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A Problem with Dynamic Nonlocal Condition for Pseudohyperbolic Equation

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Abstract—We consider an initial-boundary problem with dynamic nonlocal boundary condition for a pseudohyperbolic fourth-order equation in a cylinder. Dynamic nonlocal boundary condition represents a relation between values of a required solution, its derivatives with respect to spatial variables, second-order derivatives with respect to time variable and an integral term. The main result lies in substantiation of solvability of this problem. We prove the existence and uniqueness of a generalized solution. The proof is based on the a priori estimates obtained in this paper, Galyorkin's procedure and the properties of the Sobolev spaces.

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INTRODUCTION

Mathematicians solve problems with nonlocal conditions for the partial differential equations for several decades. The problems with nonlocal integral conditions are of particular interest due to their multifarious applications [1–5] and close relation to the inverse problems [6–8]. Now we have a number of results on the nonlocal problems with integral conditions solvability for parabolic and hyperbolic equations. The majority of works on the matter consider the nonlocal problems for the second order equations (note here the papers [9–18] and references therein). Nevertheless, mathematical models describing a lot of physical processes interesting for the modern science lead to equations of order greater than two and to the boundary conditions of more complicated structure than that described in the classical literature, particularly, to the dynamic boundary conditions. These dynamic boundary conditions comprising the values of the second order derivatives with respect to the time variable arise under consideration of the elastically fixed rod oscillations with the loaded spring ends ([19], P. 46; [20]), or of the nonstationary inner waves in anisotropic or in the rotating stratified fluid [21, 22]. Let us give an example of the problem statement from [20] whose solvability was proved in [23]: *Find in* $Q_T = (0,l) \times (0,T)$ *the solution to equation*

$$Lu \equiv \sigma(x)u_{tt} - (a(x)u_x)_x - (b(x)u_{ttx})_x = F(x,t),$$

meeting the initial

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

and the boundary

$$\begin{aligned} a(0)u_x(0,t) + b(0)u_{xtt}(0,t) - K_1u(0,t) - M_1u_{tt}(0,t) = g_1(t), \\ a(l)u_x(l,t) + b(l)u_{xtt}(l,t) + K_2u(l,t) + M_2u_{tt}(l,t) = g_2(t) \end{aligned} \tag{1}$$

conditions.

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This problem is a mathematical model of the rod oscillations that takes into account the lateral displacement effects.

The most important moment in the nonlocal problems consideration is their solvability methods study, since the existing methods of initial-boundary problems with the classical boundary and initial conditions are often inapplicable to the nonlocal conditions case [17]. It has been observed in the development of one of such methods that nonlocal conditions of the first type can be converted in the case of one spatial variable to nonlocal conditions of the second kind [18]. The transformation procedure of integral conditions of the first kind $\int_0^l K_i(x)u(x,t)dx = E(t)$ into the conditions of the second kind in the case of the fourth order pseudo-hyperbolic equation

$$\frac{\partial^2}{\partial t^2}(u - (b(x)u_x)_x) - (a(x,t)u_x)_x + c(x,t)u = f(x,t)$$

generated the dynamic nonlocal conditions containing the integral operator:

$$\begin{aligned} &a(0)u_x(0,t) + b(0)u_{xtt}(0,t) + k_{11}u(0,t) + k_{12}u(l,t) + m_{11}u_{tt}(0,t)\\ &\qquad + m_{12}u_{tt}(l,t) + \int_0^l H_1 u(x,t)dx = g_1(t),\\ &a(l)u_x(l,t) + b(l)u_{xtt}(l,t) + k_{21}u(l,t) + k_{22}u(l,t) + m_{21}u_{tt}(0,t)\\ &\qquad + m_{22}u_{tt}(l,t) + \int_0^l H_2 u(x,t)dx = g_2(t). \end{aligned} \tag{2}$$

Under certain restrictions on the functions $K_i(x)$ conditions (2) turn into (1).

This consideration allows us to state the problem for the fourth order pseudo-hyperbolic equation with dynamic nonlocal conditions whose appearance can be justified by the considerations given above.

