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The Structure of Degrees of Finite-Automaton Transformations of Prefix Decidable Superwords

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Abstract—We show that the structure of degrees of finite automaton transformations of prefix decidable superwords does not form the upper semilattice.

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In the paper we study degrees of undecidability of infinite sequences composed from symbols of finite alphabet. We investigate the degrees that are induced by reducibility, determined by using finite Mealy automata. For the first time these degrees, called degrees of finite automata transformations, have been introduced in the paper by D. Reina [1], where the first structure properties of partially ordered set of these degrees were obtained. In particular, it has been proved that the structure of degrees of finite automata transformations is an upper semilattice [1]. Later V. R. Bairasheva [2] considered two substructures of mentioned structure: a set of degrees of finite automata transformations, consisting of generalized almost periodic sequences, and a set of degrees of finite-automata transformations consisting of sequences with a decidable monadic theory. It was shown that both these substructures are not upper semilattices [2]. It has also been shown by V. R. Bairasheva [3] that and a partially ordered set of degrees of finite-automata transformations of infinite sequences, considered over an alphabet whose capacity does not exceed some positive integer, also is not an upper semilattice. In this paper we consider one more substructure of the structure of degrees of finite automata transformations consisting of degrees containing prefix decidable sequences. The concept of prefix decidability of sequences was introduced by M. N. Vyalyi and A. A. Rubtsova [4] and is a weakening of the property for the sequence to have a decidable monadic theory.

Definition 1. A finite Mealy automaton is a 5-tuple $(S, \Sigma, \Sigma', \delta, \omega)$, where S, Σ, Σ' are finite sets of states, output, and input symbols, respectively; $\delta : S \times \Sigma \to S$ is a transition function; $\omega : S \times \Sigma \to \Sigma'$ is an output function. A finite Mealy automaton with a marked initial state s_0 is called initial automaton.

In the sequel, if we consider a several automata and we need to specify a transition function or an output function, a set of input or output symbols of an automaton S, then we write δ_S, ω_S, Σ_S, Σ'_S.

Let Σ be a finite alphabet and $x = (x(n))$ be an infinite sequence of symbols over the alphabet Σ, which is called a superword over the alphabet Σ. We denote by $x(n)$ the nth letter of the superword. If $i \leq j$, then $x[i, j]$ denotes a segment of the superword x of the form $x(i)x(i+1)\dots x(j)$. The word $x[0, i]$ is a prefix of x. We denote by $\mathrm{Pref}(x)$ a set of all prefixes of the word x. An image of the superword x under the action of the automaton $(S, \Sigma, \Sigma', \delta, \omega, s_0)$ is a superword $\omega_S(s_0, x)$ over the alphabet Σ' of the form $\omega(t_0, x(0))\omega(t_1, x(1))\omega(t_2, x(2))\dots$, where $t_0 = s_0$ and $t_{i+1} = \delta(t_i, x(i))$.

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Definition 2. Let x and y be superwords over finite alphabets Σ and Σ', respectively. The superword y is finite automata reduced to the superword x if there exists a finite initial Mealy automaton $(S, \Sigma, \Sigma', \delta, \omega, s_0)$ such that $\omega(s_0, x) = Ay$, where the block A over the alphabet Σ' determines some finite delay.

Definition 3 ([1]). Let x and y be superwords over finite alphabets Σ and Σ', respectively. The superword y is finite automata equivalent to the superword x, if there exist finite initial Mealy automata $(S, \Sigma, \Sigma', \delta_S, \omega_S, s_0)$ and $(T, \Sigma', \Sigma, \delta_T, \omega_T, t_0)$ such that $\omega(s_0, x) = Ay$ and $\omega(t_0, y) = Bx$, where blocks $A \in (\Sigma')^*$ and $B \in \Sigma^*$ determine some finite delays.

A class of finite automata equivalence of a superword x is called degree of finite automata transformations of the superword x and is denoted by $[x]$. On the set of degrees of finite automata transformations, a partial order relation is naturally induced: $[y] \leq [x]$ if the superword y is finite automata reduced to the superword x.

In this paper we consider superwords and their degrees of finite automata transformations, for which the following algorithmic problem, called a prefix feasibility problem, is decidable: *By a description of a regular language to determine whether there exists a prefix of a superword belonging to the language*. Such superwords are called prefix decidable.

