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Refined Geometrically Nonlinear Equations of Motion for Elongated Rod-Type Plate

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Abstract—We derive new refined geometrically nonlinear equations of motion for elongated rod-type plates. They are based on the proposed earlier relationships of geometrically nonlinear theory of elasticity in the case of small deformations and refined S. P. Timoshenko's shear model. These equations allow to describe the high-frequency torsional oscillation of elongated rod-type plate formed in them when plate performs low-frequency flexural vibrations. By limit transition to the classical model of rod theory we carry out transformation of derived equations to simplified system of equations of lower degree.

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Cantilever beams with thickness h and width b which are much less than its length L are used as the test samples in the theoretical-experimental method of determining the logarithmic decrements (LD) of oscillation [1]. Logarithmic decrement of oscillation particularly uses to describe the damped property of material. In papers [2–4] was described the method of determining the LD of material with taking into account the aerodynamic damping. This method is based on the studies of the damped flexural vibrations of test samples in the first fundamental mode of oscillation. In the study of forced vibration of test samples on the second and third mode of flexural oscillation, the high frequency torsional vibrations, which form sound waves in the environment, are observed. The theoretical study of such flexural-torsional forms of vibrations of beams of this class based on known equations of motion for the averaged bend theory of beams and rods is impossible due to the insufficient degree of their accuracy and meaningfulness. Therefore, for the correct description of these mechanical effects the basic equations of motion will rely on consistent variant of the geometrically nonlinear theory of elasticity at small deformations, which previously was proposed and analyzed in a series of papers [5, 6]. According to them, in rectangular Cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ the elongation ε_1, ε_2, ε_3 and shear γ_{12}, γ_{13}, γ_{23} deformations can be determined by kinematic relationships in incomplete quadratic approximation

$$\varepsilon_1 = E_{11} + \left(E_{12}^2 + E_{13}^2\right)/2, \ldots, \ \gamma_{12} = E_{12} + E_{21} + E_{13}E_{23}, \ldots, \ E_{\alpha\beta} = \partial u_\beta/\partial x^\alpha, \tag{1}$$

where $u_1 = U$, $u_2 = V$, $u_3 = W$ are components of the displacement vector $\mathbf{u} = U\mathbf{e}_1 + V\mathbf{e}_2 + W\mathbf{m}$.

For plates considered in this work the inequalities $b/L \ll 1, h/L \ll 1, h/b \sim 0.06 \div 0.3$ are true. This allows to consider them as blade-type rod with free end at $x = L$ and fixed end at $x = 0$. Therefore, the displacement vector $\mathbf{U}$ is taken as [7].

$$\begin{gathered}\mathbf{U} = (u + z\psi - y\chi)\,\mathbf{e}_1 + (v - z\varphi)\,\mathbf{e}_2 + (w + y\varphi)\,\mathbf{m},\\ 0 \le x \le L, \ \ -b/2 \le y \le b/2, \ \ -h/2 \le z \le h/2,\end{gathered} \tag{2}$$

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where u, v, and w are components of the displacement vector of $0x$ axis points, φ, ψ, and χ components of the rotation vector, which use in the shear model of Timoshenko beam theory.

In accordance with representation (2) and relationships $E_{\alpha\beta} = \partial u_\beta/\partial x^\alpha$ we obtain

$$E_{11} = u' + z\psi' - y\chi', \quad E_{12} = v' - z\varphi', \quad E_{13} = w' + y\varphi',$$
$$E_{21} = -\chi, \quad E_{23} = \varphi, \quad E_{31} = \psi, \quad E_{32} = -\varphi, \quad E_{22} = E_{33} = 0.$$