1. STATEMENT OF THE PROBLEM

Let Ω be a bounded domain in R^n with the smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$. Consider the equation

$$Lu \equiv \frac{\partial^2}{\partial t^2}(u - \Delta u) - (a_{ij}(x,t)u_{x_i})_{x_j} + c(x,t)u = f(x,t), \tag{3}$$

here the redundant indexes stand for the sums from 1 to n, and consider the following problem: *Find in Q_T the solution to Eq.* (3) *meeting the initial data*

$$u(x,0) = 0, \quad u_t(x,0) = 0 \tag{4}$$

and the nonlocal condition

$$\left(\frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial N} + \alpha(t)u_{tt} + \beta(t)u + \int_\Omega K(x,y,t)u(y,t)dy\right)\Big|_{S_T} = 0. \tag{5}$$

Here $\nu(x) = (\nu_1, \dots, \nu_n)$ is the external normal vector to $\partial\Omega$ at the given point,

$$\frac{\partial u}{\partial N} = a_{ij}u_{x_i}\cos(\nu, x_j), \ \ a_{ij} = a_{ji}, \ \ \forall (x,t) \in Q_T,$$

the functions $\alpha(t)$, $\beta(t)$, and $K(x,y,t)$ are defined on $[0,T]$ and $\overline{\Omega} \times \overline{Q_T}$, respectively.

Consider the notation

$$W(Q_T) = \{u : u \in W_2^1(Q_T), \ u_t(x,t) \in W_2^1(Q_T)\},$$

$$\widehat{W}(Q_T) = \{v(x,t) : v(x,t) \in W(Q_T), \ v(x,T) = 0\}.$$

Let us now introduce the concept of the generalized solution. The standard procedure of [24] (P. 92) leads to

$$\int_0^T \int_\Omega (u_{tt}v - u_{xt}v_{xt} + u_x v_x + cuv)dx\,dt$$
$$+ \int_0^T \int_{\partial\Omega} v(x,t)[\alpha u_{tt} + \beta u + \int_\Omega K(x,y,t)u(y,t)dy]ds\,dt = \int_0^T \int_\Omega fv\,dx\,dt. \quad (6)$$

Here and in what follows we make use of the following notation:

$$u_x = (u_{x_1}, \dots, u_{x_n}),\ \ u_x^2 = \sum_{i=1}^n u_{x_i}^2,\ \ u_{xx}^2 = \sum_{i=1}^n u_{x_i x_j}^2,$$

$$u_x v_x = \sum_{i=1}^n u_{x_i} v_{x_i},\ \ u_{xt} = (u_{x_1 t}, \dots, u_{x_n t}).$$

Definition. The generalized solution to problem (3)–(5) is the function $u(x,t) \in W(Q_T)$ that meets conditions (4) and equality (6) for any $v \in \widehat{W}(Q_T)$.

2. MAIN RESULT

Theorem. *If*

$$f \in L_2(Q_T),\ \ f_t \in L_2(Q_T),\ \ c(x,t) \in C(\overline{Q}_T),\ \ c_t \in C(\overline{Q}_T),$$

$$K(x,y,t) \in C(\overline{\Omega} \times \overline{Q}_T),\ \ \alpha, \beta \in C[0,T] \cap C^1(0,T), \alpha(t) \geq 0,\ \ \beta(t) \geq 0,$$

then there exists a unique generalized solution to problem (3)–(5).

Proof. Solution uniqueness. Assume the existence of two different problem solutions, namely $u_1(x,t)$ and $u_2(x,t)$. Then their difference $u(x,t) = u_1(x,t) - u_2(x,t)$ meets the conditions $u(x,0) = 0$, $u_t(x,0) = 0$ and the equality

$$\int_0^T \int_\Omega (u_{tt}v - u_{xt}v_{xt} + u_x v_x)dx\,dt + \int_0^T \int_\Omega cuv\,dx\,dt$$
$$+ \int_0^T \int_{\partial\Omega} v(x,t)[\alpha u_{tt} + \beta u + \int_\Omega K(x,y,t)u(y,t)dy]ds\,dt = 0. \quad (7)$$

