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On Solvability of Non-Linear Semi-Periodic Boundary-Value Problem for System of Hyperbolic Equations

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Abstract—We study a non-linear semi-periodic boundary-value problem for a system of hyperbolic equations with mixed derivative. At that, the semi-periodic boundary-value problem for a system of hyperbolic equations is reduced to an equivalent problem, consisting of a family of periodic boundary-value problems for ordinary differential equations and functional relation. When solving a family of periodic boundary-value problems of ordinary differential equations we use the method of parameterization. This approach allowed to establish sufficient conditions for the existence of an isolated solution of non-linear semi-periodic boundary-value problem for a system of hyperbolic equations.

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INTRODUCTION

The mathematical modeling of phenomena and processes, which repeat after a certain time period, requires the investigation of periodic boundary-value problems of hyperbolic type. The systematic study of periodic boundary-value problems for hyperbolic equations with mixed partial derivatives began in 60ies in L. Cesari papers. O. Vejvoda, A. M. Samoilenko, B. I. Ptashnik, A. Yu. Kolesov, E. F. Mishchenko, N. Kh. Rozov, S. V. Zhestkov, Yu. A. Mitropol'skii, G. P. Khoma, M. I. Gromyak, T. I. Kiguradze and others [1–8] engaged in further research of solvability questions for such boundary-value problems. The application of different methods such as functional analysis methods, the method of successive approximations, the variation method etc allows one to obtain different solvability conditions of boundary-value problems for hyperbolic equations. In paper [9] by the method of introducing functional parameters one investigated the nonlocal boundary value problem for system of hyperbolic equations. One established sufficient conditions of the one-valued solvability in terms of coefficients and proposed an algorithm of finding a solution, each step of which is composed of two points: 1) finding introduced functional parameters, 2) finding solution to the Goursat problems on small domains. In paper [10] with the help of parameterization method [11] one proposed a new constructive algorithm of finding a solution to the quasi-linear semi-periodic boundary-value problem for a system of hyperbolic equations with mixed derivative. In the present paper, based on proposed in [10] algorithm we establish sufficient conditions of the algorithm convergence and the existence of an isolated solution to the corresponding nonlinear boundary-value problem (1)–(3).

1. PROBLEM DEFINITION

On $\overline{\Omega}=[0,\omega]\times[0,T]$ we consider the boundary-value problem

$$\frac{\partial^2 u}{\partial x\partial t}=f\left(x,t,u,\frac{\partial u}{\partial x}\right),\quad (x,t)\in\overline{\Omega},\quad u\in R^n, \tag{1}$$

$$u(0,t)=\psi(t),\quad t\in[0,T], \tag{2}$$

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$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{3}$$

where the function $f : \overline{\Omega} \times R^n \times R^n \to R^n$ is continuous, the n-vector-function $\psi(t)$ is continuously differentiable on $[0,T]$ and satisfies the condition $\psi(0) = \psi(T)$.

A function $u(x,t) \in C(\overline{\Omega}, R^n)$, which has partial derivatives $\frac{\partial u(x,t)}{\partial x} \in C(\overline{\Omega}, R^n)$, $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega}, R^n)$, is said to be a solution to problem (1)–(3), if it satisfies system (1) with all $(x,t) \in \overline{\Omega}$ and satisfies edge conditions (2), (3).

Let us introduce a new unknown function $v(x,t) = \frac{\partial u(x,t)}{\partial x}$, and write problem (1)–(3) in the form

$$\frac{\partial v}{\partial t} = f(x,t,u(x,t),v), \quad (x,t) \in \overline{\Omega}, \tag{4}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{5}$$

$$u(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi, \quad t \in [0,T], \quad x \in [0,\omega]. \tag{6}$$

Here the semi-periodic boundary-value problem for the system of hyperbolic equations is reduced to the family of periodic boundary value problems for ordinary differential equations and the functional correlation. A pair of functions $(v(x,t), u(x,t))$ belonging to $C(\overline{\Omega}, R^n)$ is said to be a solution to problem (4)–(6), if $v(x,t) \in C(\overline{\Omega}, R^n)$ is a solution to problem (4), (5), where $v(x,t)$ is connected with $u(x,t)$ by functional correlation (6).

Problems (1)–(3) and (4)–(6) are equivalent in the sense that if the function $u^*(x,t)$ is a solution to problem (1)–(3), then the pair $\big(v^*(x,t) = \frac{\partial u^*(x,t)}{\partial x}, u^*(x,t)\big)$ is a solution to problem (4)–(6) and vice versa, if a pair $(\widehat{v}(x,t), \widehat{u}(x,t))$ is a solution to problem (4)–(6), then $\widehat{u}(x,t)$ is a solution to problem (1)–(3).

2. SUFFICIENT CONDITIONS OF EXISTENCE OF AN ISOLATED SOLUTION TO SEMI-PERIODIC BOUNDARY-VALUE PROBLEM

For the solution of problem (4)–(6) we apply the parameterization method [2]. By the step $h > 0 :$ $Nh = T$ we decompose $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$, $N = 1, 2, \dots$ Here the domain Ω is decomposed into N parts. We denote by $v_r(x,t)$, $u_r(x,t)$ the narrowing of functions $v(x,t)$, $u(x,t)$ on $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$, respectively. Then problem (4)–(6) is equivalent to the boundary-value problem

$$\frac{\partial v_r}{\partial t} = f(x,t,u_r(x,t),v_r), \quad (x,t) \in \Omega_r, \tag{7}$$

$$v_1(x,0) - \lim_{t \to T-0} v_N(x,t) = 0, \quad x \in [0,\omega], \tag{8}$$

$$\lim_{t \to sh-0} v_s(x,t) = v_{s+1}(x,sh), \quad x \in [0,\omega], \quad s = \overline{1,N-1}, \tag{9}$$

$$u_r(x,t) = \psi(t) + \int_0^x v_r(\xi,t)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \tag{10}$$

where (9) are conditions of gluing of functions $v(x,t)$ in inner lines of decomposition. A solution to problem (7)–(10) is presented by systems $v(x,[t]) = (v_1(x,t),\ v_2(x,t), \dots, v_N(x,t))'$, $u(x,[t]) = (u_1(x,t), u_2(x,t), \dots, u_N(x,t))'$, where functions $v_r(x,t)$, $u_r(x,t)$, $r = \overline{1,N}$, are continuous and bounded on Ω_r, the function $v_r(x,t)$, $r = \overline{1,N}$, which has a continuous and bounded on Ω_r partial derivative $\frac{\partial v_r(x,t)}{\partial t}$, satisfies system of differential equations (7) with all $(x,t) \in \Omega_r$, $r = \overline{1,N}$, and equalities (8), (9) takes place.

We denote by $\lambda_r(x)$ the value of function $v_r(x,t)$ with $t = (r-1)h$, i.e., $\lambda_r(x) = v_r(x,(r-1)h)$ and make a substitution $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x)$, $r = \overline{1,N}$. We obtain an equivalent boundary-value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = f(x,t,u_r(x,t),\widetilde{v}_r + \lambda_r(x)), \quad (x,t)\in\Omega_r, \tag{11}$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \quad x\in[0,\omega], \quad r=\overline{1,N}, \tag{12}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t\to T-0}\widetilde{v}_N(x,t) = 0, \quad x\in[0,\omega], \tag{13}$$

$$\lambda_s(x) + \lim_{t\to sh-0}\widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, \quad x\in[0,\omega], \quad s=\overline{1,N-1}, \tag{14}$$

$$u_r(x,t) = \psi(t) + \int_0^x \widetilde{v}_r(\xi,t)d\xi + \int_0^x \lambda_r(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}. \tag{15}$$

Problems (11)–(15) and (7)–(10) are equivalent in the sense that if the system of pairs $\{v_r(x,t), u_r(x,t)\}$, $r=\overline{1,N}$, is a solution to problem (7)–(10), then the system of triples $\{\lambda_r(x)=v_r(x,(r-1)h),\ \widetilde{v}_r(x,t)=v_r(x,t)-v_r(x,(r-1)h),\ u_r(x,t)\}$, $r=\overline{1,N}$, is a solution to problem (11)–(15) and vice versa, if $\{\lambda_r(x),\ \widetilde{v}_r(x,t), u_r(x,t)\}$, $r=\overline{1,N}$, is a solution to problem (11)–(15), then the system $\{\lambda_r(x)+\widetilde{v}_r(x,t), u_r(x,t)\}$, $r=\overline{1,N}$, is a solution to problem (7)–(10).

