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Special Variants of Collocation Method for Integral Equations in a Singular Case

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Abstract—The present paper deals with a linear integral equation of the third kind with fixed singularities in its kernel. We propose and substantiate special generalized methods for its approximate solving in a space of generalized funtions.

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INTRODUCTION

We consider linear integral equation of the third kind with fixed singularities in its kernel (E3KFS)

$$Ax \equiv x(t)\prod_{j=1}^{l}(t-t_j)^{m_j} + \int_{-1}^{1} K(t,s)\,[(s+1)^{p_1}(1-s)^{p_2}]^{-1}\,x(s)ds = y(t), \qquad (1)$$

where $t \in I \equiv [-1,1]$, $t_j \in (-1,1)$, $m_j \in \mathbb{N}$, $j=\overline{1,l}$; $p_1, p_2 \in \mathbb{R}^+$, K and y are given continuous functions satisfying certain pointwise restrictions on their smoothness, $x(t)$ is the desired function, and the integral is understood as the Hadamard's finite part ([1], pp. 144–150). Equations of type (1) are becoming more widely used both in theoretical investigations and in applications. A number of important problems of theory of elasticity, neutron transfer, scattering of particles (see, e.g., [2, 3]and references in [2] and [4]), and also theory of differential equations of mixed type [5] are reducible to the equations of this type.

The intrinsic classes of solutions of the equations of third kind are, as a rule, special spaces of generalized functions (SGF) of types D and V. The space of type D (correspondingly, V) is constructed by terms of the Dirac delta-function (correspondingly, the Hadamard's finite part of an integral). The equations under consideration have explicit solutions in very rare cases. Therefore, the development of theoretically proved and effective methods for their approximate solving in SGF is actual subject of mathematical analysis and computational mathematics. A number of results of this kind is obtained in the works [6–9], where certain direct methods for solving of E3KFS (1) in the space of type D are introduced and substantiated. The first results of approximate solution of E3KFS in SGF of type V are obtained in [10], where the authors developed and substantiate a "polynomial" method for solving of Eq. (1) in certain space X of generalized functions.

In the present paper we establish certain generalized variants of collocation method, which are adapted for approximate solving E3KFS (1) in class X. The focus is on substantiation the methods in the sense of book [11] (Chap. 1). We prove theorems on existence and uniqueness of solution of

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corresponding approximate equation, establish bounds for error of approximate solution, and prove convergence of the successive approximations to the exact solution in SGF X. We consider also stability and conditionality of approximating equalities.

1. MAIN SPACES

Let $C \equiv C(I)$ be a space of all continuous on I functions with customary max-norm, and $m \in \mathbb{N}$. According to [12], we say that a function $f \in C$ belongs to the class $C\{m;0\} \equiv C_0^{\{m\}}(I)$ if at point $t=0$ function f has Taylor derivative $f^{\{m\}}(0)$ of order m (we assume that $C\{0;0\} \equiv C$). In the norm

$$\|f\|_{C\{m;0\}} \equiv \|Tf\|_C + \sum_{i=0}^{m-1} |f^{\{i\}}(0)|,$$

where

$$Tf \equiv \left[f(t) - \sum_{i=0}^{m-1} f^{\{i\}}(0) \frac{t^i}{i!} \right] \frac{1}{t^m} \equiv F(t) \in C, \quad F(0) \equiv \lim_{t\to 0} F(t),$$

the space $C\{m;0\}$ is complete and normally embedded into C (see, e.g., [13], P. 14).