Choose now a function

$$v(x,t) = \begin{cases} \int\limits_\tau^t u(x,\eta)d\eta, & 0 \leq t \leq \tau; \\ 0, & \tau \leq t \leq T, \end{cases}$$

in (7) and integrate the first summand of (7) by parts. The result is the equality

$$\int_\Omega [v_t^2(x,\tau) + a_{ij}(x,0)v_{x_i}(x,0)v_{x_j}(x,0) + v_{xt}^2(x,\tau)]dx$$
$$+ \alpha(t) \int_{\partial\Omega} v_t^2(x,\tau)ds + \beta(\tau) \int_{\partial\Omega} v^2(x,0)ds$$
$$= 2\int_0^\tau \int_\Omega c(x,t)v(x,t)v_t(x,t)dx\,dt - \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} v_{x_i} v_{x_j} + 2\int_0^\tau \int_{\partial\Omega} v \int_\Omega K v_t\,dy\,ds\,dt. \quad (8)$$

Let us now estimate the left-hand side of equality (8) applying the Cauchy and Cauchy–Bunyakovsky inequalities together with assumptins of the theorem:

$$2\left| \int_0^\tau \int_\Omega cvv_t\,dx\,dt \right| \leq c_0 \int_0^\tau \int_\Omega (v^2 + v_t^2)dx\,dt;$$

$$\left|\int_0^\tau\int_\Omega \frac{\partial a_{ij}}{\partial t} v_{x_i} v_{x_j} dx\,dt\right| \le 2a_1 \int_0^\tau \int_\Omega v_x^2 dx\,dt;$$

$$2\left|\int_0^\tau\int_{\partial\Omega} v \int_\Omega K v_t\,dy\,ds\,dt\right| \le \int_0^\tau\int_{\partial\Omega} v^2 ds\,dt + \int_0^\tau\int_{\partial\Omega}\left(\int_\Omega K v_t dy\right)^2 ds\,dt.$$

We assume next the notation

$$k = \max_{\overline{Q}_T}\int_\Omega K^2(x,y,t)dy,\ \ \omega = \int_{\partial\Omega} ds,$$

keep in mind the equality

$$\int_{\partial\Omega} v^2 ds \le c_1 \int_\Omega (v_x^2 + v^2) dx,$$

([24], P. 77) and derive from the latter inequality the relation

$$2\left|\int_0^\tau\int_{\partial\Omega} v \int_\Omega K v_t\,dy\,ds\,dt\right| \le \int_0^\tau\int_\Omega (c_1 v^2 + c_1 v_x^2 + k\omega v_t^2) dx\,dt.$$

Noting that the representation of the function $v(x,t)$ implies that

$$v_t = u,\ \ v^2 = \left(\int_\tau^t u(x,\eta)d\eta\right)^2 \le \tau\int_0^\tau u^2 dt,$$

we arrive at the inequality

$$\int_\Omega [u^2(x,\tau) + u_x^2(x,\tau) + a_{ij}(x,0)v_{x_i}(x,0)v_{x_j}(x,0)]dx \\ + \alpha(\tau)\int_{\partial\Omega} v_t^2(x,\tau)ds + \beta(\tau)\int_{\partial\Omega} v^2(x,0)ds \le c_2\int_0^\tau\int_\Omega (u^2 + v_x^2)dx\,dt,$$

here $c_2 = \max\{(c_0+c_1)T + (c_0 + k\omega), 2a_1 + c_1\}$.

In particular,

$$\int_\Omega [u^2(x,\tau) + a_{ij}(x,0)v_{x_i}(x,0)v_{x_j}(x,0)]dx \le c_2\int_0^\tau\int_\Omega (u^2 + v_x^2)dx\,dt. \tag{9}$$

Put here $W_i(x,t) = \int\limits_0^t u_{x_i} d\eta$ and arrive (by [24], P. 211) at the relation

$$v_{x_i}(x,t) = W_i(x,t) - W_i(x,\tau),\ \ v_{x_i}(x,0) = -W_i(x,\tau).$$

Then (9) turns into

$$\int_\Omega [u^2(x,\tau) + a_{ij}(x,0)W_i(x,\tau)Wj(x,\tau)]dx \\ \le c_2\int_0^\tau\int_\Omega\left(u^2 + 2\sum_{i=1}^n W_i^2\right)dx\,dt + 2c_2\tau\int_\Omega \sum_{i=1}^n W_i^2(x,\tau)dx. \tag{10}$$