Problem (11), (12) with fixed $\lambda_r(x)$, $u_r(x,t)$ is one-parameter family of the Cauchy problem for systems of ordinary differential equations, where $x\in[0,\omega]$, and equivalent to the nonlinear integral equation

$$\widetilde{v}_r(x,t) = \int_{(r-1)h}^t f(x,\tau,u_r(x,\tau),\widetilde{v}_r(x,\tau)+\lambda_r(x))d\tau. \tag{16}$$

Instead of $\widetilde{v}_r(x,\tau)$ we substitute the corresponding right-hand side of (16) and repeating this process ν times ($\nu=1,2,\dots$) we obtain

$$\widetilde{v}_r(x,t) = \int_{(r-1)h}^t f\Bigg(x,\tau_1,u_r(x,\tau_1),\int_{(r-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_r(x,\tau_2),\dots,$$
$$\int_{(r-1)h}^{\tau_{\nu-1}} f\big(x,\tau_\nu,u_r(x,\tau_\nu),\widetilde{v}_r(x,\tau_\nu)+\lambda_r(x)\big)d\tau_\nu + \dots + \lambda_r(x)\Bigg)d\tau_2 + \lambda_r(x)\Bigg)d\tau_1. \tag{17}$$

Hence, defining $\lim\limits_{t\to rh-0}\widetilde{v}_r(x,t)$, substituting them in (13), (14), we have a system of nonlinear equations with respect to $\lambda_r(x)$:

$$\lambda_1(x) - \lambda_N(x) - \int_{(N-1)h}^{Nh} f\Bigg(x,\tau_1,u_N(x,\tau_1),\int_{(N-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_N(x,\tau_2),\dots,$$
$$\int_{(N-1)h}^{\tau_{\nu-1}} f\big(x,\tau_\nu,u_N(x,\tau_\nu),\widetilde{v}_N(x,\tau_\nu)+\lambda_N(x)\big)d\tau_\nu + \dots + \lambda_N(x)\Bigg)d\tau_2 + \lambda_N(x)\Bigg)d\tau_1 = 0,$$

$$\lambda_s(x) + \int_{(s-1)h}^{sh} f\Bigg(x,\tau_1,u_s(x,\tau_1),\int_{(s-1)h}^{\tau_1} f\Bigg(x,\tau_2,u_s(x,\tau_2),\dots,$$
$$\int_{(s-1)h}^{\tau_{\nu-1}} f\big(x,\tau_\nu,u_s(x,\tau_\nu),\widetilde{v}_s(x,\tau_\nu)+\lambda_s(x)\big)d\tau_\nu + \dots + \lambda_s(x)\Bigg)d\tau_2 + \lambda_s(x)\Bigg)d\tau_1 - \lambda_{s+1}(x) = 0,$$

$x\in[0,\omega]$, $s=\overline{1,N-1}$, which we write in form

$$Q_{\nu,h}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x)) = 0. \tag{18}$$

Without decomposition ($N=1$, $h=T$), system of equations (18) has the form

$$\int_0^T f\Bigg(x,\tau_1,u(x,\tau_1),\int_0^{\tau_1} f\Bigg(x,\tau_2,u(x,\tau_2),\dots,\int_0^{\tau_{\nu-1}} f\big(x,\tau_\nu,u(x,\tau_\nu),\widetilde{v}(x,\tau_\nu)$$

$$+\lambda(x)\big)d\tau_\nu+\cdots+\lambda(x)\Big)d\tau_2+\lambda(x)\Big)d\tau_1=0,\quad x\in[0,\omega].$$

For finding the system of three functions $\{\lambda_r(x),\,\widetilde{v}_r(x,t),\,u_r(x,t)\}$, $r=\overline{1,N}$, we have a closed system composed of Eqs. (18), (17), and (15), defined by the function f, the decomposition step $h>0$, and the number of substitution ν.

We choose $h{>}0: Nh{=}T,\ N{=}1,2,...,$ a vector-function $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\lambda_2^{(0)}(x),...,\lambda_N^{(0)}(x))'\in C([0,\omega],R^{Nn})$, and assume that problem (11)–(15) with $\lambda_r(x)=\lambda_r^{(0)}(x)$, $r=\overline{1,N}$, has a solution $u_r^{(0)}(x,t)\in\widetilde{C}(\Omega_r,R^n)$, $\widetilde{v}_r^{(0)}(x,t)\in\widetilde{C}(\Omega_r,R^n)$, $r=\overline{1,N}$. We denote the set of such $\lambda^{(0)}(x)\in C([0,\omega],R^{nN})$ by $G_0(f,x,h)$, and the corresponding to $\lambda^{(0)}(x)$ system of solutions to problems (11)–(15) by $\widetilde{v}^{(0)}(x,[t])=(\widetilde{v}_1^{(0)}(x,t),\,\widetilde{v}_2^{(0)}(x,t),\ldots,\widetilde{v}_N^{(0)}(x,t))'$, $u^{(0)}(x,[t])=(u_1^{(0)}(x,t),u_2^{(0)}(x,t),\ldots,u_N^{(0)}(x,t))'$.

Taking $\lambda^{(0)}(x)\in G_0(f,x,h)$, $\widetilde{v}^{(0)}(x,[t])$, $u^{(0)}(x,[t])$ and continuous on $[0,\omega]$ functions $\rho(x)>0$, $\theta(x)>0$, we construct sets

$$S(\lambda^{(0)}(x),\rho(x))=\big\{(\lambda_1(x),\lambda_2(x),\ldots,\lambda_N(x))'\in C([0,\omega],R^{nN}):\|\lambda_r(x)-\lambda_r^{(0)}(x)\|<\rho(x),\ r=\overline{1,N}\big\},$$

$$S(\widetilde{v}^{(0)}(x,[t]),\theta(x))=\big\{(\widetilde{v}_1(x,t),\widetilde{v}_2(x,t),\ldots,\widetilde{v}_N(x,t))',\\ \widetilde{v}_r(x,t)\in\widetilde{C}(\Omega_r,R^n):\|\widetilde{v}_r(x,t)-\widetilde{v}_r^{(0)}(x,t)\|<\theta(x),r=\overline{1,N}\big\},$$

$$S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])=\{(u_1(x,t),u_2(x,t),\ldots,u_N(x,t))',\\ u_r(x,t)\in\widetilde{C}(\Omega_r,R^n):\|u_r(x,t)-u_r^{(0)}(x,t)\|<\omega[\rho(x)+\theta(x)],(x,t)\in\Omega_r,r=\overline{1,N}\},$$

$$G_1^0(\rho(x),\theta(x))=\{(x,t,u,v):(x,t)\in\overline{\Omega},\|u-u_r^{(0)}(x,t)\|<\omega[\rho(x)+\theta(x)],(x,t)\in\Omega_r,\ r=\overline{1,N},\\ \|u-\lim_{t\to Nh-0}u_N^{(0)}(x,t)\|<\omega[\rho(x)+\theta(x)],\ t=T,\ \|v-\lambda_r^{(0)}(x)-\widetilde{v}_r^{(0)}(x,t)\|<\rho(x)+\theta(x),\\ (x,t)\in\Omega_r,\ r=\overline{1,N},\ \|v-\lambda_N^{(0)}(x)-\lim_{t\to T-0}\widetilde{v}_N^{(0)}(x,t)\|<\rho(x)+\theta(x),\ t=T\}.$$

We denote by $U_0(f,L_1(x),L_2(x),x,h)$ the family $(\lambda^{(0)}(x),\widetilde{v}^{(0)}(x,[t]),u^{(0)}(x,[t]),\rho(x),\,\theta(x))$, with which the function $f(x,t,u,v)$ in $G_1^0(\rho(x),\theta(x))$ has continuous partial derivatives $f_v'(x,t,u,v)$, $f_u'(x,t,u,v)$ and $\|f_v'(x,t,u,v)\|\le L_1(x)$, $\|f_u'(x,t,u,v)\|\le L_2(x)$, where $L_1(x)$, $L_2(x)$ are continuous on $[0,\omega]$ functions.

