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Semifield Planes of Odd Order that Admit a Subgroup of Autotopisms Isomorphic to A_4

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Abstract—We develop an approach to constructing and classification of semifield projective planes with the use of a linear space and a spread set. We construct a matrix representation of the spread set of a semifield plane of odd order that admits a Baer involution in the translation complement or a subgroup of autotopisms isomorphic to the alternating group A_4. We give examples of semifield planes of order 81 satisfying the above indicated condition.

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INTRODUCTION

There is a method of constructing semifield planes, as well as other translation planes, based on a vector space of even dimension and a spread set, i.e., a family of linear transformations which gives a consistent splitting. A matrix representation of the spread set determines geometric properties of the plane and algebraic properties of the coordinatizating semifield. Of interest are the problem of investigation of the group of automorphisms (collineations) for a known representation of the spread set and the inverse problem of constructing a projective plane with definite properties of the automorphism group. In particular, in 90's, a series of papers were published ([1–4] et al.) devoted to the construction and investigation of rank two semifield planes admitting a Baer involution. In the construction, a four-dimensional vector space and a spread set represented by (2×2)-matrices over a field of order p^n were used. In this case, the functions defining the spread set of a plane are polynomials of degree $\le p^{n-1}$.

In the paper, we obtain matrix representations of the spread sets of a semifield plane of an arbitrary odd order p^N admitting a Baer collineation of order two and of a semifield plane admitting a group of autotopisms isomorphic to the alternating group A_4. The plane is presented with the use of a linear space over a field of prime order which allows ones to pass to linear functions and simplify significantly all reasonings and calculations.

In what follows we will consider the matrices of spread sets

$$\theta(V,U)=\begin{pmatrix} m(U) & f(V)\\ V & U\end{pmatrix}, \tag{1}$$

$$\theta(V_1,U_1,V_2,U_2)=\begin{pmatrix} \mu(J^{-1}U_2J) & \nu(J^{-1}V_2) & \psi(J^{-1}U_1) & \varphi(J^{-1}V_1)J^{-1}\\ \psi(JV_2) & \mu(JU_2J^{-1}) & \nu(JV_1) & \varphi(JU_1)J^{-1}\\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2)\\ V_1 & U_1 & V_2 & U_2\end{pmatrix}. \tag{2}$$

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The elements of the matrices here and elsewhere are square blocks of equal order, E is the identity matrix. To write down involutions in a group of autotopisms we use the notation

$$\tau = \begin{pmatrix} -E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}, \tag{3}$$

$$\sigma = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{4}$$

where $L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$, the order of blocks will be indicated in the text.

The main results are as follows.

Theorem 1. *Let π be a semifield plane of order p^N, $p > 2$ be a prime number, which admits a Baer involution τ in the translation complement. Then $N = 2n$ and π can be given by a $4n$-dimensional vector space over $\mathbb{Z}_p$ so that τ is defined by $(4n \times 4n)$-matrix (3). The plane π has spread set $R \subset GL_{2n}(p) \cup \{0\}$ formed by matrices of the form (1), where $\{U\} = K$ and $\{V\} = Q$ are s-spread sets in $GL_n(p) \cup \{0\}$, K is the spread set of the Baer subplane π_0 fixed by the involution τ, m and f are injective linear mappings from K and Q, respectively, to $GL_n(p) \cup \{0\}$ such that $m(E) = E$, $f(E) \neq E$.*

Theorem 2. *Let π be a semifield plane of odd order p^N, p be a prime number, whose group of autotopisms contains a subgroup $H \simeq A_4$. Then $N = 4n$ and π can be given by an $8n$-dimensional vector space over $\mathbb{Z}_p$ so that the spread set $R \subset GL_{4n}(p) \cup \{0\}$ is formed by $(4n \times 4n)$-matrices of the form (2), where $J^3 = E$; $\{V_1\} = Q_1$, $\{U_1\} = K_1$, $\{V_2\} = Q_2$, $\{U_2\} = K_2$ are s-spread sets in $GL_n(p) \cup \{0\}$; $J^{-1}K_2J = K_2$, $JK_1 = Q_2$, $JQ_1 = K_1$, $JQ_2 = Q_1$; ν, ψ, μ, and φ are injective linear mappings from K_1, Q_1, K_2, and Q_2, respectively, to $GL_n(p) \cup \{0\}$ such that*

$$\mu(E) = E, \quad \nu(E) = E, \quad \varphi(E) \neq E, \quad \psi(E) \neq E.$$

To illustrate the obtained results, we construct semifield planes of order 81. The results were announced partially in [5, 6].

1. BASIC DEFINITIONS AND NOTATION

Below we give some basic definitions and notation according to [7, 8].

For points and lines of a finite projective plane, a coordinate system can be introduced with the use of elements of some *coordinatizating set*. The properties of the incidence relation in a projective plane allow one to introduce on the coordinatizating set operations of multiplication and addition. Algebraic properties of the coordinatizating set are closely related to geometric properties of the corresponding projective plane. In particular, a classical or *Desargues* projective plane is coordinatized by a field, and a translation plane is coordinatized by a quasifield. The coordinatizating set of a semifield plane is a division ring, or a semifield.