Definition 4 ([4]). A superword x over an alphabet Σ is called prefix decidable if for any regular language L over the alphabet Σ the problem $L \cap \mathrm{Pref}(x) \neq \varnothing$ is decidable.

A regular language can be defined by a finite deterministic or a finite nondeterministic automaton that recognizes this language or by a regular expression. Since there is no difference for us between various ways for defining a regular language, we suppose that a regular language is defined by a finite deterministic automaton recognizing this language. Thereby, a superword x is prefix decidable if there exists an algorithm which by any finite deterministic automaton determines whether this automaton goes through an accepting state when reading the superword x.

Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of partially computable functions and $S_0, S_1, S_2, \ldots$ be an enumeration of deterministic finite automata.

A superword x is prefix decidable if

$$\exists n \forall k [[\varphi_n(S_k) = 1 \leftrightarrow \exists l [\delta_{S_k}(s_0^k, x[0, l]) \in \mathcal{F}_{S_k}]] \wedge [\varphi_n(S_k) = 0 \leftrightarrow \forall l [\delta_{S_k}(s_0^k, x[0, l]) \notin \mathcal{F}_{S_k}]]],$$

where $\varphi_n(S_k)$ is a value of the nth partially computable function for the argument which equals the number of the deterministic finite automaton S_k.

A property of decidability by Buchi is stronger. A superword if called decidable by Buchi [4] if the following algorithmic problem is decidable for this word: *By a description of a regular language to determine whether an intersection of the language and a set of prefixes of the superword is infinite*. Besides, the property of decidability by Buchi of a superword is equivalent to decidability of a monadic theory of the superword [4]. For decidable by Buchi superwords, the closure property with respect to finite automata transformations has already been proved [2]. Also it was proved that a set of degrees of finite automata transformations of decidable by Buchi superwords is not an upper semilattice [2].

In this paper we prove the similar statements for prefix decidable superwords.

The following theorem is a consequence of a more general result about closure (with respect to asynchronous-automata transformations) of the property of prefix decidability of superwords, which was proved in [5].

Theorem 1. *Let x be a prefix decidable superword over an alphabet Σ, $(T, \Sigma, \Sigma', \delta, \omega, t_0)$ be a finite initial Mealy automaton. Then $y = \omega(t_0, x)$ is a prefix decidable superword over the alphabet Σ'.*

Sketch of the proof. We need to prove that for an arbitrary deterministic finite automaton $(S, \Sigma', \delta_S, s_0, \mathcal{F}_S)$ it is possible to determine whether it goes through an accepting state, when reading a superword y, or not.

Let us construct an automaton $(\widetilde{S}, \Sigma, \delta_{\widetilde{S}}, \widetilde{s}_0, \mathcal{F}_{\widetilde{S}})$, where $\widetilde{S} = T \times S$, an initial state $\widetilde{s}_0 = (t_0, s_0)$. A transition function is defined as follows: $\delta_{\widetilde{S}}((t,s),a) = (\delta_T(t,a), \delta_S(s, \omega_T(t,a)))$. A set of accepting states $\mathcal{F}_{\widetilde{S}} = \{(t,s) | s \in \mathcal{F}_S\}$.

The automaton $\widetilde{S}$ goes through an accepting state, when reading the input superword x, if and only if S goes through an accepting state when reading the superword y. Since for the superword x the prefix feasibility problem is decidable, for the superword y this problem is also decidable.

From Theorem 1 it follows that a degree of finite automata transformations, containing a prefix decidable superword, consists only of prefix decidable superwords. Next we prove that a set of degrees of finite automata transformations of prefix decidable superwords is not an upper semilattice. First we construct a superword that is not prefix decidable.

