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On the Solvability of a Nonlocal Problem for a Hyperbolic Equation of the Second Kind

O. A. Repin[1*] and S. K. Kumykova[2]**

[1]*Samara State Economic University*
ul. Sovetskoi Armii 141, Samara, 443090 Russia
[2]*Kabardino-Balkarian State University*
ul. Chernyshevskogo 173, Nalchik, 360004 Russia
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Abstract—In the characteristic triangle for a hyperbolic equation of the second kind we study a nonlocal problem, where the boundary value condition contains a linear combination of Riemann–Liouville fractional integro-differentiation operators. We establish variation intervals of orders of fractional integro-differentiation operators, taking into account parameters of the considered equation with which the mentioned problem has either a unique solution or more than one solution.

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INTRODUCTION

Consider the equation

$$u_{xx}-(-y)^m u_{yy}+\alpha(-y)^{m-1}u_y=0, \tag{1}$$

where $0<m<2$ and $\alpha=\text{const}$, in the domain Ω bounded by characteristics

$$AC: x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=0, \quad BC: x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}}=1$$

and the segment $\overline{I}\equiv[0,1]$ of the axis $y=0$.

Problem. *Find a regular in the domain Ω solution $u(x,y)$ to Eq. (1) in the class $C\left(\overline{\Omega}\right)\cap C^1(\Omega\cup I)$ subject to*

$$u(x,0)=\tau(x)\ \ \forall x\in\overline{I}, \tag{2}$$

$$A(x)D_{0x}^a u\left[\theta_0(x)\right]+B(x)D_{x1}^b u\left[\theta_1(x)\right]=C(x)\ \ \forall x\in I, \tag{3}$$

where $\tau(x)$, $A(x)$, $B(x)$, and $C(x)$ are given continuous functions such that $A^2(x)+B^2(x)\neq 0$; $\theta_0(x)$ and $\theta_1(x)$ are points of intersection of characteristics of Eq. (1) originating at the point $(x,0)\in I$ and characteristics AC and BC, respectively; $\left(D_{0x}^a f\right)(x)$ and $\left(D_{x1}^b f\right)(x)$ are Riemann–Liouville fractional integro-differential operators ([1], pp. 9–10; [2], pp. 42–44).

Problem (1)–(3) is a shift problem [3]. Shift problems for hyperbolic equations were studied by many authors. See [3–5] for references to papers where one studies the mentioned problems in the case of a nonlocal condition stated on the characteristic part of the domain boundary. Namely, this condition

*E-mail: Matstat@mail.ru.
**E-mail: bsk@rect.kbsu.ru.

pointwisely connects fractional derivatives of the desired solution of a certain order which ensures the unique solvability of the problem and depends on the order of degeneration of the equation.

There are only few papers [6–8] devoted to problems for hyperbolic equations, where boundary-value conditions contain Riemann–Liouville fractional integro-differential operators of an arbitrary order or generalized integrals and derivatives with a Gaussian hypergeometric function. Moreover, all hyperbolic equations considered in the mentioned papers are of the first kind.

A specific feature of Eq. (1), which is a hyperbolic equation of the second kind, consists in the fact that the envelope of its characteristics is the axis $y = 0$; at the same time, it is the line of parabolic degeneration and a characteristic of Eq. (1). The behavior of a solution to Eq. (1) near the line of parabolic degeneration depends on the coefficient at u_y and on m. Both the solution and its derivative u_y can turn into infinity on the parabolic line.

Therefore, instead of the classical Cauchy problem, which may be ill-posed, it is natural to study the modified Cauchy problem

$$u(x,0) = \tau(x), \quad \lim_{y\to 0}(-y)^{\alpha} u_y(x,y) = \nu(x). \tag{4}$$

We study problem (1)–(3) in the following cases:

(1) $m-1 < \alpha < \frac{m}{2}$, i.e., $-\frac{1}{2} < \beta < 0$, where $\beta = \frac{2\alpha - m}{2(2-m)}$;

(2) $\alpha = \frac{m}{2}$ or, which is the same, $\beta = 0$;

(3) $\frac{m}{2} < \alpha < 1$ or $0 < \beta < \frac{1}{2}$.

In this paper we continue the research described in [6–8].