Note now that by assumption $a_{ij}W_iW_j \ge \gamma\sum\limits_{i=1}^n W_i^2$; so, because τ is arbitrary, we choose it so that the condition $\gamma - 2c_2\tau > 0$ holds. It holds, for instance, in the case of $\tau \in [0, \frac{\gamma}{4c_2}]$, for which (10) yields

$$\int_\Omega \left(u^2(x,\tau) + \sum_{i=1}^n W_i^2(x,\tau)\right)dx \le c_3\int_0^\tau\int_\Omega\left(u^2 + \sum_{i=1}^n W_i^2\right)dx\,dt,$$

here $c_3 = \frac{2c_2}{m_0}$, $m_0 = \min\{1, \frac{\gamma}{2}\}$. The resulting inequality together with the Gronwall lemma allow us to state that we have $u(x,\tau) = 0$ for the chosen τ. We repeat the considerations for $\tau \in \left[\frac{\gamma}{4c_2}, \frac{\gamma}{2c_2}\right]$, and continue the process that results in $u(x,t) = 0$ for all $t \in [0,T]$.

Existence of the solution. Let the functions $w_k(x) \in C^3[0,l]$ comprise the complete and linearly independent in $W_2^1(0,l)$ system. We search for the approximate solution in the form of

$$u^m(x,t) = \sum_{k=1}^{m} c_k(t) w_k(x)$$

from the relations

$$\int_\Omega (u^m_{tt} w_l + u^m_{x_i tt} w_{lx_i} + a_{ij} u^m_{x_i} w_{lx_j} + cu^m w_j)dx$$

$$\int_{\partial\Omega} w_l(x)[\alpha u^m_{tt} + \beta u^m + \int_\Omega K u^m dy]ds = \int_\Omega f w_l dx. \quad (11)$$

We complete relations (11) that constitute the ordinary differential equation system on $c_k(t)$ with the initial conditions $c_k(0) = 0$, $c_k'(0) = 0$, and arrive at the Cauchy problem with initial data (13) on ordinary differential equation system (11) that can be rewritten as follows:

$$\sum_{k=1}^{m} c''(t) A_{kj} + \sum_{k=1}^{m} c_k(t) B_{kj}(t) = f_j(t), \quad (12)$$

$$c_k(0) = 0, \quad c_k'(0) = 0, \quad (13)$$

$$A_{kl}(t) = (w_k, w_l)_{W_2^1(\Omega)} + \alpha(t)(w_k, w_l)_{L_2(\partial\Omega)},$$

$$B_{kl}(t) = \int_\Omega [a_{ij} \nabla w_k \nabla w_l + c w_k w_l]dx + \beta(t)(w_k, w_l)_{L_2(\partial\Omega)} + \int_{\partial\Omega} w_l \int_\Omega K w_k \, dy \, ds,$$

$$f_l(t) = \int_\Omega f(x,t) w_l(x) dx.$$

It seems clear that system (12) is solvable with respect to the senior derivatives. Indeed, consider the quadric $q = \sum\limits_{k,l=1}^{m} A_{kl}\xi_k\xi_l$, here ξ_i are components of the vector $\xi = \sum\limits_{i=1}^{m} \xi_i w_i(x)$. We put into this quadric the expressions for coefficients A_{kl} and obtain

$$q = \int_\Omega (|\xi|^2 + |\nabla\xi|^2)dx + \alpha(t) \int_{\partial\Omega} |\xi|^2 ds \geq 0.$$

Since $q = 0$ only if $\xi = 0$, therefore due to linear independence of $w_k(x)$ all $\xi_i = 0$. Thus, the quadric q and, consequently, the matrix with the senior derivatives is positive. So system (12) is resolvable with respect to the senior derivatives. By assertion, coefficients of the system are bounded and the free summands $f_l \in L_1(0,T)$. Hence, there exists a solution to the Cauchy problem (12)—(13), moreover, $c_k'' \in L_1(0,T)$. Thus we have a sequence of approximate solutions.