Theorem 2. *There exists a superword which is not prefix decidable.*

Sketch of the proof. Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be an enumeration of partially computable functions, $S_0, S_1, S_2, \dots$ be an enumeration of deterministic finite automata over the alphabet $\Sigma = \{0,1\}$. A superword x over the alphabet $\Sigma = \{0,1\}$ is not prefix decidable if

$$\forall n \exists k [[\varphi_n(S_k) \notin \{0,1\}] \vee [[\varphi_n(S_k) = 1] \wedge \forall l [\delta_{S_k}(s_0^k, x[0,l]) \notin \mathcal{F}_{S_k}]] \vee [[\varphi_n(S_k) = 0] \\ \wedge \exists l [\delta_{S_k}(s_0^k, x[0,l]) \in \mathcal{F}_{S_k}]]].$$

For constructing a superword x which is not prefix decidable we use the method of initial segments. We construct this superword from blocks a_i of length $(i+2)$, beginning in 1, ending in 0, and not containing within themselves any blocks of such form of shorter length. For example, let $a_i = 11^i0$.

Let us consider automata $S^{(i)}$ that recognize languages containing the word a_i and words ending in $0a_i$.

Step 0. We determine I_0 as an empty block (of length 0).

Step $(n+1)$. Let us make $(n+1)$ steps in calculating functions $\varphi_0(S^{(0)}), \varphi_1(S^{(1)}), \dots, \varphi_n(S^{(n)})$. Let i be the least positive integer such that $\varphi_i(S^{(i)})$ is determined. If such a number does not exist, then we set $I_{n+1} = I_n$. If such a number exists, then we proceed as follows:

1) if $\varphi_i(S^{(i)}) \notin \{0,1\}$, then we set $I_{n+1} = I_n$;

2) if $\varphi_i(S^{(i)}) = 0$, then we set $I_{n+1} = I_n a_i$;

3) if $\varphi_i(S^{(i)}) = 1$, then we set $I_{n+1} = I_n$.

We remove the function φ_i and the block a_i from further consideration and move to the step $(n+2)$.

The superword x that is a limit of the sequence of blocks I_i as $i \to \infty$ is not prefix decidable because the nth partially computable function φ_n fails when checking whether the automaton $S^{(n)}$ goes through an accepting state while reading x.

A superword that is not prefix decidable can also be constructed using blocks $b_i = 10^i0$. The construction will be similar to the construction presented in Theorem 2 with replacing automata $S^{(i)}$ by automata that recognize languages containing the word b_i and words ending in $0b_i$.

By using blocks a_i and b_i, we can also construct a prefix decidable superword x.

Proposition. *There exists a prefix decidable superword constructed from blocks a_i (or b_i).*

Sketch of the proof. Let $S_0, S_1, S_2, \ldots$ be an enumeration of deterministic finite automata over the alphabet $\Sigma = \{0, 1\}$ such that an automaton with a number n has at most $(n+1)$ states.

A prefix decidable superword x over the alphabet $\Sigma = \{0, 1\}$ is constructed from blocks $a_i = 11^i0$ by the method of initial segments. (For blocks $b_i = 10^i0$ the construction is similar.)

Step 0. We determine I_0 as an empty block.

Step $(n+1)$. Let us consider an automaton $S_n = (S, \{0, 1\}, \delta, s_0, \mathcal{F})$ with the number n in our enumeration of deterministic finite automata. Let I_n be its input. If the automaton goes through an accepting state when reading this input, then we set $I_{n+1} = I_n$. Otherwise we consider a sequence of sets:

$$Q_0 = \{\overline{s} \mid \overline{s} = \delta(s_0, I_n)\},\ Q_1 = \{\delta(\overline{s}, a_i) \mid i \in \mathbb{N}\} \cup \{\overline{s}\},$$
$$Q_2 = \{\delta(s, a_i) \mid s \in Q_1,\ i \in \mathbb{N}\} \cup \{\overline{s}\}, \ldots, Q_{k+1} = \{\delta(s, a_i) | s \in Q_k, i \in \mathbb{N}\} \cup \{\overline{s}\}, \ldots$$

By construction, it is obvious that $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_k \subseteq Q_{k+1} \subseteq \cdots$. Hence there exists the least positive integer $l(\leq n)$ such that $Q_l = Q_{l+1} = Q_{l+2} = \cdots$.

If $Q_l \cap \mathcal{F} \neq \varnothing$, then there exists a word b, consisting of words a_i $(i \leq n)$ such that, when reading this word from the state $\overline{s}$, the automaton goes through an accepting state. Besides, we can take the word b whose length does not exceed $(n+1)(n+2)$. We set $I_{n+1} = I_n b$.