By the system $\{\lambda_r(x),\widetilde{v}_r(x,t),u_r(x,t)\}$, $r=\overline{1,N}$, we compose the triple $\{\lambda(x),\,\widetilde{v}(x,[t]),\,u(x,[t])\}$, where $\lambda(x)=(\lambda_1(x),...,\lambda_N(x))'$, $\widetilde{v}(x,[t])=(\widetilde{v}_1(x,t),...,\widetilde{v}_N(x,t))'$, $u(x,[t])=(u_1(x,t),...,u_N(x,t))'$. Assuming the existence of $\lambda^{(0)}(x)\in G_0(f,x,h)$, we take the triple $\{\lambda^{(0)}(x),\,\widetilde{v}^{(0)}(x,[t]),\,u^{(0)}(x,[t])\}$ as an initial approximation of problem (11)–(15), and construct successive approximations by the following algorithm.

Step 1. A) Assuming that $u_r(x,t)=u_r^{(0)}(x,t)$, $r=\overline{1,N}$, we find the first approximations by $\lambda_r(x)$, $\widetilde{v}_r(x,t)$, solving problem (11)–(14). Taking $\lambda^{(1,0)}(x)=\lambda^{(0)}(x)$, $\widetilde{v}_r^{(1,0)}(x,t)=\widetilde{v}_r^{(0)}(x,t)$, we find the system of pairs $\{\lambda_r^{(1)}(x),\,\widetilde{v}_r^{(1)}(x,t)\}$ as a limit of sequence $\{\lambda_r^{(1,m)}(x),\,\widetilde{v}_r^{(1,m)}(x,t)\}$, $r=\overline{1,N}$, which is defined by the following method.

Step 1.1. a) Substituting $\widetilde{v}_r^{(1,0)}(x,t)$, $r=\overline{1,N}$, in (18) from the system of functional equations $Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(1,0)}(x,[\cdot]),\lambda(x))=0$ we define $\lambda_r^{(1,1)}(x)$, $r=\overline{1,N}$. b) Sunstituting $\widetilde{v}_r^{(1,0)}(x,t)$, $\lambda_r^{(1,1)}(x)$ instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, in the right-hand side of (17), we find $\widetilde{v}_r^{(1,1)}(x,t)$, $r=\overline{1,N}$.

Step 1.2. a) Substituting $\widetilde{v}_r^{(1,1)}(x,t)$, $r=\overline{1,N}$, in (18) from the system of functional equations $Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(1,1)}(x,[\cdot]),\lambda(x))=0$ we define $\lambda_r^{(1,2)}(x)$, $r=\overline{1,N}$. b) Substituting in the right-hand side of (17) instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(1,1)}(x,t)$, $\lambda_r^{(1,2)}(x)$, we find $\widetilde{v}_r^{(1,2)}(x,t)$, $r=\overline{1,N}$.

On the $(1,m)$th step we obtain the system of pairs $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$, $r=\overline{1,N}$. We assume that the solution to problem (11)–(14) (a sequence of systems of pairs $\{\lambda_r^{(1,m)}(x),\ \widetilde{v}_r^{(1,m)}(x,t)\}$) is defined and with $m\to\infty$ it converges to $\{\lambda_r^{(1)}(x),\widetilde{v}_r^{(1)}(x,t)\}$, $r=\overline{1,N}$.

B) Functions $u_r^{(1)}(x,t)$, $r=\overline{1,N}$, are defined from correlation (15), where $\lambda_r(x)=\lambda_r^{(1)}(x)$, $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1)}(x,t)$, $r=\overline{1,N}$.

Step 2. A) Assuming that $u_r(x,t)=u_r^{(1)}(x,t)$, $r=\overline{1,N}$, we find the second approximations by $\lambda_r(x),\widetilde{v}_r(x,t)$ solving problem (11)–(14). Taking $\lambda^{(2,0)}(x)=\lambda^{(1)}(x)$, $\widetilde{v}_r^{(2,0)}(x,t)=\widetilde{v}_r^{(1)}(x,t)$, we find the system of pairs $\{\lambda_r^{(2)}(x),\ \widetilde{v}_r^{(2)}(x,t)\}$ as the limit of sequence $\{\lambda_r^{(2,m)}(x),\ \widetilde{v}_r^{(2,m)}(x,t)\}$, $r=\overline{1,N}$, defined as follows.

Step 2.1. a) Substituting $\widetilde{v}_r^{(2,0)}(x,t)$, $r=\overline{1,N}$, in (18) from the system of functional equations $Q_{\nu,h}(x,u^{(1)}(x,[\cdot]),\widetilde{v}^{(2,0)}(x,[\cdot]),\lambda(x))=0$ we define $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$. b) Substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(2,0)}(x,t)$, $\lambda_r^{(2,1)}(x)$ in the right-hand side of (17), we find $\widetilde{v}_r^{(2,1)}(x,t)$, $r=\overline{1,N}$.

Step 2.2. a) Substituting $\widetilde{v}_r^{(2,1)}(x,t)$, $r=\overline{1,N}$, in (18) from the system of functional equations $Q_{\nu,h}(x,u^{(1)}(x,[\cdot]),\widetilde{v}^{(2,1)}(x,[\cdot]),\lambda(x))=0$ we define $\lambda_r^{(2,2)}(x)$, $r=\overline{1,N}$. b) Substituting instead of $\widetilde{v}_r(x,t)$, $\lambda_r(x)$, respectively, $\widetilde{v}_r^{(2,1)}(x,t)$, $\lambda_r^{(2,2)}(x)$ in the right-hand side of (17), we find $\widetilde{v}_r^{(2,2)}(x,t)$, $r=\overline{1,N}$.

On the $(2,m)$th step we obtain the system of pairs $\{\lambda_r^{(2,m)}(x),\ \widetilde{v}_r^{(2,m)}(x,t)\}$, $r=\overline{1,N}$. We assume that the solution to problem (11)–(14) (the sequence of systems of pairs $\{\lambda_r^{(2,m)}(x),\ \widetilde{v}_r^{(2,m)}(x,t)\}$) is defined and with $m\to\infty$ it converges to $\{\lambda_r^{(2)}(x),\widetilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$.

B) Functions $u_r^{(2)}(x,t)$, $r=\overline{1,N}$, are defined from correlation (15), where $\lambda_r(x)=\lambda_r^{(2)}(x)$, $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(2)}(x,t)$, $r=\overline{1,N}$.

Continuing the process, on the kth step we obtain the system of triples $\{\lambda_r^{(k)}(x),\widetilde{v}_r^{(k)}(x,t),u_r^{(k)}(x,t)\}$, $r=\overline{1,N}$.

The following theorem establishes sufficient conditions of the feasibility, the algorithm convergence, and the existence of solution to the multi-characteristic boundary-value problem with functional parameters (11)–(15).

Theorem 1. *Let there exist* $h>0:Nh=T$ $(N=1,2,\dots)$, $\nu\in\mathbb{N}$, $(\lambda^{(0)}(x),\widetilde{v}^{(0)}(x,[t]),\ u^{(0)}(x,[t]),\ \rho(x),\theta(x))\in U_0(f,L_1(x),L_2(x),x,h)$, *with which the Jacobi matrix* $\frac{\partial Q_{\nu,h}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))}{\partial\lambda}$ *is invertible for all* $(x,\lambda(x),\ \widetilde{v}(x,[\cdot]),u(x,[\cdot]))$, *where* $x\in[0,\omega]$, $(\lambda(x),\widetilde{v}(x,[t]),u(x,[t]))\in S(\lambda^{(0)}(x),\rho(x))\times S(\widetilde{v}^{(0)}(x,[t]),\theta(x))\times S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])$ *and the following inequalities are fulfilled*:

$$\Big\|\Big[\frac{\partial Q_{\nu,h}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x),)}{\partial\lambda}\Big]^{-1}\Big\|\le\gamma_\nu(x,h),$$

$$q_\nu(x,h)=\frac{(L_1(x)h)^\nu}{\nu!}\Big[1+\gamma_\nu(x,h)\sum_{j=1}^{\nu}\frac{(L_1(x)h)^j}{j!}\Big]\le\mu<1,$$