Further on in order to prove the existence of generalized solution we need a certain a priori estimate.

We multiply each of equalities (11) by $c_l'(t)$, sum them over l from 1 to m, and then integrate the result from 0 to τ. Finally, we obtain

$$\int_0^\tau \int_\Omega (u^m_{tt} u^m_t + a_{ij} u^m_{x_i} u^m_{x_i t} + u^m_{x_i tt} u^m_{x_i t} + cu^m u^m_t) dx\, dt$$

$$+ \int_0^\tau \int_{\partial\Omega} u^m_t \int_\Omega K u^m \, dy \, ds \, dt + \int_0^\tau \int_{\partial\Omega} u^m_t [\alpha u^m_{tt} + \beta u^m] ds \, dt = \int_0^\tau \int_\Omega f(x,t) u^m_t(x,t) dx \, dt. \quad (14)$$

Integration by part allows us to transform (14) into

$$\int_\Omega [(u_t^m(x,\tau))^2 + a_{ij}u_{x_i}^m(x,\tau))u_{x_j}^m(x,\tau) + (u_{xt}^m(x,\tau))^2]dx$$
$$+ \int_{\partial\Omega} [\alpha(u_t^m(x,\tau))^2 + \beta(u^m(x,\tau))^2] = 2\int_0^\tau \int_\Omega f u_t^m \, dx\, dt$$
$$- 2\int_0^\tau \int_\Omega c u^m u_t^m \, dx\, dt + \int_\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} u_{x_i}^m u_{x_j}^m dx\, dt - 2\int_0^\tau \int_{\partial\Omega} u_t^m \int_\Omega K u^m \, dy\, ds\, dt.$$

The estimate of right-hand side of the latter equality leads to the *first a priori estimate*

$$\|u^m\|^2_{W_2^1(Q_T)} + \|u^m_{xt}\|^2_{L_2(Q_T)} + \|u_t^m\|^2_{L_2(S_T)} + \|u^m\|^2_{L_2(S_T)} \le P_1. \tag{15}$$

In order to find the second a priori estimate we differentiate relation (11) with respect to t, multiply the result by $c_l''(t)$, sum it over l from 1 to m and integrate from 0 to τ. Thus, we arrive at the equation

$$\int_0^\tau \int_\Omega (u_{ttt}^m u_{tt}^m + a_{ij}u_{tx_i}^m u_{ttx_j}^m + u_{tttx_i}^m u_{ttx_i}^m + c u_t^m u_{tt}^m + c_t u^m u_{tt}^m + a_{ijt}u_{x_i}^m u_{ttx_j}^m) dx\, dt$$
$$+ \int_0^\tau \int_{\partial\Omega} u_{tt}^m \int_\Omega (K u_t^m + K_t u^m) dy\, ds\, dt$$
$$+ \int_0^\tau \int_{\partial\Omega} [\alpha_t (u_{tt}^m)^2 + \beta_t u_{tt}^m u^m] ds\, dt + \int_0^\tau \int_{\partial\Omega} u_{tt}^m(\alpha u_{ttt}^m + \beta u_t^m) ds\, dt = \int_0^\tau \int_\Omega f_t u_{tt}^m dx\, dt.$$

We now integrate by parts the first three summands of the left-hand side and transform this identity into

$$\int_\Omega [(u_{tt}^m)^2 + (u_{xtt}^m)^2]\Big|_{t=\tau} dx + \int_{\partial\Omega} [\alpha(u_{tt}^m)^2 + \beta(u_t^m)^2]\Big|_{t=\tau} ds$$
$$= \int_\Omega [(u_{tt}^m)^2 + (u_{xtt}^m)^2]\Big|_{t=0} dx + \int_{\partial\Omega} \alpha(u_{tt}^m(x,0))^2 ds + 2\int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} u_{x_i t}^m u_{x_j t}^m dx\, dt$$
$$- \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} u_{x_i}^m u_{x_j tt}^m dx\, dt - 2\int_0^\tau \int_\Omega (cu)_t u_{tt} dx\, dt + \int_0^\tau \int_{\partial\Omega} [\alpha_t (u_{tt}^m)^2 + \beta_t (u_t^m)^2] ds\, dt$$
$$- 2\int_0^\tau \int_{\partial\Omega} u_{tt}^m \int_\Omega (Ku)_t \, dy\, ds\, dt + 2\int_0^\tau \int_\Omega f_t u_{tt}^m dx\, dt. \tag{16}$$