When use them it is reasonable to rewrite expression (1) as

$$\varepsilon_1 = u' + \left[(v')^2 + (w')^2\right]/2 \ - y\left(\chi' - w'\varphi'\right) + z\left(\psi' - v'\varphi'\right), \tag{3}$$

$$\gamma_{12} = v' - \chi + w'\varphi - z\varphi', \quad \gamma_{13} = \psi + w' - v'\varphi + y\varphi', \tag{4}$$

$$\gamma_{23} \approx 0, \quad \varepsilon_2 \approx 0, \quad \varepsilon_3 \approx 0. \tag{5}$$

In accordance with relations (3)–(5), the variation of the strain energy will be equal

$$\delta\Pi = \int_0^L \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} (\sigma_{11}\delta\varepsilon_1 + \sigma_{12}\delta\gamma_{12} + \sigma_{13}\delta\gamma_{23})\,dx\,dy\,dz$$
$$= \int_0^L (Q_x\delta u' + Q_y^*\delta v' + Q_z^*\delta w' + M_y\delta\psi' + M_z\delta\chi' + Q_z\delta\psi - Q_y\delta\chi + M_x^*\delta\varphi' + N^*\delta\varphi)dx, \tag{6}$$

where we accept denotations

$$Q_y^* = Q_y + Q_x v' - Q_z\varphi - M_y\varphi', \ \ Q_z^* = Q_z + Q_x w' + Q_y\varphi - M_z\varphi',$$
$$M_x^* = M_x - M_z w' - M_y v', \ \ N^* = Q_y w' - Q_z v',$$

and define internal forces and torques in the cross-section $x = \text{const}$ which are expressed from stress $\sigma_{11}, \sigma_{12}, \sigma_{13}$ by the formulas $\Big(\int\limits_{-b/2}^{b/2}\int\limits_{-h/2}^{h/2} (\cdots)dy\,dz = \iint\limits_F (\cdots)dF\Big)$.

$$Q_x = \iint_F \sigma_{11}dF, \ \ Q_y = \iint_F \sigma_{12}dF, \ \ Q_z = \iint_F \sigma_{13}dF,$$
$$M_y = \iint_F \sigma_{11}zdF, \ \ M_z = -\iint_F \sigma_{11}ydF, \ \ M_x = \iint_F (\sigma_{13}y - \sigma_{12}z)dF. \tag{7}$$

With $z = 0$ expression (2) takes the form

$$\mathbf{U}|_{z=0} = (u - y\chi)\,\mathbf{e}_1 + v\mathbf{e}_2 + (w + y\varphi)\,\mathbf{m}. \tag{8}$$

Considering that aerodynamic forces found in the form of $\mathbf{p}\,(x, y) = p\,(x, y)\,\mathbf{m}$ and using the (8) we derive the expression for the variation of work of the external force

$$\delta A = \int_0^L \int_{-b/2}^{b/2} \mathbf{p}\delta\mathbf{U}|_{z=0} dx\,dy = \int_0^L (p\delta w + m_x\delta\varphi)\,dx, \tag{9}$$

where $P = \int\limits_{-b/2}^{b/2} p\,(x, y)\,dy$, $m_x = \int\limits_{-b/2}^{b/2} p\,(x, y)\,y\,dy$.

Using expression (2) we derive the next expression for the variation of kinetic energy of a body (t is a time)

$$\delta K = -\rho \int_0^L \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} \delta\mathbf{U} dx\,dy\,dx = -\int_0^L [\rho hb\,(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w) + \rho J_p\ddot{\varphi}\delta\varphi]dx, \tag{10}$$

where $\ddot{\varphi} = \partial^2\varphi/\partial t^2$, ρ is the density of plate material, $J_p = \left(hb^3 + bh^3\right)/12$ is the polar moment of inertia of the cross-sectional area.