1. THE UNIQUE SOLVABILITY OF THE PROBLEM

The solution to Eq. (1) subject to (4) with $-\frac{1}{2} < \beta < 0$ takes the form ([5], P. 113)

$$\begin{aligned} u(x,y) &= k_1 \int_0^1 \tau\left[x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}}(2t-1)\right] t^{\beta}(1-t)^{\beta}\,dt \\ &+ \frac{2k_1}{(1+2\beta)(2-m)}(-y)^{\frac{2-m}{2}} \int_0^1 \tau'\left[x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}}(2t-1)\right] t^{\beta}(1-t)^{\beta}(2t-1)\,dt \\ &- \left(\frac{2-m}{4}\right)^{2\beta-1} k_2 (-y)^{1-\alpha} \int_0^1 \nu\left[x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}}(2t-1)\right] t^{-\beta}(1-t)^{-\beta}\,dt, \end{aligned} \tag{5}$$

where

$$k_1 = \frac{\Gamma(2+2\beta)}{\Gamma^2(1+\beta)}, \quad k_2 = \left(\frac{2-m}{4}\right)^{1-2\beta} \frac{\Gamma(2-2\beta)}{\Gamma(1-\alpha)\Gamma^2(1-\beta)},$$

and $\Gamma(z)$ is the Euler gamma function ([9], pp. 11–13).

Using (5), we get

$$\begin{aligned} u\left[\theta_0(x)\right] &= k_1 x^{-1-2\beta} \int_0^x \frac{\tau(\xi)\,d\xi}{\xi^{-\beta}(x-\xi)^{-\beta}} \\ &+ \frac{k_1 x^{-1-2\beta}}{2(1+2\beta)}\left[\int_0^x \frac{\xi^{1+\beta}\tau'(\xi)\,d\xi}{(x-\xi)^{-\beta}} - \int_0^x \frac{\tau'(\xi)(x-\xi)^{1+\beta}\,d\xi}{\xi^{-\beta}}\right] - k_2 \int_0^x \xi^{-\beta}(x-\xi)^{-\beta}\nu(\xi)\,d\xi, \end{aligned}$$

$$\begin{aligned} u\left[\theta_1(x)\right] &= k_1 (1-x)^{-1-2\beta} \int_x^1 \frac{\tau(\xi)\,d\xi}{(1-\xi)^{-\beta}(\xi-x)^{-\beta}} \\ &+ \frac{k_1(1-x)^{-1-2\beta}}{2(1+2\beta)}\left[\int_x^1 \frac{\tau'(\xi)(\xi-x)^{1+\beta}\,d\xi}{(1-\xi)^{-\beta}} - \int_x^1 \frac{\tau'(\xi)(1-\xi)^{1+\beta}\,d\xi}{(\xi-x)^{-\beta}}\right] \\ &- k_2 \int_x^1 (1-\xi)^{-\beta}(\xi-x)^{-\beta}\nu(\xi)\,d\xi. \end{aligned}$$

Theorem 1. *Let $a = 1 - \beta$ and either $b < -\beta$ or $-\beta < b \le 1 - \beta$. Then if*

$$\tau(x) = x\tau_1(x), \quad \tau_1(x) \in C^3\left(\overline{I}\right) \cap C^5\left(I\right), \quad A(x) \neq 0, \quad B(x) = x^{-\beta}(1-x)b_1(x),$$

where $b_1(x) \in C\left(\overline{I}\right) \cap C^2\left(I\right)$ and $A(x), C(x) \in C\left(\overline{I}\right) \cap C^2\left(I\right)$, then a solution to problem (1)–(3) *exists and is unique.*

Indeed, assuming that $u\left[\theta_0(x)\right]$ and $u\left[\theta_1(x)\right]$ satisfy condition (3), by certain transformations we obtain the equation

$$\Gamma(1-\beta)A(x)\nu(x) + \int_x^1 K_i(x,\xi)\nu(\xi)\,d\xi = F_i(x,\beta), \tag{6}$$

where $i = 1$ with $b < 0$, $i = 2$ with $0 < b < -\beta$, $i = 3$ with $-\beta < b < 1 - \beta$;

$$K_1(x,\xi) = -\frac{\Gamma(1-\beta)}{\Gamma(1-b-\beta)} x^{\beta}(1-\xi)^{-\beta}(\xi - x)^{-\beta} b_1(x), \quad \beta < 0;$$

$$K_2(x,\xi) = \frac{\Gamma(1-\beta)}{\Gamma(1-b-\beta)} x^{\beta}(1-\xi)^{-\beta}(\xi - x)^{-b-\beta} b_1(x), \quad \beta < 0, \ \ b + \beta < 0;$$