Moreover, let $p \in \mathbb{R}^+$ and $g \in C$. According to [12], we say that $g \in C\{p;1\} \equiv C_1^{\{p\}}(I)$ if there exist left Taylor derivatives $g^{\{j\}}(1)$, $j=\overline{1,[p]}$, at the point $t=1$, and for $p \neq [p]$ (here $[\cdot]$ stands for entire part) the limit exists

$$\lim_{t\to 1-} \left\{ \left[g(t) - \sum_{j=0}^{[p]} g^{\{j\}}(1) \frac{(t-1)^j}{j!} \right] (1-t)^{-p} \right\}.$$

We equip the space $C\{p;1\}$ with the norm

$$\|g\|_{\{p\}} \equiv \|g\|_{C\{p;1\}} \equiv \|Sg\|_C + \sum_{i=0}^{\lambda} |g^{\{i\}}(1)|, \tag{2}$$

where

$$Sg \equiv \left[g(t) - \sum_{i=0}^{\lambda} g^{\{i\}}(1) \frac{(t-1)^i}{i!} \right] (1-t)^{-p} \equiv G(t) \in C, \tag{3}$$

$\lambda = \lambda(p) \equiv [p] - (1 + \mathrm{sign}([p]-p))$, $G(1) \equiv \lim\limits_{t\to 1-} G(t)$. Note that the space $C\{p;1\}$ consists of functions $g(t) = (1-t)^p G(t) + \sum\limits_{i=0}^{\lambda} b_i (t-1)^i$, where $G = Sg \in C$, $b_i = g^{\{i\}}(1)/i!$, $i = \overline{0,\lambda}$. Clearly, the space $C\{p;1\}$ with norm (2) is complete and embedded into C.

Now we define the main for our investigations space

$$Y \equiv C_{0;1}^{\{m\};\{p\}} \equiv C_{0;1}^{\{m\};\{p\}}(I) \equiv \{ y \in C\{m;0\} \mid Ty \in C\{p;1\} \}.$$

We equip it with norm

$$\|y\|_Y \equiv \|Ty\|_{\{p\}} + \sum_{i=0}^{m-1} |y^{\{i\}}(0)|, \quad y \in Y. \tag{4}$$

Lemma 1 ([6]). i) *There is valid the relation*

$$\varphi \in Y \Leftrightarrow \varphi(t) = (UV\Phi)(t) + t^m \sum_{j=0}^{\lambda} d_j (t-1)^j + \sum_{i=0}^{m-1} e_i t^i, \tag{5}$$

where $\Phi = ST\varphi \in C$, $\varphi^{\{i\}}(0) = e_i i!$, $i = \overline{0, m-1}$, $(T\varphi)^{\{j\}}(1) = d_j j!$, $j = \overline{0, \lambda}$, $Uf \equiv t^m f(t)$, $Vf \equiv (1-t)^p f(t)$.

ii) *The space* Y *with norm* (4) *is complete and embedded into* $C\{m; 0\}$.

Let $v \in C(I^2)$, and for any fixed $s \in I$ the function $v(t,s)$ belongs to the space $C\{p; 1\}$. We write $v \in C_t^{\{p\}}(I^2)$ if $S_t v \in C$, where S_t is operator (3) applied in variable t. In analogous way we define the class $C_s^{\{p\}}(I^2)$. Then

$$C_1^{\{p\}}(I^2) \equiv C_t^{\{p\}}(I^2) \cap C_s^{\{p\}}(I^2).$$

Then we consider on the main space Y the family $X \equiv V^{\{p\}}\{m; 0\}$ of generalized functions $x(t)$ representable as

$$x(t) \equiv z(t) + \sum_{i=0}^{m-1} \gamma_i \text{ P. F. } t^{-k}, \tag{6}$$

where $t \in I$, $z(t) \in C\{p; 1\}$, $\gamma_i \in \mathbb{R}$ are arbitrary constant values, and P. F. t^{-k} are generalized functions defined on the space Y by rule

$$(\text{P. F. } t^{-k}, y) \equiv \text{P. F.} \int_{-1}^{1} y(t) t^{-k} dt, \quad y \in Y, \ \ k = \overline{1, m}.$$

