

Proof. Cauchy–Schwartz inequality turns into equality if and only if there exists a constant λ such that $A = \lambda\omega_k$. We multiply this relation scalarly by ω_k. Then $[A, \omega_k] = \lambda[\omega_k, \omega_k]$. In the proof of Theorem we can see that $[A, \omega_k] = I_k$, $[\omega_k, \omega_k] = 1/\widetilde{c}_k$. Hence $\lambda = \widetilde{c}_k I_k$. The equality $A = \widetilde{c}_k I_k \omega_k$ in expanded form is (8). □

3. AN EXAMPLE OF THE EQUALITY

Let $U = \Omega_n$ be a unit sphere in $\mathbb{C}^n$, $\mu = \omega$ be a rotation invariant measure on the sphere, $\omega(\Omega_n) = 1$. Then we may replace z with $\|z\|e_1$ in integral (8), here $e_1 = (1, 0, \ldots, 0)$. Then $\langle e_1, w\rangle = w_1$ and for any $z \in \mathbb{C}^n$

$$\int_{\Omega_n} |\langle z, w\rangle|^{2k}\, d\omega(w) = \|z\|^{2k} \int_{\Omega_n} |w_1|^{2k}\, d\omega(w)$$

$$= \|z\|^{2k} \int_{\Omega_n} w_1^k \overline{w}_1^k \, d\omega(w) = \|z\|^{2k} \frac{(n-1)!k!}{(n+k-1)!} = \widetilde{c}_k \|z\|^{2k}.$$

Here we apply formula (4). Moreover,

$$I_k = \int_{\Omega_n} \|w\|^{2k} \, d\omega(w) = \int_{\Omega_n} d\omega(w) = 1.$$

So we have equality (8). Thus in the case of $U = \Omega_n$, $\mu = \omega$ inequality (2) turns into an equality.

Real inequality (1) turns into equality if $U = S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. This fact can be found in [1].

4. DISCRETE ANALOG OF SIDELNIKOV INEQUALITY

The discrete variant appears for finite set $U = \{z_1, \dots, z_m\}$ in $\mathbb{C}^n$, $\mu(\{z_i\}) = W_i > 0$. Without loss of generality assume that $\sum_{i=1}^{m} W_i = 1$. The inequality appears more expressive if we assume that the points $\{z_i\}$ belong to the unit sphere Ω_n. Here inequality (2) turns into

$$\sum_{i=1}^{m} \sum_{j=1}^{m} W_i W_j |\langle z_i, z_j \rangle|^{2k} \ge \widetilde{c}_k, \tag{9}$$

here k is an arbitrary natural number, $\widetilde{c}_k$ is given by (3). The points $\{z_i\}$ on the sphere Ω_n are not necessarily mutually different. Inequality (9) was proved in [5] in the equal weight case and with the wrong constant.

Condition of equality for (9) follows from Proposition 1. In our case $I_k = (W_1 + \dots + W_m)^2 = 1$.

Proposition 2. *Inequality* (9) *is an equality if and only if the sequence* $\{z_i\}_{i=1}^m$ *on the sphere* Ω_n *and the weight sequence* $\{W_i\}_{i=1}^m$, $W_1 + \dots + W_m = 1$ *admit the relation*

$$\sum_{i=1}^{m} W_i |\langle z_i, z \rangle|^{2k} = \widetilde{c}_k \|z\|^{2k}, \quad z \in \mathbb{C}^n. \tag{10}$$

Definition. The sequence of points $\{z_i\}_{i=1}^m$ on the sphere Ω_n is called the complex spherical semi-design of order $2k$ with weights $W_i > 0$, $W_1 + \dots + W_m = 1$, if (10) holds.

So discrete Sidelnikov inequality (9) turns into equality on the spherical semi-designs of order $2k$ and only on them. Vectors $\{z_i\}$ and weights $\{w_i\}$ constitute the semi-design of order $2k$ if the following equality holds:

$$\sum_{i=1}^{m} \sum_{j=1}^{m} W_i W_j \big|\langle z_i, z_j \rangle\big|^{2k} = \widetilde{c}_k.$$

In the real case the notion of the weighted spherical semi-design appeared in [6], and the notion of weighted design was introduced in [7]. The sequence $\{x_i\}_{i=1}^m$ of vectors on the sphere S^{n-1} in $\mathbb{R}^n$ is said to be a spherical weighted semi-design of order $2k$ if

$$\sum_{i=1}^{m} W_i \langle x_i, x \rangle^{2k} = c_k \|x\|^{2k}, \quad x \in \mathbb{R}^n,$$

here $c_k = (2k-1)!/\big(n(n+2)\cdots(n+2k-2)\big)$. The constants c_k and $\widetilde{c}_k$ are subject to the relations $c_1 = \widetilde{c}_1 = 1/n$, and for $k \ge 2$ we have the inequalities

$$\widetilde{c}_k = \prod_{s=0}^{k-1} \frac{s+1}{n+s} < c_k = \prod_{s=0}^{k-1} \frac{2s+1}{n+2s} \ \text{ for } \ n \ge 2.$$

Thus for $k = 1$ real semi-designs are simultaneously complex semi-designs (of the same weights W_i), but for $k > 1$ this does not hold true.

The simplest semi-design is the triple of vectors x_1, x_2, and x_3 on the circle S^1, with the angles equal 120°. Such vectors constitute semi-design of order 2 in both real and complex cases (of weights $W_1 = W_2 = W_3 = \frac{1}{3}$).