$$K_3(x,\xi) = -\frac{\Gamma(1-\beta)}{\Gamma(1-b-\beta)} \frac{(1-x)(1-\xi)^{-\beta} b_1(x)}{(\xi - x)^{b+\beta}}, \qquad \beta < 0, \ \ 0 < b + \beta < 1;$$

$$F_i(x,\beta) = -\frac{1}{k_2} x^{\beta} F(x,\beta),$$

$$\begin{aligned} F(x,\beta) = C(x) - \frac{k_1 A(x)}{2(1+2\beta)} \Bigg[D_{0x}^{1-\beta} x^{-1-2\beta} \int_0^x \frac{\xi^{1+\beta}\tau'(\xi)\,d\xi}{(x-\xi)^{-\beta}} \\ - D_{0x}^{1-\beta} x^{-1-2\beta} \int_0^x \frac{(x-\xi)^{1+\beta}\tau'(\xi)\,d\xi}{\xi^{-\beta}} \Bigg] - k_1 A(x) D_{0x}^{1-\beta} x^{1-2\beta} \int_0^x \frac{\tau(\xi)\,d\xi}{\xi^{-\beta}(x-\xi)^{-\beta}} \\ - k_1 B(x) D_{x1}^{b} (1-x)^{-1-2\beta} \int_x^1 \frac{\tau(\xi)\,d\xi}{(1-\xi)^{-\beta}(\xi - x)^{-\beta}} \\ - \frac{k_1}{2(1+2\beta)} \Bigg[D_{x1}^{b} (1-x)^{-1-2\beta} \int_x^1 \frac{\tau'(\xi)(\xi - x)^{1+\beta}\,d\xi}{(1-\xi)^{-\beta}} \\ - D_{x1}^{b} (1-x)^{-1-2\beta} \int_x^1 \frac{\tau'(\xi)(1-\xi)^{1+\beta}\,d\xi}{(\xi - x)^{-\beta}} \Bigg]. \end{aligned}$$

One can easily see that kernels $K_1(x,\xi)$ and $K_2(x,\xi)$ are continuously differentiable in the square $0 < x, \xi < 1$, and with $x = 0$ they can become infinite of the order $(-\beta)$. The kernel $K_3(x,\xi)$ is continuously differentiable with $0 < x, \xi < 1$, $\xi \neq x$, and with $\xi = x$ it has a weak singularity of the order $b + \beta$.

Under assumptions of Theorem 1 by a chain of transformations and calculations we conclude that $F_i(x,\beta) \in C\left(0,1\right] \cap C^2\left(0,1\right)$, $i = 1,2,3$; moreover, with $x = 0$ they can become infinite of the order (-2β), and with $x = 1$ they are bounded.

Therefore, Eq. (6) is a Volterra equation of the second kind, whose unique solution in the given function class can be calculated by the method of successive approximations ([10], pp. 14–18).

Remark. In the case when $b = 1 - \beta$ and, additionally, $A(x)x^{-\beta} + B(x)(1-x)^{-\beta} \neq 0$ one can immediately find the function $\nu(x)$ from the correlation

$$\Gamma(1-\beta)\left[A(x)x^{-\beta} + B(x)(1-x)^{-\beta}\right]\nu(x) = -\frac{1}{k_2}F(x,\beta).$$

2. CASES OF MULTIPLE SOLUTIONS TO THE PROBLEM

Theorem 2. *If* $a = 1 - \beta\, k - \beta < b < k + 1 - \beta$, $k = 1, 2, 3, \dots$,

$$\tau(x) = (1-x)^{\sigma}\tau_1(x), \quad \tau_1(x) \in C^{k+3}\left(\overline{I}\right) \cap C^{k+5}\left(I\right), \quad \sigma \geq b,$$

$$\nu(x) = (1-x)^{b+2\beta-2}\nu_1(x), \quad \nu_1(x) \in C^{k}\left(I\right), \quad \nu_1(1) \neq 0,$$

$$A(x) = (1-x)^{k} a_1(x), \quad a_1(x)B(x) \neq 0,$$

and $a_1(x), B(x), C(x) \in C^1\left(\overline{I}\right)$*, then problem* (1)–(3) *has infinitely many linearly independent solutions.*