Here the symbol P. F. stands for the Hadamard finite part of integral ([1], pp. 144–150). In what follows we omit this symbol for brevity. Clearly, the vector space X is the Banach space with respect to the norm

$$\|x\|_X \equiv \|z\|_{\{p\}} + \sum_{i=0}^{m-1} |\gamma_i|. \tag{7}$$

2. COLLOCATION IN TERMS OF THE BERNSTEIN POLYNOMIALS

Let E3KFS (1) be given. We put for simplicity $l = 1$, $t_1 = 0$, $p_1 = 0$, i.e., we consider the equation

$$Ax \equiv (Ux)(t) + (Kx)(t) = y(t), \ \ t \in I, \tag{8}$$

$$Kx \equiv \int_{-1}^{1} K(t,s)(1-s)^{-p} x(s) ds,$$

where $m \in \mathbb{N}$, $p \in \mathbb{R}^+$; $y \in Y$, K is a given function satisfying restrictions

$$\begin{gathered} K \in C_s^{\{p\}}(I^2), \ \ \psi_i(t) \equiv K_s^{\{i\}}(t,1) \in Y, \ \ i = \overline{0, \lambda}, \\ u \equiv S_s K \in C_t^{\{m\}}(I^2), \ \ \theta_i(s) \equiv u_t^{\{i\}}(0,s) \in C\{m; 0\}, \ \ i = \overline{0, m-1}, \\ v \equiv T_t u \in C_t^{\{p\}}(I^2), \ \ \varphi_i(s) \equiv v_t^{\{i\}}(1,s) \in C\{m; 0\}, \ \ i = \overline{0, \lambda}, \\ h \equiv S_t v \in C_s^{\{m\}}(I^2), \end{gathered} \tag{9}$$

and $x \in X$ is the desired generalized function. Fredholm properties and sufficient conditions for continuous invertibility of operator $A : X \to Y$ are established in [10]; a method for evaluation of exact solution of E3KFS (1) in class X is described in the same paper.

We seek approximate solution to Eq. (8) in the form

$$x_n \equiv x_n(t; \{c_j\}) \equiv g_n(t) + \sum_{i=0}^{m-1} c_{i+\lambda+n+2} t^{-i-1}, \tag{10}$$

$$g_n(t) \equiv (Vz_n)(t) + \sum_{i=0}^{\lambda} c_{i+n+1} (t-1)^i, \tag{11}$$

$$z_n(t) \equiv \frac{1}{2^n} \sum_{i=0}^{n} c_i \binom{n}{i} (t+1)^i (1-t)^{n-i}, \ \ n \in \mathbb{N}, \tag{12}$$

where $\binom{n}{i}$, $i = \overline{0,n}$, are binomial coefficients. Unknown parameters $c_j = c_j^{(n)}$, $j = \overline{0, n+m+\lambda+1}$, are found from the system of linear algebraic equations (SLAE)

$$\rho_n^{\{i\}}(0) = 0, \ \ (T\rho_n)^{\{j\}}(1) = 0, \ \ c_k = (STy - STKx_n)\,(\nu_k), \tag{13}$$

$$i = \overline{0, m-1}, \ \ j = \overline{0, \lambda}, \ \ k = \overline{0, n},$$

where $\rho_n(t) \equiv \rho_n^A(t) \equiv (Ax_n - y)(t)$ is discrepancy of the approximate solution, and points $\nu_k = \nu_k^{(n)} \in I$ are given by the formula

$$\nu_k = -1 + \frac{2k}{n}, \ \ k = \overline{0, n}. \tag{14}$$

For computation algorithm (8), (10)–(14) there is valid

Theorem 1. *If the homogeneous E3KFS $Ax = 0$ has in X only null solution (for instance, under assumptions of the theorem 2 from [10]), then for any $n \in \mathbb{N}$ $(n \geqslant n_0)$ SLAE (13) has a unique solution $\{c_j^*\}$, and the sequence of approximate solutions $x_n^* \equiv x_n(t; \{c_j^*\})$ converges to exact solution $x^* = A^{-1}y$ in norm of space X with the rate*