$$[c_0(x)c_1(x)+c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)e^{\int\limits_0^{\xi}c(\xi_1)d\xi_1}\int_0^{\xi}[c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]$$

$$\times \gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,u^{(0)}(\xi_1,[\cdot]),\widetilde{v}^{(0)}(\xi_1,[\cdot]),\lambda^{(0)}(\xi_1))\|d\xi_1 d\xi$$
$$+[c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<\rho(x), \quad (19)$$

$$[c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)e^{\int\limits_0^\xi c(\xi_1)d\xi_1}\int_0^\xi [c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]$$
$$\times \gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,u^{(0)}(\xi_1,[\cdot]),\widetilde{v}^{(0)}(\xi_1,[\cdot]),\lambda^{(0)}(\xi_1))\|d\xi_1 d\xi$$
$$+c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<\theta(x), \quad (20)$$

where

$$c(x)=[c_0(x)c_1(x)+2c_0(x)+2]c_2(x)L_2(x), \ \ c_0(x)=\frac{1}{1-q_\nu(x,h)}\sum_{j=1}^{\nu}\frac{(L_1(x)h)^j}{j!},$$

$$c_1(x)=\gamma_\nu(x,h)\frac{(L_1(x)h)^\nu}{\nu!}, \ \ c_2(x)=\gamma_\nu(x,h)\sum_{j=0}^{\nu}\frac{(L_1(x)h)^j}{j!}h,$$

then the sequence of functions $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]),u^{(k)}(x,[t]))$, $k=1,2,\dots,$ *defined by the algorithm is contained in* $S(\lambda^{(0)}(x),\rho(x))\times S(\widetilde{v}^{(0)}(x,[t]),\theta(x))\times S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])$, *converges to* $(\lambda^*(x),\widetilde{v}^*(x,[t]),\ u^*(x,[t]))$, *which is a solution to problem* (11)–(15), *and estimates are true*:

a) $$\max_{r=\overline{1,N}}\|\lambda_r^*(x)-\lambda_r^{(0)}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{(0)}(x,t)\|$$

$$\le c(x)e^{\int\limits_0^\xi c(\xi_1)d\xi_1}\int_0^x [c_0(\xi)c_1(\xi)+c_0(\xi)+1]\gamma_\nu(\xi,h)\|Q_{\nu,h}(\xi,u^{(0)}(\xi,[\cdot]),\widetilde{v}^{(0)}(\xi,[\cdot]),\lambda^{(0)}(\xi))\|d\xi$$

$$+[c_0(x)c_1(x)+c_0(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|,$$

b) $$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^*(x,t)-u_r^{(0)}(x,t)\|$$

$$\le\int_0^x\max_{r=\overline{1,N}}\|\lambda_r^*(\xi)-\lambda_r^{(0)}(\xi)\|d\xi+\int_0^x\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{(0)}(\xi,t)\|d\xi.$$

In addition, any solution $(\lambda(x),\widetilde{v}(x,[t]),u(x,[t]))$ *to problem* (11)–(15) *in* $S(\lambda^{(0)}(x),\rho(x))S(\widetilde{v}^{(0)}\times(x,[t]),\theta(x))\times S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])$ *is isolated.*

Proof. In view of theorem 1 from [12] (P. 91) with $u_r(x,t)=u_r^{(0)}(x,t)$, $r=\overline{1,N}$, problem (11)–(15) has an isolated solution $(\lambda^{(1)}(x),\widetilde{v}^{(1)}(x,t))\in S(\lambda^{(0)}(x),\rho(x))\times S(\widetilde{v}^{(0)}(x,[t]),\theta(x))$ and the estimates hold

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|\le\frac{1}{1-q_\nu(x,h)}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\le\frac{(L_1(x)h)^\nu}{\nu!}\frac{\gamma_\nu(x,h)}{1-q_\nu(x,h)}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|$$
$$+\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|.$$

Since functions $\widetilde{v}_r^{(1,1)}(x,t)$, $\widetilde{v}_r^{(1,0)}(x,t)$, $r=\overline{1,N}$, are defined from correlations

$$\widetilde{v}_r^{(1,1)}(x,t)=\int_{(r-1)h}^{t} f\bigg(x,\tau_1,u_r^{(0)}(x,\tau_1),\int_{(r-1)h}^{\tau_1} f\bigg(x,\tau_2,u_r^{(0)}(x,\tau_2),\dots,$$
$$\int_{(r-1)h}^{\tau_{\nu-1}} f\big(x,\tau_\nu,u_r^{(0)}(x,\tau_\nu),\widetilde{v}_r^{(1,0)}(x,\tau_\nu)+\lambda_r^{(1,1)}(x)\big)d\tau_\nu+\dots+\lambda_r^{(1,1)}(x)\bigg)d\tau_2+\lambda_r^{(1,1)}(x)\bigg)d\tau_1,$$

$$\widetilde{v}_r^{(1,0)}(x,t)=\int_{(r-1)h}^{t} f\bigg(x,\tau_1,u_r^{(0)}(x,\tau_1),\int_{(r-1)h}^{\tau_1} f\bigg(x,\tau_2,u_r^{(0)}(x,\tau_2),\dots,$$
$$\int_{(r-1)h}^{\tau_{\nu-1}} f\big(x,\tau_\nu,u_r^{(0)}(x,\tau_\nu),\widetilde{v}_r^{(1,0)}(x,\tau_\nu)+\lambda_r^{(1,0)}(x)\big)d\tau_\nu+\dots+\lambda_r^{(1,0)}(x)\bigg)d\tau_2+\lambda_r^{(1,0)}(x)\bigg)d\tau_1,$$

$r=\overline{1,N}$, we have that the estimate is true

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1,1)}(x,t)-\widetilde{v}_r^{(1,0)}(x,t)\|\le\sum_{j=1}^{\nu}\frac{(L_1(x)h)^j}{j!}\max_{r=\overline{1,N}}\|\lambda_r^{(1,1)}(x)-\lambda_r^{(1,0)}(x)\|.$$

Taking into account the inequality

$$\max_{r=\overline{1,N}}\|\lambda_r^{(1,1)}(x)-\lambda_r^{(1,0)}(x)\|\le\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|,$$

we have

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|$$
$$\le c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<\theta(x),$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\le[c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<\rho(x).$$

Then

$$\Delta^{(1)}(x)=\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|$$
$$\le[c_0(x)c_1(x)+c_0(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(1)}(x,t)-u_r^{(0)}(x,t)\|\le\int_0^x\Delta^{(1)}(\xi)d\xi.$$

Hence we see that $u^{(1)}(x,[t])\in S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])$. We assume that the triple is defined $(\lambda_r^{(k-1)}(x),\widetilde{v}_r^{(k-1)}(x,t),u_r^{(k-1)}(x,t))\in S(\lambda^{(0)}(x),\rho(x))\times S(\widetilde{v}^{(0)}(x,[t]),\theta(x))S(u^{(0)}(x,[t]),\omega[\rho(x)+\theta(x)])$, we find the kth approximation by $\lambda_r(x)$, $\widetilde{v}_r(x,t)$ solving problem (11)–(14). Conditions of this theorem provide the fulfillment of conditions of theorem 1 from [12], if we take $\lambda_r^{(k-1)}(x)$, $\widetilde{v}_r^{(k-1)}(x,t)$ as $\lambda_r^{(k,0)}(x)$, $\widetilde{v}_r^{(k,0)}(x,t)$, respectively. Then due to theorem 1 from [12] there exists an isolated solution $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]))\in S(\lambda^{(k-1)}(x),\rho^{(k-1)}(x))\times S(\widetilde{v}^{(k-1)}(x,[t]),\theta^{(k-1)}(x))$ and estimates hold true

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k)}(x,t)-\widetilde{v}_r^{(k-1)}(x,t)\|$$
$$\le\frac{1}{1-q_\nu(x,h)}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k,1)}(x,t)-\widetilde{v}_r^{(k,0)}(x,t)\|,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k)}(x)-\lambda_r^{(k-1)}(x)\|$$

$$\leq \frac{(L_1(x)h)^\nu}{\nu!}\frac{\gamma_\nu(x,h)}{1-q_\nu(x,h)}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k,1)}(x,t)-\widetilde{v}_r^{(k,0)}(x,t)\|$$
$$+\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k-1)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))\|.$$