If $Q_l \cap \mathcal{F} = \varnothing$, then for any word, consisting of blocks a_i, that we add to I_n, when reading this word from the state $\overline{s}$, the automaton does not go through any accepting state. We can set $I_{n+1} = I_n$.

A superword x that is not a limit of a sequence of blocks I_i as $i \to \infty$ is prefix decidable.

Further, along with superwords $x = (x(n))$ and $y = (y(n))$ we consider a superword $(x, y) = (x(n), y(n))$.

Theorem 3. *There exist prefix decidable superwords x and y such that a superword (x, y) is not prefix decidable.*

Sketch of the proof. Superwords x and y over the alphabet $\Sigma = \{0, 1\}$ are constructed from blocks $a_i = 11^i0$ and $b_i = 10^i0$, respectively, a superword (x, y) over the alphabet $\Sigma \times \Sigma = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is constructed from blocks, beginning in $(1, 1)$, ending in $(0, 0)$, and not containing within themselves any blocks of such form of shorter length.

Let $S_0, S_1, S_2, \ldots$ be an enumeration of deterministic finite automata over the alphabet Σ such that an automaton with a number n has at most $(n+1)$ states, and $T_0, T_1, T_2, \ldots$ be an enumeration of deterministic finite automata over the alphabet $\Sigma \times \Sigma$.

Let us define automata $T^{(i)}$ that recognize languages containing a word $c_i = (1, 1)(1, 0)^i(0, 0)$ and words ending in $(0, 0)c_i$.

Step 0. We determine I_0 to be an empty block.

Step $(n+1)$. Repeat the construction that is similar to construction of Theorem 2 with automata $T^{(i)}$ and blocks c_i. As a result we construct a block $I'_{n+1} = (X'_{n+1}, Y'_{n+1})$.

For the automaton S_n and the block X'_{n+1}, we repeat the construction from the previous sentence. We get a word $X''_{n+1} = X'_{n+1}U'0$. We set $I''_{n+1} = (X''_{n+1}, Y''_{n+1})$, where $Y''_{n+1} = Y'_{n+1}V'0$ and V' is obtained from U' by replacing zeroes by unities and unities by zeroes.

For the automaton S_n and the block Y''_{n+1} we repeat the construction that is similar to the construction from the previous sentence but using blocks b_i. We get the word $Y'''_{n+1} = Y''_{n+1}V'0$. We set $I_{n+1} = (X'''_{n+1}, Y'''_{n+1})$, where $X'''_{n+1} = X''_{n+1}U'0$ and U' is obtained from the word V' by replacing zeroes by unities and unities by zeroes.

Let us move to the step $n+2$.

We define a superword (x, y) as a limit of a sequence of blocks I_i as $i \to \infty$, $x = \mathrm{pr}_1(x, y)$, and $y = \mathrm{pr}_2(x, y)$. By Theorem 2, (x, y) is not prefix decidable and according to the previous statement x and y are prefix decidable.

It is known that the structure of degrees of finite automata transformations is an upper semilattice, and besides the degree of finite automata transformations of the superword (x, y) is the least upper bound of degrees $[x]$ and $[y]$ [1]. This result and the result from the previous theorem allows us to prove the following statement.

Theorem 4. *The set of degrees of finite-automata transformations of prefix decidable superwords is not an upper semilattice.*

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REFERENCES

1. Reina, G. “Degrees of Finite-State Transformations”, Kiberneticheskii Sborn., No. 14, 95–106 (1977).
2. Bairasheva, V. R. “Degrees of Automata Transformations of Almost Periodic Superwords and Superwords with the Decidable Monadic Theory”, Available from VINITI, No. 3103-B89 (Kazan, 1989).
3. Bairasheva, V. R. “Structure Properties of Automata Transformations”, Soviet Mathematics (Iz. VUZ) **32**, No. 7, 54–64 (1988).
4. Vyalyi, M. N., Rubtsov, A. A. “Decidability Conditions for Problems About Automata Reading Infinite Words”, Diskretn. Anal. Issled. Oper. **19**, No. 2, 3–18 (2012).
5. Korneeva, N. N. “Automata Transformations of Prefix Decidable and Decidable by Buchi Superwords, Russian Mathematics (Iz. VUZ) **60**, No. 7, 47–55 (2016).

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