$$\|x_n^* - x^*\| = O\Big\{\omega_t(h; n^{-1/2}) + \sum_{j=0}^{\lambda} \omega(\alpha_j; n^{-1/2}) + \sum_{i=0}^{m-1} \omega(\beta_i; n^{-1/2}) + \omega(STy; n^{-1/2})\Big\}, \tag{15}$$

where $\omega(f; \Delta)$ is modulus of continuity of function $f \in C$ with step Δ $(0 < \Delta \leqslant 2)$, and $\omega_t(h; \Delta)$ is partial modulus of continuity of function $h(t,s)$ in variable t; $h \equiv S_t v$, $\alpha_j \equiv ST\psi_j$, $j = \overline{0, \lambda}$, $\beta_i \equiv ST\Phi_i$, $i = \overline{0, m-1}$, and

$$\Phi_i(t) \equiv \int_{-1}^{1} K(t,s)(1-s)^{-p} s^{-i-1} ds \in Y, \ \ i = \overline{0, m-1}.$$

Proof. We consider E3KFS (8) as linear operator equation

$$Ax \equiv Ux + Kx = y, \ \ x \in X \equiv V^{\{p\}}\{m; 0\}, \ \ y \in Y \equiv C_{0;1}^{\{m\};\{p\}}, \tag{16}$$

where the operator $A : X \to Y$ is continuously invertible.

Let $X_n \subset X$ be an $(n+m+\lambda+2)$-dimensional subspace consisting of elements (10), and $Y_n \subset Y$ is the class $H_{n+m+\lambda+1} \equiv UV(\Pi_n) \oplus \Pi_{m+\lambda}$, where $\Pi_l \equiv \operatorname{span}\{t^i\}_0^l$. Then we introduce linear operator $\Gamma_n \equiv \Gamma_{n+m+\lambda+1} : Y \to Y_n$ by the rule

$$\Gamma_n y \equiv \Gamma_{n+m+\lambda+1}(y; t) \equiv (UVB_nSTy)(t) + \sum_{j=0}^{\lambda} (Ty)^{\{j\}}(1) t^m (t-1)^j \frac{1}{j!} + \sum_{i=0}^{m-1} y^{\{i\}}(0) \frac{t^i}{i!}, \tag{17}$$

where $B_n : C \to \Pi_n$ is the Bernstein operator ([14], P. 407) with nodes (14). Let us show first that system (10)–(13) is equivalent to the following functional equation:

$$A_n x_n \equiv Ux_n + \Gamma_n K x_n = \Gamma_n y, \quad x_n \in X_n, \ \ \Gamma_n y \in Y_n. \tag{18}$$

Indeed, let $x_n^* \equiv x_n(t; \{c_j^*\})$ be a solution to Eq. (18), i.e., $Ux_n^* + \Gamma_n \rho_n^K = 0$, $\rho_n^K \equiv Kx_n^* - y$. By virtue of (10) and (17) the latter means that

$$(UV(z_n^* + B_nST\rho_n^K))(t) + \sum_{j=0}^{\lambda} \left[c_{j+n+1}^* + (T\rho_n^K)^{\{j\}}(1)/j!\right]$$

$$\times t^m(t-1)^j + \sum_{i=0}^{m-1}\left[c^*_{i+\lambda+n+2}t^{m-i-1} + (\rho^K_n)^{\{i\}}(0)t^i/i!\right] \equiv 0. \quad (19)$$

Representation (5) implies the equivalence of identity (19) and the system

$$\begin{gathered} z^*_n(t) \equiv \left(B_n(STy - STKx^*_n)\right)(t), \quad (T\rho^K_n)^{\{j\}}(1) = -c^*_{j+n+1}j!, \quad j=\overline{0,\lambda}, \\ (\rho^K_n)^{\{i\}}(0) = -c^*_{m-1-i+\lambda+n+2}i!, \quad i=\overline{0,m-1}. \end{gathered} \quad (20)$$