Since the functions $\widetilde{v}_r^{(k,1)}(x,t)$, $\widetilde{v}_r^{(k,0)}(x,t)$, $r=\overline{1,N}$, are defined from correlations

$$\widetilde{v}_r^{(k,1)}(x,t)=\int_{(r-1)h}^{t} f\biggl(x,\tau_1,u_r^{(k-1)}(x,\tau_1),\int_{(r-1)h}^{\tau_1} f\biggl(x,\tau_2,u_r^{(k-1)}(x,\tau_2),\dots,$$
$$\int_{(r-1)h}^{\tau_{\nu-1}} f\bigl(x,\tau_\nu,u_r^{(k-1)}(x,\tau_\nu),\widetilde{v}_r^{(k,0)}(x,\tau_\nu)+\lambda_r^{(k,1)}(x)\bigr)d\tau_\nu+\ldots+\lambda_r^{(k,1)}(x)\biggr)d\tau_2+\lambda_r^{(k,1)}(x),\biggr)d\tau_1,$$

$$\widetilde{v}_r^{(k,0)}(x,t)=\int_{(r-1)h}^{t} f\biggl(x,\tau_1,u_r^{(k-2)}(x,\tau_1),\int_{(r-1)h}^{\tau_1} f\biggl(x,\tau_2,u_r^{(k-2)}(x,\tau_2),\dots,$$
$$\int_{(r-1)h}^{\tau_{\nu-1}} f\bigl(x,\tau_\nu,u_r^{(k-2)}(x,\tau_\nu),\widetilde{v}_r^{(k,0)}(x,\tau_\nu)+\lambda_r^{(k,0)}(x)\bigr)d\tau_\nu+\ldots+\lambda_r^{(k,0)}(x)\biggr)d\tau_2+\lambda_r^{(k,0)}(x)\biggr)d\tau_1,$$

$r=\overline{1,N}$, we have that estimates are true

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k,1)}(x,t)-\widetilde{v}_r^{(k,0)}(x,t)\|$$
$$\leq\sum_{j=0}^{\nu}\frac{(L_1(x)h)^j}{j!}L_2(x)h\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k-1)}(x,t)-u_r^{(k-2)}(x,t)\|$$
$$+\sum_{j=1}^{\nu}\frac{(L_1(x)h)^j}{j!}\max_{r=\overline{1,N}}\|\lambda_r^{(k,1)}(x)-\lambda_r^{(k,0)}(x)\|\leq\sum_{j=0}^{\nu}\frac{(L_1(x)h)^j}{j!}L_2(x)h\int_0^x\Delta^{(k-1)}(\xi)d\xi$$
$$+\sum_{j=1}^{\nu}\frac{(L_1(x)h)^j}{j!}\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k-1)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))\|.$$

Therefore, we obtain

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{v}_r^{(k)}(x,t)-\widetilde{v}_r^{(k-1)}(x,t)\|$$
$$\leq c_2(x)L_2(x)\int_0^x\Delta^{(k-1)}(\xi)d\xi+c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k-1)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))\|.$$

Using $Q_{\nu,h}(x,u^{(k-2)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))=0$, we establish the inequality

$$\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k-1)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))\|$$
$$\leq\gamma_\nu(x,h)\sum_{j=0}^{\nu}\frac{(L_1(x)h)^j}{j!}L_2(x)h\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k-1)}(x,t)-u_r^{(k-2)}(x,t)\|$$
$$\leq c_2(x)L_2(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|u_r^{(k-1)}(x,t)-u_r^{(k-2)}(x,t)\|\leq c_2(x)L_2(x)\int_0^x\Delta^{(k-1)}(\xi)d\xi,$$

$$\max_{r=\overline{1,N}}\|\lambda_r^{(k)}(x)-\lambda_r^{(k-1)}(x)\|\leq c_0(x)c_2(x)L_2(x)\int_0^x\Delta^{(k-1)}(\xi)d\xi$$
$$+[c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k-1)}(x,[\cdot]),\widetilde{v}^{(k-1)}(x,[\cdot]),\lambda^{(k-1)}(x))\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k)}(x,t)-u_r^{(k-1)}(x,t)\| \\ \le \int_0^x \max_{r=\overline{1,N}} \|\lambda_r^{(k)}(\xi)-\lambda_r^{(k-1)}(\xi)\|d\xi + \int_0^x \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k)}(\xi,t)-\widetilde{v}_r^{(k-1)}(\xi,t)\|d\xi.$$

We take

$$\theta^{(k)}(x) = c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi + c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k)}(x,[\cdot]),\widetilde{v}^{(k)}(x,[\cdot]),\lambda^{(k)}(x))\|,$$

$$\rho^{(k)}(x) = c_0(x)c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi \\ + [c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k)}(x,[\cdot]),\widetilde{v}^{(k)}(x,[\cdot]),\lambda^{(k)}(x))\|,$$

and show that $S(\lambda^{(k)}(x),\rho^{(k)}(x))\subset S(\lambda^{(0)}(x),\rho(x))$, $S(\widetilde{v}^{(k)}(x,[t]),\theta^{(k)}(x))\subset S(\widetilde{v}^{(0)}(x,[t]),\theta(x))$. Using theorem 1 from [12], with $u(x,[t])=u^{(k)}(x,[t])$ we obtain that problem (11)–(14) has an isolated solution $(\lambda^{(k+1)}(x),\ \widetilde{v}^{(k+1)}(x,[t]))\in S(\lambda^{(k)}(x),\rho^{(k)}(x))\times S(\widetilde{v}^{(k)}(x,[t]),\theta^{(k)}(x))$ and estimates hold true

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\| \\ \le c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi + c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k)}(x,[\cdot]),\widetilde{v}^{(k)}(x,[\cdot]),\lambda^{(k)}(x))\|, \quad (21)$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x)-\lambda_r^{(k)}(x)\| \le c_0(x)c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi \\ + [c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(k)}(x,[\cdot]),\widetilde{v}^{(k)}(x,[\cdot]),\lambda^{(k)}(x))\|. \quad (22)$$

Then we can write inequalities (21), (22) in the form

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\| \le [c_0(x)+1]c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi, \quad (23)$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x)-\lambda_r^{(k)}(x)\| \le [c_0(x)c_1(x)+c_0(x)+1]c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi. \quad (24)$$

By summing the left-hand and right-hand sides of inequalities (23), (24), respectively, we obtain

$$\Delta^{(k+1)}(x) = \max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x)-\lambda_r^{(k)}(x)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(k)}(x,t)\| \\ \le [c_0(x)c_1(x)+2c_0(x)+2]c_2(x)L_2(x)\int_0^x \Delta^{(k)}(\xi)d\xi. \quad (25)$$

From inequality (25) it follows that $\Delta^{(k+1)}(x)\le c(x)\int\limits_0^x \Delta^{(k)}(\xi)d\xi$. Then

$$\Delta^{(k+1)}(x) \\ \le \frac{c(x)}{k!}\left(\int_0^x c(\xi)d\xi\right)^k \int_0^x [c_0(\xi)c_1(\xi)+c_0(\xi)+1]\gamma_\nu(\xi,h)\|Q_{\nu,h}(\xi,u^{(0)}(\xi,[\cdot]),\widetilde{v}^{(0)}(\xi,[\cdot]),\lambda^{(0)}(\xi))\|d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k+1)}(x,t)-u_r^{(k)}(x,t)\| \le \int_0^x \Delta^{(k+1)}(\xi)d\xi,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|$$

$$\leq [c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)\sum_{j=0}^{k-1}\frac{1}{j!}\Big(\int_0^\xi c(\xi_1)d\xi_1\Big)^j \Delta^{(1)}(\xi)d\xi$$

$$+\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\| \leq [c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)\sum_{j=0}^{k-1}\frac{1}{j!}\Big(\int_0^\xi c(\xi_1)d\xi_1\Big)^j$$

$$\times\int_0^\xi [c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,u^{(0)}(\xi_1,[\cdot]),\widetilde{v}^{(0)}(\xi_1,[\cdot]),\lambda^{(0)}(\xi_1))\|d\xi_1 d\xi$$