Proof. Let $k = 1$, then $1 - \beta < b < 2 - \beta$. Under assumptions of Theorem 2, assuming that (5) satisfies condition (3), we obtain an equation analogous to (6) with respect to $\nu(x)$; introducing a new unknown function

$$\varphi(x) = \int_x^1 \frac{(1-\xi)^{-\beta}\nu(\xi)\, d\xi}{(\xi - x)^{b+\beta-1}} \tag{7}$$

and applying the inversion formula for the Abel integral equation, we turn it to

$$b_1(x)\frac{d}{dx}\varphi(x) + x^{-\beta}(1-x)^{1+\beta}a_2(x)\frac{d}{dx}\int_x^1 \frac{\varphi(t)\, dt}{(t-x)^{2-b-\beta}} = \frac{F(x,\beta)}{k_2};$$

here

$$b_1(x) = \frac{\Gamma(1-\beta)}{\Gamma(2-b-\beta)}B(x), \quad a_2(x) = \frac{\Gamma(1-\beta)}{\pi}\sin\left[\pi(b+\beta-1)\right]a_1(x),$$

$$\varphi(1) = \nu_1(1)B(2-b-\beta, b+\beta-1) = C^* = \text{const} \neq 0,$$

and $B(x,y)$ is the Beta function ([9], P. 25).

Denote

$$\psi(x) = \frac{d}{dx}\varphi(x). \tag{8}$$

Then in view of (7) we have

$$\varphi(x) = C^* - \int_x^1 \psi(t)\, dt. \tag{9}$$

Substituting (8) and (9) in (7), with $b > 1 - \beta$ we get

$$b_1\psi(x) + x^{-\beta}(1-x)^{1+\beta}a_2(x)\int_x^1 \frac{\psi(t)\, dt}{(t-x)^{2-b-\beta}} - C^* x^{-\beta}(1-x)^{b+\beta-1}a_2(x) = \frac{1}{k_2}F(x,\beta). \tag{10}$$

To prove that the problem is not uniquely solvable, it suffices to show that the homogeneous equation that corresponds to (10) has a nontrivial solution.

Let $b_1(x) \neq 0$ and $F(x,\beta) = 0$. Then (10) takes the form

$$\psi(x) + \int_x^1 \frac{K(x,\beta)\psi(t)\, dt}{(t-x)^{2-b-\beta}} = (1-x)^{b+\beta-1}x^{-\beta}\gamma(x), \tag{11}$$

where

$$K(x,\beta) = \frac{(1-x)^{1+\beta}a_2(x)}{x^{\beta}b_1(x)}, \quad \gamma(x) = \frac{C^* a_2(x)}{b_1(x)}.$$

Therefore, with $k = 1$ and $b_1(x) \neq 0$ the homogeneous problem is equivalent in the sense of solvability to the Volterra equation of the second kind (11).

Using the method of successive approximations, one can prove that Eq. (11) has a nontrivial solution in the class of functions $\psi(x) = x^{-\beta}(1-x)^{b+\beta-1}\psi_1(x)$, where $\psi_1(x) \in C\left(\overline{I}\right) \cap C^2\left(I\right)$, so the solution to the problem is not unique.

With $b_1(x) \neq 0$ and $F(x,\beta) \neq 0$ Eq. (11) takes the form

$$\psi(x) + \int_x^1 \frac{K(x,\beta)\psi(t)\,dt}{(t-x)^{2-b-\beta}} = F_1(x,\beta), \tag{12}$$

where

$$F_1(x,\beta) = x^{-\beta}(1-x)^{b+\beta-1}\gamma(x) + \frac{F(x,\beta)}{k_2 b_1(x)}.$$

Therefore, the right-hand side $F_1(x,\beta)$ of Eq. (12) is representable in the form

$$F_1(x,\beta) = x^{-\beta}(1-x)^{b+\beta-1}F^*(x,\beta),$$

where $F^*(x,\beta) \in C\left(\overline{I}\right) \cap C^2\left(I\right)$.

In this class of functions Eq. (12) has a nontrivial solution $\psi(x)$. With the help of calculated $\psi(x)$ one can find $\varphi(x)$ and then do $\nu(x)$ and a solution to problem (1)–(3).

Hereinafter we understand a regular solution to Eq. (1) in the domain Ω as a function $u(x,y) \in C\left(\overline{\Omega}\right) \cap C^2\left(\Omega\right)$ satisfying Eq. (1) such that $\nu(x) = (1-x)^{b+2\beta-2}\nu_1(x)$, where the function $\nu_1(x)$ is sufficiently many times differentiable in some neighborhood $(1-\delta, 1)$ of the point $x = 1$, and $\nu_1(1) \neq 0$.