Both sides of the first equality of system (20) contain Bernstein polynomials of certain functions with meanings correspondingly c^*_i and $(STy - STKx^*_n)(\nu_i)$, $i=\overline{0,n}$, at points (14). Then due to (16), (10), (11) and (3) we have $(T\rho^A_n)^{\{j\}}(1) = c^*_{j+n+1}j! + (T\rho^K_n)^{\{j\}}(1)$, $j=\overline{0,\lambda}$, and $(\rho^A_n)^{\{i\}}(0) = (\rho^K_n)^{\{i\}}(0) + c^*_{m-1-i+\lambda+n+2}i!$, $i=\overline{0,m-1}$. Hence, system (20) implies relations

$$\begin{gathered} c^*_i = (STy - STKx^*_n)(\nu_i), \ \ i=\overline{0,n}, \quad (T\rho^A_n)^{\{j\}}(1) = 0, \ \ j=\overline{0,\lambda}, \\ (\rho^A_n)^{\{i\}}(0) = 0, \quad i=\overline{0,m-1}. \end{gathered} \quad (21)$$

Thus, SLAE (13) has solution $c_i = c^*_i$, $i=\overline{0,n+m+\lambda+1}$, i.e., a solution to Eq. (18) is a solution to system (10)–(13).

The inverse proposition becomes clear after multiplication of the corresponding equations in (21) by factors $2^{-n}\binom{n}{i}(t+1)^i(1-t)^{n-i}$, $i=\overline{0,n}$, and their term-by-term addition.

Consequently, for the proof of Theorem 1 it suffices to establish existence, uniqueness and convergence of solutions to Eqs. (18). For this purpose we need the following approximative property of the operator Γ_n.

Lemma 2. *Any function $y \in Y$ satisfies the estimate*

$$\|y - \Gamma_n y\|_Y \leqslant d_1\omega(STy; n^{-1/2}). \quad (22)$$

Here and in what follows d_i, $i=\overline{1,3}$, stand for constants, which are independent of the parameter n.

Lemma 2 follows from relations (5), (17), (4) and bound ([14], P. 245)

$$\|f - B_n f\|_C \leqslant d_1\omega(f; n^{-1/2}), \ \ f \in C. \quad (23)$$

Let us find now a characteristic of closeness of operators A and A_n on X_n. By virtue of (16), (18), (5), (17), (4) and (2) we consequently find for arbitrary element $x_n \in X_n$

$$\|Ax_n - A_nx_n\|_Y = \|Kx_n - \Gamma_nKx_n\|_Y = \|STKx_n - B_nSTKx_n\|_C. \quad (24)$$

As is known [10],

$$STKx_n = \int_{-1}^{1} h(t,s)g_n(s)ds + \sum_{j=0}^{\lambda}\lambda_j(g_n)\alpha_j(t) + \sum_{i=0}^{m-1} c_{i+\lambda+n+2}\beta_i(t), \quad (25)$$

where

$$\lambda_j(g) \equiv \int_{-1}^{1}(Sg)(s)(s-1)^j\frac{1}{j!}ds + \sum_{k=0}^{\lambda} g^{\{k\}}(1)\beta_{jk},$$

$$\beta_{jk} \equiv \int_{-1}^{1}(s-1)^{j+k}\frac{1}{j!k!}(1-s)^{-p}ds, \quad j,k=\overline{0,\lambda}.$$