$$+c_0(x)\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\| < \theta(x),$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x)-\lambda_r^{(0)}(x)\|$$

$$\leq [c_0(x)c_1(x)+c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)\sum_{j=0}^{k-1}\frac{1}{j!}\Big(\int_0^\xi c(\xi_1)d\xi_1\Big)^j \Delta^{(1)}(\xi)d\xi$$

$$+\max_{r=\overline{1,N}} \|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\| \leq [c_0(x)c_1(x)+c_0(x)+1]c_2(x)L_2(x)\int_0^x c(\xi)\sum_{j=0}^{k-1}\frac{1}{j!}\Big(\int_0^\xi c(\xi_1)d\xi_1\Big)^j$$

$$\times\int_0^\xi [c_0(\xi_1)c_1(\xi_1)+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,u^{(0)}(\xi_1,[\cdot]),\widetilde{v}^{(0)}(\xi_1,[\cdot]),\lambda^{(0)}(\xi_1))\|d\xi_1 d\xi$$

$$+[c_0(x)c_1(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\| < \rho(x),$$

$$\max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(x)-\lambda_r^{(0)}(x)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(x,t)-\widetilde{v}_r^{(0)}(x,t)\|$$

$$\leq \Delta^{(k+1)}(x)+\Delta^{(k)}(x)+\cdots+\Delta^{(1)}(x) \leq \sum_{j=0}^{k}\frac{c(x)}{j!}\Big(\int_0^x c(\xi)d\xi\Big)^j\int_0^x \Delta^{(1)}(\xi)d\xi+\Delta^{(1)}(x)$$

$$\leq c(x)\sum_{j=0}^{k}\frac{1}{j!}\Big(\int_0^x c(\xi)d\xi\Big)^j\int_0^x [c_0(\xi)c_1(\xi)+c_0(\xi)+1]\gamma_\nu(\xi,h)$$

$$\times\|Q_{\nu,h}(\xi,u^{(0)}(\xi,[\cdot]),\widetilde{v}^{(0)}(\xi,[\cdot]),\lambda^{(0)}(\xi))\|d\xi$$

$$+[c_0(x)c_1(x)+c_0(x)+1]\gamma_\nu(x,h)\|Q_{\nu,h}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|,$$

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^{(k+1)}(x,t)-u_r^{(0)}(x,t)\|$$

$$\leq \int_0^x \max_{r=\overline{1,N}} \|\lambda_r^{(k+1)}(\xi)-\lambda_r^{(0)}(\xi)\|d\xi+\int_0^x \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^{(k+1)}(\xi,t)-\widetilde{v}_r^{(0)}(\xi,t)\|d\xi.$$

With $k\to\infty$ we obtain conditions (23), (24) and estimates a), b) of Theorem 1.

Let us show the isolation of solution. Let a triple of functions $(\lambda^*(x),\widetilde{v}^*(x,[t]),u^*(x,[t]))$ be a solution to problem (11)–(15), belonging to $S(\lambda^*(x),\rho(x))\times S(\widetilde{v}^*(x,[t]),\theta(x))\times S(u^*(x,[t]),\omega[\rho(x)+\theta(x)])$. Then there exists a continuous on $[0,\omega]$ function $\delta(x)>0$ such that

$$\|\lambda^*(x)-\lambda^{(0)}(x)\|+\delta(x)<\rho(x),\ \|\widetilde{v}^*(x,[t])-\widetilde{v}^{(0)}(x,[t])\|+\delta(x)<\theta(x),$$

$$\|u^*(x,[t])-u^{(0)}(x,[t])\|+\delta(x)<\omega[\rho(x)+\theta(x)],$$

$$\lambda(x) \in S(\lambda^*(x), \delta(x)), \ \ \widetilde{v}(x,[t]) \in S(\widetilde{v}^*(x,[t]), \delta(x)), \ \ u(x,[t]) \in S(u^*(x,[t]), \delta(x)).$$

In view of inequalities

$$\|\lambda(x) - \lambda^{(0)}\| \le \|\lambda(x) - \lambda^*(x)\| + \|\lambda^*(x) - \lambda^{(0)}\| \le \delta(x) + \|\lambda^*(x) - \lambda^{(0)}\| < \rho(x),$$

$$\|\widetilde{v}(x,[\cdot]) - \widetilde{v}^{(0)}(x,[\cdot])\| \le \|\widetilde{v}(x,[\cdot]) - \widetilde{v}^*(x,[\cdot])\| + \|\widetilde{v}^*(x,[\cdot]) - \widetilde{v}^{(0)}(x,[\cdot])\| \\ \le \delta(x) + \|\widetilde{v}^*(x,[\cdot]) - \widetilde{v}^{(0)}(x,[\cdot])\| < \theta(x),$$

$$\|u(x,[\cdot]) - u^{(0)}(x,[\cdot])\| \le \|u(x,[\cdot]) - u^*(x,[\cdot])\| + \|u^*(x,[\cdot]) - u^{(0)}(x,[\cdot])\| \\ \le \delta(x) + \|u^*(x,[\cdot]) - u^{(0)}(x,[\cdot])\| < \omega[\rho(x) + \theta(x)],$$

we have

$$\lambda(x) \in S(\lambda^{(0)}(x), \rho(x)), \widetilde{v}(x,[t]) \in S(\widetilde{v}^{(0)}(x,[t]), \theta(x)), u(x,[t]) \in S(u^{(0)}(x,[t]), \omega[\rho(x) + \theta(x)]),$$

i.e., $S(\lambda^*(x), \delta(x)) \subset S(\lambda^{(0)}, \rho(x))$, $S(\widetilde{v}^*(x,[t]), \delta(x)) \subset S(\widetilde{v}^{(0)}(x,[t]), \theta(x))$, $S(u^*(x,[t]), \delta(x)) \subset S(u^{(0)}(x,[t]), \omega[\rho(x) + \theta(x)])$. We take a number $\varepsilon > 0$ such that

$$\varepsilon\gamma_\nu(x,h) < 1, \ \ \frac{\gamma_\nu(x,h)}{1 - \varepsilon\gamma_\nu(x,h)} \cdot \frac{(L_1(x)h)^\nu}{\nu!} \sum_{j=1}^{\nu} \frac{(L_1(x)h)^j}{j!} < 1 - \frac{(L_1(x)h)^\nu}{\nu!}.$$

From the uniform continuity of $f'_v(x,t,u,v)$ in $G_1^0(\rho(x), \theta(x))$ and the structure of the Jacobian matrix $\frac{\partial Q_{\nu,h}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))}{\partial\lambda}$ its uniform continuity in $S(\lambda^*(x), \delta(x)) \times S(\widetilde{v}^*(x,[t]), \delta(x)) \times S(u^*(x,[t]), \delta(x))$ follows. Therefore there exists a continuous on $[0,\omega]$ function $\widetilde{\delta}(x) \in (0, \delta(x)]$, with which

$$\left\| \frac{\partial Q_{\nu,h}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))}{\partial\lambda} - \frac{\partial Q_{\nu,h}(x,u^*(x,[\cdot]),\widetilde{v}^*(x,[\cdot]),\lambda^*(x))}{\partial\lambda} \right\| < \varepsilon$$

for all $x \in [0,\omega]$, $(\lambda(x),\ \widetilde{v}(x,[t]), u(x,[t])) \in S(\lambda^*(x), \widetilde{\delta}(x)) \times S(\widetilde{v}^*(x,[t]), \widetilde{\delta}(x)) \times S(u^*(x,[t]), \widetilde{\delta}(x))$. We note that if $\{\lambda^*(x), \widetilde{v}^*(x,[t]), u^*(x,[t])\}$ is a solution to problem (11)–(15), then

$$Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)) = 0$$

with any $\nu \in \mathbb{N}$.

Let $(\widehat{\lambda}(x), \widehat{\widetilde{v}}(x,[t]), \widehat{u}(x,[t])) \in S(\lambda^*(x), \widetilde{\delta}(x)) \times S(\widetilde{v}^*(x,[t]), \widetilde{\delta}(x)) \times S(u^*(x,[t]), \widetilde{\delta}(x))$ be another solution to problem (11)–(15).