Consider the example of the semi-design of complex order 4 with equal weights.

Assume that $n = 2$, $m = 4$. Consider the following vectors in the space $\mathbb{C}^2$:

$$v_1 = \begin{bmatrix} 1 \\ -\lambda(1+i) \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ \lambda(1+i) \end{bmatrix}, \ v_3 = \begin{bmatrix} -\lambda(1+i) \\ 1 \end{bmatrix}, \ v_4 = \begin{bmatrix} \lambda(1+i) \\ 1 \end{bmatrix},$$

here λ is yet unknown parameter. The norms of all the vectors v_k equal $\|v_k\| = \sqrt{1+2\lambda^2}$. Put $z_k = v_k/\|v_k\|$, $k = 1, \ldots, 4$. We now compute the inner products

$$\langle v_1, v_2 \rangle = 1 - 2\lambda^2, \ \langle v_1, v_3 \rangle = -2\lambda, \ \langle v_1, v_4 \rangle = -2i\lambda, \ \langle v_2, v_3 \rangle = 2\lambda, \ \langle v_2, v_4 \rangle = 2\lambda, \ \langle v_3, v_4 \rangle = 1 - 2\lambda^2.$$

Choose λ so that $2\lambda^2 - 1 = 2\lambda$, i.e., $\lambda = (1+\sqrt{3})/2$. Then for $k \neq l$

$$\left|\langle v_k, v_l \rangle\right| = 2\lambda = 1 + \sqrt{3}, \ \left|\langle z_k, z_l \rangle\right| = \frac{1}{\sqrt{3}}.$$

The system of vectors $\{z_1, \ldots, z_4\}$ is complex semi-design of order 4 if we have equal values

$$S_4 = \sum_{k,l=1}^{4} \left|\langle z_k, z_l \rangle\right|^4 \ \text{and} \ \widetilde{c}_2 m^2 = 16\widetilde{c}_2 = \frac{16}{3}.$$

We have $S_4 = 4 + 12(\sqrt{3})^{-4} = \frac{16}{3}$. Hence $\{z_1, \ldots, z_4\}$ is the semi-design of order 4. At the same time this system is the semi-design of order 2 abut not the semi-design of order 6.

5. INTEGRAL CHARACTERISTICS OF SPHERICAL SEMI-DESIGNS

Proposition 3. *The sequence $\{z_i\}_{i=1}^m$ on the sphere Ω_n is complex spherical semi-design of order $2k$ with weights $W_i > 0$, $W_1 + \cdots + W_m = 1$ if and only if the equality*

$$\int_{\Omega_n} F(z)\, d\omega(z) = \sum_{i=1}^{m} W_i F(z_i) \tag{11}$$

holds for any generalized polynomial F of $\mathrm{Hom}\,(k)$.

Proof. Assume that (10) holds. We can rewrite it as follows:

$$\sum_{i=1}^{m} W_i \langle z_i, z \rangle^k \langle z, z_i \rangle^k = \widetilde{c}_k \langle z, z \rangle^k, \quad z \in \mathbb{C}^n.$$

Expand the inner products and obtain

$$\sum_{i=1}^{m} W_i \sum_{|\alpha|=k} \frac{k!}{\alpha!} z_i^{\alpha} \overline{z}^{\alpha} \sum_{|\beta|=k} \frac{k!}{\beta!} z^{\beta} \overline{z}_i^{\beta} = \widetilde{c}_k \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^{\alpha} \overline{z}^{\alpha}.$$

The system of functions $\{z^{\beta}\overline{z}^{\alpha} \mid |\alpha| = |\beta| = k\}$ is linearly independent on $\mathbb{C}^n$, so we may equate the coefficients with $z^{\beta}\overline{z}^{\alpha}$. This leads us to identities

$$\sum_{i=1}^{m} W_i z_i^{\alpha} \overline{z}_i^{\beta} = 0 \qquad \text{for } \alpha \neq \beta, \tag{12}$$

$$\sum_{i=1}^{m} W_i z_i^{\alpha} \overline{z}_i^{\alpha} = \widetilde{c}_k \frac{\alpha!}{k!} \text{ for } |\alpha| = k. \tag{13}$$

These relations are equivalent to condition (11). Indeed, consider the functions $F(z) = z^{\alpha}\overline{z}^{\beta}$, here $|\alpha| = |\beta| = k$. Then condition (11) turns into

$$I(\alpha, \beta) := \int_{\Omega_n} z^{\alpha}\overline{z}^{\beta}\, d\omega(z) = \sum_{i=1}^{m} W_i z_i^{\alpha} \overline{z}_i^{\beta}. \tag{14}$$

We obtain for $\alpha \neq \beta$ by (4) and (12) the relation $I(\alpha, \beta) = 0 = \sum\limits_{i=1}^{m} W_i z_i^{\alpha} \overline{z}_i^{\beta}$. If $\alpha = \beta$, then

$$I(\alpha, \alpha) = \frac{(n-1)!\alpha!}{(n+k-1)!} = \sum_{i=1}^{m} W_i z_i^{\alpha} \overline{z}_i^{\alpha}.$$

The latter equality holds because it is simply another form of equality (13). This establishes equality (14) and sufficiency of the statement. The necessity of statement can be established simply by “reversing”the order of consideration of arguments. □

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