We deduce by virtue of (25) and (23)

$$\|STKx_n - B_nSTKx_n\|_C = \max_{t\in I}\left|\int_{-1}^{1}(h - B^t_nh)(t,s)g_n(s)ds\right.$$

$$+\sum_j \lambda_j(g_n)(\alpha_j - B_n\alpha_j)(t) + \sum_i c_{i+\lambda+n+2}(\beta_i - B_n\beta_i)(t)\Big|$$

$$\leqslant 2d_1\|g_n\|_C\omega_t(h;n^{-1/2}) + d_1\sum_j |\lambda_j(g_n)|\omega(\alpha_j;n^{-1/2}) + d_1\sum_i |c_{i+n+\lambda+2}|\omega(\beta_i;n^{-1/2})$$

$$\leqslant 2^{p+1}d_1\|g_n\|_{\{p\}}\omega_t(h;n^{-1/2})+$$

$$+ d_1\|g_n\|_{\{p\}}(2^{\lambda+1}+\beta)\sum_j \omega(\alpha_j;n^{-1/2}) + d_1\|x_n\|_X\sum_i \omega(\beta_i;n^{-1/2})$$

$$\leqslant d_1\|x_n\|\Big\{2^{p+1}\omega_t(h;n^{-1/2}) + (2^{p+1}+\beta)\sum_j \omega(\alpha_j;n^{-1/2}) + \sum_i \omega(\beta_i;n^{-1/2})\Big\}$$

$$\leqslant d_2\Big\{\omega_t(h;n^{-1/2}) + \sum_j \omega(\alpha_j;n^{-1/2}) + \sum_i \omega(\beta_i;n^{-1/2})\Big\}\|x_n\|. \quad (26)$$

We use here the notation $\beta \equiv \max\limits_{0\leqslant j,k\leqslant\lambda} |\beta_{jk}|$, $d_2 \equiv d_1(2^{p+1}+\beta)$. Hence, relations (24) and (26) yield

$$\varepsilon_n \equiv \|A - A_n\|_{X_n\to Y} \leqslant d_2\Big\{\omega_t(h;n^{-1/2}) + \sum_j \omega(\alpha_j;n^{-1/2}) + \sum_i \omega(\beta_i;n^{-1/2})\Big\}. \quad (27)$$

We conclude the proof of Theorem 1 by means of inequalities (22), (27) and theorem 7 from [11] (P. 19). □

Corollary. If derivatives $h_t^{(r)}, \alpha_j^{(r)}, \beta_i^{(r)}$ and $(STy)^{(r)}$ $(-1 \leqslant t, s \leqslant 1, r \geqslant 2)$ exist and are bounded, then under assumptions of Theorem 1 we have $\|x_n^* - x_n\| = O(1/n)$.

3. COLLOCATION IN TERMS OF HERMITE–FEJÉR INTERPOLATION POLYNOMIALS

Let us seek approximate solution to problem (8), (9) in the form

$$x_n(t) \equiv (1-t)^p \sum_{i=0}^{2n-1} c_i t^i + \sum_{i=0}^{\lambda} c_{i+2n}(t-1)^i + \sum_{i=0}^{m-1} c_{i+\lambda+2n+1} t^{-i-1}, \quad (28)$$

where $c_i = c_i^{(n)}$, $i = \overline{0, m+\lambda+2n}$, are undetermined coefficients. We find them from the SLAE

$$\rho_n^{\{i\}}(0) = 0,\ \ (T\rho_n)^{\{j\}}(1) = 0,\ \ (ST\rho_n)(\nu_k) = 0,\ \ (d(STUx_n)/dt)(\nu_k) = 0, \quad (29)$$
$$i = \overline{0, m-1},\ j = \overline{0,\lambda},\ k = \overline{1,n},$$

where $\{\nu_k\}$ are the Chebyshev nodes of the first kind.

Let $H_{2n+m+\lambda} \equiv UV(\Pi_{2n-1}) \oplus \Pi_{m+\lambda}$, and $F_n \equiv F_{2n+m+\lambda} : Y \to H_{2n+m+\lambda}$ is a linear operator mapping any function $y \in Y$ onto element $F_n y$ uniquely defined by conditions

$$(STF_ny - STy)(\nu_i) = 0,\ \ i = \overline{1,n},\ \ (d(STF_ny)/dt)(\nu_i) = 0,\ \ i = \overline{1,n},$$
$$(TF_ny - Ty)^{\{j\}}(1) = 0,\ \ j = \overline{0,\lambda},\ \ (F_ny - y)^{\{i\}}(0) = 0,\ \ i = \overline{0,m-1}.$$