According to the Hamilton–Ostrogradski principle and expressions (6), (9), (10) we derive in a standard way the equations of motion

$$\begin{gathered}\frac{\partial Q_x}{\partial x}-\rho bh\frac{\partial^2 u}{\partial t^2}=0,\ \ \frac{\partial Q_y^*}{\partial x}-\rho bh\frac{\partial^2 v}{\partial t^2}=0,\ \ \frac{\partial Q_z^*}{\partial x}+P-\rho bh\frac{\partial^2 w}{\partial t^2}=0,\\ \frac{\partial M_y}{\partial x}-Q_z=0,\ \ \frac{\partial M_z}{\partial x}+Q_y=0,\ \ \frac{\partial M_x^*}{\partial x}-N^*+m_x-\rho J_p\frac{\partial^2\varphi}{\partial t^2}=0,\end{gathered}\tag{11}$$

for which we formulate the kinematic boundary conditions at $x=0$

$$u=v=w=\psi=\chi=\varphi=0,$$

and the boundary conditions for force at $x=L$

$$Q_x=0,\quad Q_y^*=0,\quad Q_z^*=0,\quad M_y=0,\quad M_z=0,\quad M_x^*=0.\tag{12}$$

In addition to Eqs. (5), we accept the hypothesis of a static theory of thin plates and shells $\sigma_{33}=0$. This allows to obtain physical relations for stress σ_{11}, σ_{12}, σ_{13} according to linear elastic deformations under static deformation and the equality $\varepsilon_2=0\,(E_*=E_1/(1-\nu_{12}\nu_{21}))$

$$\sigma_{11}=E_*\varepsilon_1,\ \ \sigma_{12}=G_{12}\gamma_{12},\ \ \sigma_{13}=G_{13}\gamma_{13},\tag{13}$$

where E_1, ν_{12}, ν_{21}, G_{12}, G_{13} are the module of elasticity, the Poisson ratios and shear modulus of orthotropic material.

After the substitution of expressions (3), (4) into Eqs. (13) according to (7) we obtain

$$\begin{gathered}Q_x=\widetilde{E}_*bh\left(u'+\frac{(v')^2}{2}+\frac{(w')^2}{2}\right),\quad Q_y=\widetilde{G}_{12}bh\left(v'-\chi+w'\varphi\right),\\ Q_z=\widetilde{G}_{13}bh\left(w'+\psi-v'\varphi\right),\ \ M_y=D_y\left(\psi'-v'\varphi'\right),\\ M_z=D_z\left(\chi'-w'\varphi'\right),\ \ M_x=B_p\varphi',\end{gathered}\tag{14}$$

where

$$B_p=\frac{\widetilde{G}_{13}hb^3+\widetilde{G}_{12}bh^3}{12},\quad D_y=\frac{\widetilde{E}_*bh^3}{12},\quad D_z=\frac{\widetilde{E}_*hb^3}{12}.\tag{15}$$

The first of formulas (15), which characterizes the torsional stiffness of the cross-section, requires the addition of a correction term that depends on the parameter h/b. It is known that this term for a beam of isotropic material in the case of $h/b\le 0.3$ takes the form $B_p=Gbh^3/3$, where $G=G_{12}=G_{13}$.

Shear strain γ_{12}^0, γ_{13}^0 at the points of the planes $x0z$ and $x0y$ according to expression (4) take the form

$$\gamma_{12}^0=v'-\chi+w'\varphi,\quad \gamma_{13}^0=\psi+w'-v'\varphi.\tag{16}$$

Application of the Bernoulli hypothesis requires the adoption of the equalities $\gamma_{12}^0=\gamma_{13}^0=0$. According to them, from (16) it follows

$$\chi=v'+w'\varphi,\ \ \psi=-w'+v'\varphi,\tag{17}$$

and relations (3), (4) are transformed to the following

$$\begin{gathered}\varepsilon_1=u'+\frac{1}{2}\left[\left(v'\right)^2+\left(w'\right)^2\right]-y\left(v''+w''\varphi\right)-z\left(w''-v''\varphi\right),\\ \gamma_{12}=-z\varphi',\quad \gamma_{13}=y\varphi'.\end{gathered}\tag{18}$$