We have proved that Theorem 2 is valid with $k = 1$. Let us assume that it is also valid with $k = n-1$ and prove that the desired assertion is valid with $k = n$.

With $k = n$ the condition $0 < b - n + \beta < 1$ is fulfilled and the equation takes the form

$$\Gamma(1-\beta)A(x)x^{-\beta}\nu(x) - \frac{\Gamma(1-\beta)B(x)}{\Gamma(n+1-b-\beta)}\frac{d^n}{dx^n}\int_x^1 \frac{(1-\xi)^{-\beta}\nu(\xi)\,d\xi}{(\xi-x)^{b-n+\beta}} = -\frac{F_n(x,\beta)}{k_2}.$$

Hence by introducing a new unknown function

$$\varphi(x) = \int_x^1 \frac{(1-\xi)^{-\beta}\nu(\xi)\,d\xi}{(\xi-x)^{b-n+\beta}}$$

we deduce

$$b_n(x)\frac{d^n}{dx^n}\varphi(x) + a_n(x)x^{-\beta}(1-x)^{1-\beta} \times \left[-(b+\beta-n)\int_x^1 \frac{\varphi(t)\,dt}{(t-x)^{n+1-b-\beta}} + \int_x^1 \frac{(1-t)\varphi'(t)\,dt}{(t-x)^{n+1-b-\beta}}\right] = \frac{F_n(x,\beta)}{k_2},$$

where

$$b_n(x) = \frac{\Gamma(1-\beta)B(x)}{\Gamma(n+1-b+\beta)}, \quad a_n(x) = \frac{\Gamma(1-\beta)}{\pi}\sin\left[\pi(b-n+\beta)\right]A(x).$$

Putting $\frac{d^n}{dx^n}\varphi(x) = \psi(x)$, by certain transformations we obtain

$$\psi(x) + \int_x^1 K_n^*(x,\beta)\psi(\xi)\,d\xi + \gamma_1(x)(1-x)^{b+\beta-2} = \frac{F_n(x,\beta)}{k_2 b_n(x)}, \tag{13}$$

where

$$\gamma_1(x) = \frac{\Gamma(b+\beta-n)}{\Gamma(b+\beta)}\left[\frac{1}{b+\beta} - (1-x)(b+\beta-n)\right]\frac{a_n(x)}{b_n(x)}x^{-\beta}(1-x)^{-1-\beta},$$

$$K_n^*(x,\beta) = x^{-\beta}(1-x)^{-1-\beta}(\xi-x)^{b+\beta-2}\left[\frac{2\Gamma(b+\beta-n+1)}{\Gamma(b+\beta)}(\xi-x) - \frac{\Gamma(b+\beta-n)}{\Gamma(b+\beta-1)}(1-x)\right]\frac{a_n(x)}{b_n(x)}.$$

Assumptions of the theorem imply that $\gamma_1(x) \in C\left(\overline{I}\right) \cap C^2(I)$, $K_n^*(x,\beta) \in C\left(\overline{I} \times \overline{I}\right) \cap C^2(I \times I)$. With $F_n(x,\beta) = 0$ the homogeneous problem is reduced to the Volterra equation of the second kind

$$\psi(x) + \int_x^1 K_n^*(x,\beta)\psi(\xi)\,d\xi = -\gamma_1(x)(1-x)^{b+\beta-2}. \tag{14}$$

Using the method of successive approximations, one can prove that Eq. (14) has a nontrivial solution, so the problem is not uniquely solvable. In the case, when $F_n(x,\beta) \neq 0$, Eq. (13) takes the form

$$\psi(x) + \int_x^1 K_n^*(x,\beta)\psi(\xi)\,d\xi = F_n^*(x,\beta), \tag{15}$$

where

$$F_n^*(x,\beta) = \frac{F_n(x,\beta)}{k_2 b_n(x)} - \gamma_1(x)(1-x)^{b+\beta-2}.$$

The formula

$$\psi(x) = F_n^*(x,\beta) + \int_x^1 R_n(x,t,\beta)F_n^*(t,\beta)\,dt, \tag{16}$$

where $R_n(x,t,\beta)$ is a resolvent of the kernel $K^*(x,\beta)$, defines a solution to Eq. (15) in the class of desired functions.