Consider the summands $\int\limits_\Omega (u_{ttx}^m(x,0))^2 dx$, $\int\limits_{\partial\Omega} (u_{tt}^m(x,0))^2 dx$ that are somewhat difficult to estimate because we know nothing of their behavior for $t = 0$. In (11) put $t = 0$, multiply it by $c_l''(0)$ and sum the resulting equalities over l from 1 to m. Since $c_l(0) = c_l'(0) = 0$, we have also $u^m(x,0) = 0$, $u_{x_i}^m(x,0) = 0$. Under these circumstances we have the identity

$$\int_\Omega [(u_{tt}^m(x,0))^2 + (u_{ttx_i}^m(x,0))^2] dx + \alpha(0)\int_{\partial\Omega} (u_{tt}^m(x,0))^2 ds = \int_\Omega f_t(x,0) u_{tt}^m(x,0) dx.$$

By the Cauchy inequality

$$\left| \int_\Omega f_t u_t^m dx \right| \le \frac{1}{2}\int_\Omega f_t^2(x,0) dx + \frac{1}{2}\int_\Omega (u_{tt}^m(x,0))^2 dx$$

we have

$$\int_\Omega [(u_{tt}^m(x,0))^2 + (u_{ttx_i}^m(x,0))^2] dx + \alpha(0)\int_{\partial\Omega} (u_{tt}^m(x,0))^2 ds \le \int_\Omega (f_t(x,0))^2 dx,$$

this expression allows us to infer from (16) the *second a priori estimate* by elementary inequalities and assumptions of the theorem:

$$\|u_{tt}^m\|^2_{L_2(Q_T)} + \|u_{xtt}^m\|^2_{L_2(Q_T)} + \|u_{tt}^m\|^2_{L_2(S_T)} \le P_2. \tag{17}$$

Estimates (15) and (17) give us the a priori estimate in the space $W(Q_T)$

$$\|u^m\|_{W(Q_T)} \le P, \tag{18}$$

that allows us to make the next step in the proof of the theorem. Note that the constant P in (18) does not depend on m. Hence, the constructed sequance of approximate solutions $\{u^m(x,t)\}$ contains a weakly convergent in $W(Q_T)$ subsequence for which for the sake of brevity we conserve the same notation.

Let us show now that the limit of this subsequence $u \in W(Q_T)$ is exactly the desired approximate solution.

We multiply each equality of (11) by $d_l \in W_2^1(0,T)$, $d_l(T) = 0$, sum over l from 1 to m, and then integrate the result from 0 to T. After we integrate the second summand of the left-hand side of the identity we denote $\sum\limits_{l=1}^{m} d_l(t)w_l(x)$ by $\eta(x,t)$ and obtain

$$\int_0^T \int_\Omega (u_{tt}^m \eta - u_{xt}^m \eta_{xt} + a_{ij} u_{x_i}^m \eta_{x_j} + cu^m \eta)dx\,dt + \int_0^T \int_{\partial\Omega} \eta(x,t) \int_\Omega K u^m\,dy\,ds\,dt$$
$$+ \int_0^T \int_{\partial\Omega} \eta(x,t)[\alpha u_{tt}^m + \beta u^m]ds\,dt = \int_0^T \int_\Omega f\eta\,dx\,dt.$$

Estimates (15) and (17) ensure the possibility of passing to the limit for $m \to \infty$ for the fixed $\eta(x,t)$. This gives us relation (6) for any $\eta(x,t) = \sum\limits_{l=1}^{m} d_l(t)w_l(x)$. Since the set of such functions is dense in $W_2^1(Q_T)$, identity (6) holds for any function of $\widehat{W}_2^1(Q_T)$. This ensures the generalized solution existence and completes the proof of the theorem. □

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