Since $Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)) = 0$ and $Q_{\nu,h}(x, \widehat{u}(x,[\cdot]), \widehat{\widetilde{v}}(x,[\cdot]), \widehat{\lambda}(x)) = 0$, from inequalities

$$\lambda^*(x) = \lambda^*(x) - \left[\frac{\partial Q_{\nu,h}(x,u^*(x,[\cdot]),\widetilde{v}^*(x,[\cdot]),\lambda^*(x))}{\partial\lambda}\right]^{-1} Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)),$$

$$\widehat{\lambda}(x) = \widehat{\lambda}(x) - \left[\frac{\partial Q_{\nu,h}(x,u^*(x,[\cdot]),\widetilde{v}^*(x,[\cdot]),\lambda^*(x))}{\partial\lambda}\right]^{-1} Q_{\nu,h}(x, \widehat{u}(x,[\cdot]), \widehat{\widetilde{v}}(x,[\cdot]), \widehat{\lambda}(x))$$

it follows that

$$\lambda^*(x) - \widehat{\lambda}(x) = \lambda^*(x) - \widehat{\lambda}(x) - \left[\frac{\partial Q_{\nu,h}(x,u^*(x,[\cdot]),\widetilde{v}^*(x,[\cdot]),\lambda^*(x))}{\partial\lambda}\right]^{-1} \\ \times \big[Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)) - Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \widehat{\lambda}(x)) \\ + Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \widehat{\lambda}(x)) - Q_{\nu,h}(x, \widehat{u}(x,[\cdot]), \widehat{\widetilde{v}}(x,[\cdot]), \widehat{\lambda}(x))\big].$$

Applying the Lagrange formula of finite increments ([13], P. 375) to the difference $Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)) - Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \widehat{\lambda}(x))$, we obtain

$$\begin{aligned}
&\|\lambda^*(x) - \widehat{\lambda}(x)\| \\
&\quad \le \frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \|Q_{\nu,h}(x, u^*(x,[\cdot]), \widetilde{v}^*(x,[\cdot]), \lambda^*(x)) - Q_{\nu,h}(x, \widehat{u}(x,[\cdot]), \widehat{\widetilde{v}}(x,[\cdot]), \widehat{\lambda}(x))\| \\
&\qquad \le \frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \max_{r=\overline{1,N}} \left\{ \frac{(L_1(x)h)^\nu}{\nu!} \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \right. \\
&\qquad\qquad \left. + \sum_{j=0}^{\nu} \frac{(L_1(x)h)^j}{j!} L_2(x)h \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - \widehat{u}_r(x,t)\| \right\}. \qquad (26)
\end{aligned}$$

Since

$$\begin{aligned}
&\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \le \frac{(L_1(x)h)^\nu}{\nu!} \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \\
&+ \sum_{j=1}^{\nu} \frac{(L_1(x)h)^j}{j!} \max_{r=\overline{1,N}} \|\lambda_r^*(x) - \widehat{\lambda}_r(x)\| + \sum_{j=0}^{\nu} \frac{(L_1(x)h)^j}{j!} L_2(x)h \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - \widehat{u}_r(x,t)\|,
\end{aligned} \qquad (27)$$

after substitution of (27) in the right-hand side of (26), we get

$$\begin{aligned}
&\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \\
&\le q_\nu(x,h,\varepsilon) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| + d_\nu(x,h,\varepsilon) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\|,
\end{aligned}$$

where $q_\nu(x,h,\varepsilon) = \frac{(L_1(x)h)^\nu}{\nu!}\Big[1 + \frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \sum\limits_{j=1}^{\nu} \frac{(L_1(x)h)^j}{j!}\Big]$,

$$d_\nu(x,h,\varepsilon) = \sum_{j=0}^{\nu} \frac{(L_1(x)h)^j}{j!} L_2(x)h \left[1 + \frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \sum_{j=1}^{\nu} \frac{(L_1(x)h)^j}{j!}\right].$$

Therefore,

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \le \frac{d_\nu(x,h,\varepsilon)}{1-q_\nu(x,h,\varepsilon)} \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - \widehat{u}_r(x,t)\|, \qquad (28)$$

$$\begin{aligned}
\max_{r=\overline{1,N}} \|\lambda_r^*(x) - \widehat{\lambda}_r(x)\| \le \Bigg[&\frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \frac{(L_1(x)h)^\nu}{\nu!} \frac{d_\nu(x,h,\varepsilon)}{1-q_\nu(x,h,\varepsilon)} \\
&+ \sum_{j=0}^{\nu} \frac{(L_1(x)h)^j}{j!} L_2(x)h\Bigg] \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - \widehat{u}_r(x,t)\|, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\Delta^*(x) &= \max_{r=\overline{1,N}} \|\lambda_r^*(x) - \widehat{\lambda}_r(x)\| + \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{v}_r^*(x,t) - \widehat{\widetilde{v}}_r(x,t)\| \\
&\le \Bigg(\left[1 + \frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)} \frac{(L_1(x)h)^\nu}{\nu!}\right] \frac{d_\nu(x,h,\varepsilon)}{1-q_\nu(x,h,\varepsilon)} \\
&\qquad + \sum_{j=0}^{\nu} \frac{(L_1(x)h)^j}{j!} L_2(x)h\Bigg) \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|u_r^*(x,t) - \widehat{u}_r(x,t)\|. \qquad (30)
\end{aligned}$$

Using functional correlation (15) and inequality (30), we obtain

$$\Delta^*(x) \le \left(\left[1+\frac{\gamma_\nu(x,h)}{1-\varepsilon\gamma_\nu(x,h)}\frac{(L_1(x)h)^\nu}{\nu!}\right]\frac{d_\nu(x,h,\varepsilon)}{1-q_\nu(x,h,\varepsilon)}+\sum_{j=0}^{\nu}\frac{(L_1(x)h)^j}{j!}L_2(x)h\right)\int_0^x \Delta^*(\xi)d\xi.$$

Hence it follows that $u_r^*(x,t)=\widehat{u}_r(x,t)$, $r=\overline{1,N}$. Due to (28), (29), equalities take place $\lambda_r^*(x)=\widehat{\lambda}_r(x), \widetilde{v}_r^*(x,t)=\widehat{\widetilde{v}}_r(x,t)$, $r=\overline{1,N}$. □

We define functions

$$v^{(k)}(x,t)=\begin{cases}\lambda_r^{(k)}(x)+\widetilde{v}_r^{(k)}(x,t) & \text{with } (x,t)\in\Omega_r,\ r=\overline{1,N},\\ \lambda_N^{(k)}(x)+\lim\limits_{t\to T-0}\widetilde{v}_N^{(k)}(x,t) & \text{with } t=Nh,\end{cases}$$

$$u^{(k)}(x,t)=\psi(t)+\int_0^x v^{(k)}(\xi,t)d\xi,\ \ k=1,2,\dots,$$

denote by $S_1(u^{(0)}(x,[t]),\rho(x),\theta(x))$ the set of piecewise-continuously differentiable with respect to x, t functions $u:\overline{\Omega}\to R^n$, satisfying inequalities

$$\|u(x,t)-u^{(0)}(x,t)\|<\omega[\rho(x)+\theta(x)],\quad \|u(x,T)-u^{(0)}(x,T)\|<\omega[\rho(x)+\theta(x)],$$
$$\|u_x(x,t)-u_x^{(0)}(x,t)\|<\rho(x)+\theta(x),\qquad \|u_x(x,T)-u_x^{(0)}(x,T)\|<\rho(x)+\theta(x),$$
$$\|u_t(x,t)-u_t^{(0)}(x,t)\|<\omega[\rho(x)+\theta(x)],\ \ \|u_t(x,T)-u_t^{(0)}(x,T)\|<\omega[\rho(x)+\theta(x)].$$

Due to the equivalence of problems (1)–(3) and (11)–(15), the next theorem follows from Theorem 1.