Therefore, it is proved that under assumptions of Theorem 2 a solution to problem (1)–(3) exists, but is not unique, solutions to the problem obey formula (16).

Theorem 3. *If* $a = 1-\beta$, $b = n+1-\beta$, $n = 1,2,3,\ldots$, $A(x) = (1-x)a_2(x)$, $A(x), B(x), C(x) \in C^1\left(\overline{I}\right)$, $\tau(x) = (1-x)^\sigma \tau_1(x)$, $\tau_1(x) \in C^{n+3}\left(\overline{I}\right) \cap C^{n+5}(I)$, $\sigma \geq b$; $a_2(x)B(x) \neq 0$, $\nu(x) = (1-x)^{b+2\beta-2}\nu_1(x)$, $\nu_1(x) \in C^n(I)$, *and* $\nu_1(1) \neq 0$, *then problem* (1)–(3) *has more than one regular solution.*

Indeed, in this case we obtain the ordinary differential equation

$$A(x)x^{-\beta}\nu(x) - B(x)\frac{d^n}{dx^n}\left[(1-x)^{-\beta}\nu(x)\right] = \frac{C(x)}{\Gamma(1-\beta)}.$$

If we put $(1-x)^{-\beta}\nu(x) = \nu_2(x)$, then

$$A_1(x)\nu_2(x) - B(x)\frac{d^n}{dx^n}\nu_2(x) = \frac{C(x)}{\Gamma(1-\beta)}, \tag{17}$$

where $A_1(x) = x^{-\beta}(1-x)^\beta A(x)$.

Since with $x = 1$ the function $\nu(x)$ turns into zero of the order $b + 2\beta - 2$ and is sufficiently many times differentiable near the point $x = 1$, while $\nu_1(1) \neq 0$, we have

$$\nu_2(1) = 0,\ \nu_2'(1) = 0,\ \ldots,\ \nu_2^{(n-1)}(1) = (-1)^{n-1}\nu_1(1)(n-1)! = C_{n-1}. \tag{18}$$

With $C(x) = 0$ the linear differential equation (17) with initial conditions (18) has a continuous on $\overline{I}$ nonzero solution, so problem (1)–(3) is not uniquely solvable.

Let us prove the existence of a solution to the problem with $b = n+1-\beta$. Let us write (17) in the form

$$\frac{d^n}{dx^n}\nu_2(x) - B_1(x)\nu_2(x) = \gamma_2(x), \tag{19}$$

where $B_1(x) = A_1(x)/B(x)$ and $\gamma_2(x) = -C(x)/\left[\Gamma(1-\beta)B(x)\right]$.

Denote

$$\frac{d^n}{dx^n}\nu_2(x) = \psi(x). \tag{20}$$

Taking into account initial conditions (18), sequentially integrating (20), we find

$$\nu_2(x) = \frac{C_{n-1}}{(n-1)!}(1-x)^{n-1} - \frac{1}{(n-1)!}\int_x^1 (\xi - x)^{n-1}\psi(\xi)\,d\xi. \tag{21}$$

Substituting (20) and (21) in (19), we obtain the Volterra equation of the second kind

$$\psi(x) + \int_x^1 \overline{K}(x,\xi)\psi(\xi)\,d\xi = \overline{f}(x), \tag{22}$$

where

$$\overline{f}(x) = \gamma_2(x) + \frac{C_{n-1}}{(n-1)!}(1-x)^{n-1}B_1(x),$$

$$\overline{K}(x,\xi) = \frac{1}{(n-1)!}(\xi - x)^{n-1}B_1(x).$$

One can easily see that $\overline{K}(x,\xi) \in C\left(\overline{I}\times\overline{I}\right) \cap C^2\left(I\times I\right)$, $\overline{f}(x) = C\left(\overline{I}\right)\cap C^2\left(I\right)$, and the formula

$$\psi(x) = \overline{f}(x) + \int_x^1 \overline{R}(x,t,b)\overline{f}(t)\,dt,$$

where $\overline{R}(x,t,b)$ is a resolvent of the kernel $\overline{K}(x,\xi)$, defines a solution to Eq. (22).

With the help of the known function $\psi(x)$ we can find $\nu(x)$ and a solution to problem (1)–(3) which solves the Cauchy problem (5).

We have also studied cases $\beta = 0$ and $0 < \beta < 1/2$ and obtained assertions analogous to Theorems 1–3.

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