Clearly,

$$F_ny \equiv F_{2n+m+\lambda}(y;t) \equiv (UV\Phi_n STy)(t) + \sum_{j=0}^{\lambda}(Ty)^{\{j\}}(1)t^m(t-1)^j\frac{1}{j!} + \sum_{i=0}^{m-1} y^{\{i\}}(0)\frac{t^i}{i!}, \quad (30)$$

where $\Phi_n \equiv \Phi_{2n-1} : C \to \Pi_{2n-1}$ is the Hermite–Fejér operator ([14], P. 549) with respect to system $\{\nu_i\}$.

Lemma 3. *Let $y \in Y$, and $STy \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$. Then*
$$\|y - F_n y\|_Y \leqslant d_3 n^{-\alpha/2}.$$

Proof easily follows from equalities (5), (30), (4) and known bound (see, e.g., [15])
$$\|f - \Phi_n f\|_C \leqslant d_3 n^{-\alpha/2}, \ \ f \in \mathrm{Lip}\,\alpha, \ \ 0 < \alpha \leqslant 1.$$

For algorithm (8), (9), (28), (29) there is valid

Theorem 2. *Let equation $Ax = 0$ have only trivial solution in X, $h \equiv S_t v$ (by t), $\alpha_j \equiv ST\psi_j$, $\beta_i \equiv ST\Phi_i$, $j = \overline{0, \lambda}$, $i = \overline{0, m-1}$, $STy \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$. Then for sufficiently large n the approximate solutions x_n^*, which are determined by means of relations* (28), (29), *exist, are unique, and converge to exact solution $x^* = A^{-1}y$ in the space X with the rate $\|x_n^* - x^*\| = O(n^{-\alpha/2})$.*

Proof can be obtained by means of repeating of the scheme of the proof of Theorem 1. In the present case SLAE (29) is equivalent to linear operator equation $A_n x_n \equiv F_n A x_n = F_n y$, $x_n \in X$, $F_n y \in Y_n$, where X_n is set of all x_n of the form (28) such that $(d(STUx_n)/dt)(\nu_i) = 0$, $i = \overline{1, n}$, and the subspace Y_n consists of all elements $y_n \in H_{2n+m+\lambda}$ with property $(d(STy_n)/dt)(\nu_i) = 0$, $i = \overline{1, n}$.

4. REMARKS

1. Our norm in $X \equiv V^{\{p\}}\{m; 0\}$ ensures that the convergence of a sequence (x_n^*) to $x^* = A^{-1}y$ in the norm of the space X implies its customary convergence in the space of generalized functions, i.e., the weak convergence.

2. The approximation of solutions to operator equations $Ax = y$ causes a natural question on a rate of convergence of the discrepancy $\rho_n^A(t) \equiv (Ax_n^* - y)(t)$ of the method under consideration. One of that results follows easily from Theorems 1 and 2. These theorems imply the following simple consequences: 1) if initial data (h, α_j, β_i, STy) of Eq. (8) belong to class $C^{(r)}$ $(r - 1 \in \mathbb{N})$, then under assumptions of Theorem 1 there is valid the bound $\|\rho_n^A\|_Y = O(n^{-1})$; 2) if the initial data belong to class $\mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, then under assumptions of Theorem 2 we have $\|\rho_n^A\|_Y = O(n^{-\alpha/2})$.

3. Since under assumptions of Theorems 1 and 2 the approximating operators A_n satisfy estimates $\|A_n^{-1}\| = O(1)$, $A_n^{-1} : Y_n \to X_n$, $n \geqslant n_1$, obviously (see [11], pp. 23–24), our direct methods Eq. (8) are stable with respect to small disturbances of initial data. Furthermore, if the equation is well-conditioned, then SLAE (13) and (29) are well-conditioned, too.

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