Due to the established relations (18) for calculating the $\delta\Pi$ instead of (6) we arrive at the expression

$$\delta\Pi=\int_0^L(Q_x\delta u'+Q_y^*\delta v'+Q_z^*\delta w'+M_z^*\delta v''+M_y^*\delta w''+M_x\delta\varphi'+N^*\delta\varphi)dx,$$

where

$$Q_y^*=Q_xv',\ \ Q_z^*=Q_xw',\ \ M_z^*=M_z-M_y\varphi,\ \ M_y^*=M_y+M_z\varphi,$$

$$N^* = M_z w'' - M_y v''$$

and instead of (14),

$$M_y = -\iint_F \sigma_{11} z dF = D_y \left(w'' - \varphi v''\right), \quad M_z = -\iint_F \sigma_{11} y dF = D_z \left(v'' + \varphi w''\right).$$

Thus, the use of (17) allow to reduce the system of six equations of motion (11) to the system of four equations

$$\begin{gathered}\frac{\partial Q_x}{\partial x} - \rho bh \frac{\partial^2 u}{\partial t^2} = 0, \;\; \frac{\partial S_y^*}{\partial x} + \rho bh \frac{\partial^2 v}{\partial t^2} = 0, \;\; \frac{\partial S_z^*}{\partial x} - P + \rho bh \frac{\partial^2 w}{\partial t^2} = 0, \\ \frac{\partial M_x}{\partial x} - N^* + m_x - \rho J_p \frac{\partial^2 \varphi}{\partial t^2} = 0,\end{gathered} \tag{19}$$

where

$$S_y^* = \frac{\partial M_z^*}{\partial x} - Q_x v', \;\; S_z^* = \frac{\partial M_y^*}{\partial x} - Q_x w'.$$

For obtained Eq. (19) according to (17) we formulate the kinematic boundary conditions at $x = 0$

$$u = v = w = v' = w' = \varphi = 0,$$

and the boundary conditions for force at $x = L$

$$Q_x = 0, \;\; S_y^* = 0, \;\; S_z^* = 0, \;\; M_y^* = 0, \;\; M_z^* = 0, \;\; M_x = 0.$$

In Eqs. (19) for the considered beams $(h/b \ll 1)$ of plate type under the action of aerodynamic loads one can neglect the bending in the plane $x0y$, assuming $v \equiv 0$. According to this equality for unknowns S_z^* and N^* in the remaining three equations of motion

$$\frac{\partial Q_x}{\partial x} - \rho bh \frac{\partial^2 u}{\partial t^2} = 0, \;\; \frac{\partial S_z^*}{\partial x} - P + \rho bh \frac{\partial^2 w}{\partial t^2} = 0, \;\; \frac{\partial M_x}{\partial x} - N^* + m_x - \rho J_p \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{20}$$

the expression will take place

$$\begin{aligned}S_z^* = \frac{\partial M_y^*}{\partial x} - Q_x w' = (M_y + M_z \varphi)' - Q_x w' = D_y w''' \\ + D_z \big(\varphi^2 w''\big)' - Q_x w', \;\; N^* = M_z w'' = D_z \varphi \big(w''\big)^2.\end{aligned} \tag{21}$$

It can be shown that in (21) terms containing the force Q_x can be neglected. As a result, the system of Eqs. (20) with the use of relations (21) is reduced to the system

$$\begin{gathered}\rho bh \frac{\partial^2 w}{\partial t^2} + \frac{Ebh^3}{12} w^{(IV)} + \frac{Eb^3 h}{12} \big(\varphi^2 w''\big)'' = 0, \\ \frac{\rho b^3 h}{12} \frac{\partial^2 \varphi}{\partial t^2} - \frac{Gbh^3}{3} \varphi'' + \frac{Eb^3 h}{12} \big(w''\big)^2 \varphi = 0.\end{gathered}$$

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