Theorem 2. *If conditions of Theorem 1 are fulfilled, then the sequence of functions $\{u^{(k)}(x,t)\}$, $k=1,2,\dots$, is contained in $S_1(u^{(0)}(x,[t]),\rho(x),\theta(x))$, converges to $u^*(x,t)$, which is a solution to problem (1)–(3) in $S_1(u^{(0)}(x,[t]),\rho(x),\theta(x))$, and the inequality is fulfilled*

$$\|u^*(x,t)-u^{(k)}(x,t)\|\le\int_0^x c(\xi)\sum_{j=k-1}^{\infty}\frac{1}{j!}\left(\int_0^\xi c(\xi_1)d\xi_1\right)^j\int_0^\xi[c_0(\xi_1)c_1(\xi_1)$$
$$+c_0(\xi_1)+1]\gamma_\nu(\xi_1,h)\|Q_{\nu,h}(\xi_1,u^{(0)}(\xi_1,[\cdot]),\widetilde{v}^{(0)}(\xi_1,[\cdot]),\lambda^{(0)}(\xi_1))\|d\xi_1 d\xi,\quad (x,t)\in\overline{\Omega}.$$

In addition, any solution to problem (1)–(3) in $S_1(u^{(0)}(x,[t]),\rho(x),\theta(x))$ is isolated.

Example. Let us consider on $[0,\ 0.5]\times[0,\ 0.5]$ the semi-periodic boundary-value problem

$$\frac{\partial^2 u_1}{\partial x\partial t}=\frac{1}{2}\frac{\partial u_2}{\partial x}+f_1(x,t), \tag{31}$$

$$\frac{\partial^2 u_2}{\partial x\partial t}=\frac{1}{3}\left(\frac{\partial u_1}{\partial x}-1\right)\frac{\partial u_1}{\partial x}+u_1(x,t)+f_2(x,t), \tag{32}$$

$$u_1(x,0)=u_1(x,0.5),\quad u_2(x,0)=u_2(x,0.5), \tag{33}$$

$$u_1(0,t)=0,\quad u_2(0,t)=0, \tag{34}$$

$$f_1(x,t)=0.2\pi\cos 2\pi t-0.1t(t-0.5),$$

$$f_2(x,t)=0.4t-0.1-\frac{1}{3}(0.1\sin 2\pi t-1)0.1\sin 2\pi t-0.1x\sin 2\pi t.$$

We introduce a new unknown function $v_i(x,t)=\frac{\partial u_i(x,t)}{\partial x}$, $i=1,2$, and write problem (31)–(34) in the form

$$\frac{\partial v_1}{\partial t}=\frac{1}{2}v_2+f_1(x,t),\quad \frac{\partial v_2}{\partial t}=\frac{1}{3}(v_1-1)v_1+u_1(x,t)+f_2(x,t),$$
$$v_1(x,0)=v_1(x,0.5),\quad v_2(x,0)=v_2(x,0.5),$$

$$u_1(x,t)=\int_0^x v_1(x,\xi)d\xi,\quad u_2(x,t)=\int_0^x v_2(x,\xi)d\xi.$$

Taking $h=0.5$, $\nu=1$, we denote $\lambda_i(x)\widehat{=}v_i(x,0)$, $i=1,2$. We replace $\widetilde{v}_i(x,t)=v_i(x,t)-\lambda_i(x)$, $i=1,2$, $t\in[0,\,0.5]$, and obtain the problem with a parameter

$$\frac{\partial\widetilde{v}_1}{\partial t}=\frac{1}{2}\widetilde{v}_2+\frac{1}{2}\lambda_2(x)+f_1(x,t),$$

$$\frac{\partial\widetilde{v}_2}{\partial t}=\frac{1}{3}\big(\widetilde{v}_1+\lambda_1(x)-1\big)\big(\widetilde{v}_1+\lambda_1(x)\big)+u_1(x,t)+f_2(x,t),$$

$$\widetilde{v}_1(x,0)=0,\quad \widetilde{v}_2(x,0)=0,$$

$$u_1(x,t)=\int_0^x(\widetilde{v}_1(\xi,t)+\lambda_1(\xi))d\xi,\quad u_2(x,t)=\int_0^x(\widetilde{v}_2(\xi,t)+\lambda_2(\xi))d\xi, \tag{35}$$

$$\lim_{t\to0.5-0}\widetilde{v}_1(x,t)=0,\qquad \lim_{t\to0.5-0}\widetilde{v}_2(x,t)=0. \tag{36}$$

For definition of the triple $(\lambda(x)=(\lambda_1(x),\lambda_2(x))'$, $\widetilde{v}(x,t)=(\widetilde{v}_1(x,[t]),\widetilde{v}_2(x,[t]))'$, $u(x,t)=(u_1(x,[t]),$ $u_2(x,[t]))')$ we have the system of equations with respect to the introduced functional parameter $\lambda(x)\in C([0,\omega],R^2)$

$$Q_{1;0.5}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))=0, \tag{37}$$

where

$$Q_{1;0,5}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))$$
$$\equiv\begin{pmatrix}\displaystyle\int_0^{0.5}\Big(\frac{1}{2}\widetilde{v}_2(x,\tau)+\frac{1}{2}\lambda_2(x)+f_1(x,\tau)\Big)d\tau\\ \displaystyle\int_0^{0.5}\Big(\frac{1}{3}\big(\widetilde{v}_1(x,\tau)+\lambda_1(x)-1\big)\big(\widetilde{v}_1(x,\tau)+\lambda_1(x)\big)+u_1(x,t)+f_2(x,\tau)\Big)d\tau\end{pmatrix},$$

the system of integral equations

$$\widetilde{v}_1(x,t)=\int_0^t\Big(\frac{1}{2}\widetilde{v}_2(x,\tau)+\frac{1}{2}\lambda_2(x)+f_1(x,\tau)\Big)d\tau,$$

$$\widetilde{v}_2(x,t)=\int_0^t\Big(\frac{1}{3}\big(\widetilde{v}_1(x,\tau)+\lambda_1(x)-1\big)\big(\widetilde{v}_1(x,\tau)+\lambda_1(x)\big)+u_1(x,t)+f_2(x,\tau)\Big)d\tau$$

and functional correlations (35). The application of parameterization method begins from the choice of an initial approximation by the functional parameter $\lambda^{(0)}(x)$. We find an initial approximation by the functional parameter $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\,\lambda_2^{(0)}(x))'\in C([0,\omega],R^2)$ from Eq. (37) with $\widetilde{v}_1(x,t)=0$, $\widetilde{v}_2(x,t)=0$, $u_1(x,t)=\int\limits_0^x\lambda_1^{(0)}(\xi)d\xi$, $u_2(x,t)=\int\limits_0^x\lambda_2^{(0)}(\xi)d\xi$. Then $\|\lambda_1^{(0)}(x)\|\le0.023$, $\|\lambda_2^{(0)}(x)\|\le$ 0.00836, $\max\limits_{t\in[0;0,5]}\|\widetilde{v}_1^{(0)}(x,t)\|<0.0071085$, $\max\limits_{t\in[0;0,5]}\|\widetilde{v}_2^{(0)}(x,t)\|<0.0117142$, $\max\limits_{t\in[0;0,5]}\|u_1^{(0)}(x,t)\|<$ 0.0150543, $\max\limits_{t\in[0;0,5]}\|u_2^{(0)}(x,t)\|<0.0100371$, $\|Q_{1;0.5}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))\|<0.0117142$; $L_1(x,t)=$ $\frac{1}{2}$, $L_2(x,t)=1$. We verify the fulfillment of conditions of Theorem 1:

1) $\|\big[\frac{\partial Q_{1;0.5}(x,u(x,[\cdot]),\widetilde{v}(x,[\cdot]),\lambda(x))}{\partial\lambda}\big]^{-1}\|\le\gamma_1(x,0.5)=3$,

2) $q_1(x,0.5)=\frac{1}{4}\big(1+\frac{3}{4}\big)=\frac{7}{16}<1$,

3) $2.285\cdot\gamma_1(x,0.5)\|Q_{1,0.5}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<0.160602<0.2$,

4) $1.7213\cdot\gamma_1(x,0.5)\|Q_{1,0.5}(x,u^{(0)}(x,[\cdot]),\widetilde{v}^{(0)}(x,[\cdot]),\lambda^{(0)}(x))\|<0.1209819<0.2$.

Then the sequence of functions $u^{(k)}(x,[t])$, $k=1,2,\dots$, defined by the algorithm is contained in the set $S(u^{(0)}(x,[t]),0.2)$ and converges to an isolated solution to the considered problem in $S(u^{(0)}(x,[t]),0.2)$.

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