Let π be a translation plane of order q^n ($q = p^k$, p is a prime number), and let W be an n-dimensional linear space over the field $GF(q)$. Then affine points of π can be represented ([7], P. 160) by vectors (x, y), $x, y \in W$, and affine lines by cosets of the subgroups

$$V_i = \{(x, x\theta_i) \mid x \in W\}, \;\; i = 1, 2, \ldots, q^n, \;\; V_0 = \{(0, y) \mid y \in W\}.$$

Here θ_i are $(n \times n)$-matrices with elements from $GF(q)$ which form the *spread set* R of π ([8]).

Definition 1. The set R consisting of q^n $n \times n$-matrices over $GF(q)$, $R = \{\theta_i \mid i = 1, 2, \dots, q^n\}$, is called a *spread set* if the following conditions hold:

1) R contains the zero and the identity matrices,

2) $\det(\theta_i - \theta_j) \neq 0$ for all $i \neq j$.

Thus, we can write down $R = \{\theta(w) \mid w \in W\}$, where $\theta : W \to GL(W) \cup \{0\}$ and $\theta(0) = 0$.

Let $*$ be the operation on W defined by $x * y = x \cdot \theta(y)$, $x, y \in W$. Then $\langle W, +, * \rangle$ is a quasifield.

Definition 2. A spread set $R \subset GL(W) \cup \{0\}$ is called an s-*spread* set if it is closed with respect to the addition.

It was proved in [8] that if R is an s-spread set, then $\langle W, +, * \rangle$ is a semifield.

Definition 3. The following subsets

$$W_r = \{x \in W | (ab)x = a(bx)\ \forall a, b \in W\},$$
$$W_m = \{x \in W | (ax)b = a(xb)\ \forall a, b \in W\},$$
$$W_l = \{x \in W | (xa)b = x(ab)\ \forall a, b \in W\}$$

are called, respectively, the right, the middle, and the left nuclei of a semifield W.

These sets are subfields in W, and it is known that a semifield plane can be considered as a linear space over any of the nuclei of the semifield ([7], P. 169). As a rule, it is convenient to use the left nucleus W_l.

Let $[\infty]$ be a translation line of a plane π and (∞) its translation point. The subgroup Λ formed by collineations that fix the triangle with vertices P_1, $P_2 = (\infty)$, $P_3 \in [\infty]$ and sides l_1, $l_2 = [\infty]$, $l_3 \ni (\infty)$ is called the *autotopism group*. By virtue of the $((\infty), (\infty))$-transitivity and the $([\infty], [\infty])$-transitivity of a semifield plane, one can assume without loss of generality that $P_1 = (0, 0)$, $P_3 = (0)$, $l_1 = [0, 0]$, $l_3 = [0]$ (for notation, see [7]).

There is the conjecture ([7], P. 178) on solvability of the full group of collineations for any semifield non-Desargues plane of finite order (see also [9], Question 11.76). By now, this conjecture has been proved only for some classes of semifield planes ([2, 4, 10] et al.). As was proved in ([7], P. 174), the conjecture on solvability of the full group of automorphisms for a non-Desargues semifield plane is reduced to solvability of the autotopism group. Further, if the autotopism group Λ has odd order, it is solvable by the Feit–Thompson theorem. Therefore, discussing the question on solvability, one should consider only semifield planes admitting autotopisms of order two.

According to a classical result on projective planes ([7], P. 91), a collineation of order two is either a perspectivity (a central collineation) or a Baer collineation.

Definition 4. A collineation of a projective plane is called central if it fixes pointwise a line (the axis), a point (the center) and all lines passing through the center (not pointwise). If the center is incident to the axis, the collineation is called an elation, otherwise it is called a homology.

Definition 5. A collineation of a projective plane of order m is called a Baer collineation if it fixes pointwise a maximal subplane of order $\sqrt{m}$ (the Baer subplane).

It was proved [4] that central collineations form in the autotopism group the following cyclic subgroups:

1) $H_r \simeq W_r^*$, the group of homologies with axis $[0, 0]$ and center (∞),

2) $H_l \simeq W_l^*$, the group of homologies with axis $[\infty]$ and center $(0, 0)$,

3) $H_m \simeq W_m^*$, the group of homologies with axis $[0]$ and center (0).

The matrices that form the indicated subgroups of homologies are as follows:

$$H_r = \left\{ \begin{pmatrix} E & 0 \\ 0 & A \end{pmatrix} \right\}, \quad H_m = \left\{ \begin{pmatrix} B & 0 \\ 0 & E \end{pmatrix} \right\}, \quad H_r = \left\{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right\}.$$

The matrices A, B, and C form in $GL(W)$ multiplicative subgroups isomorphic to W_r^*, W_m^*, and W_l^*, respectively.

Lemma 1 ([7], P. 181). *Let D be a finite semifield. If the plane $P(D)$ has even order and contains no Baer subplanes or has odd order and the dimension of D over at least one of the nuclei is odd, then the autotopism group of D is solvable.*

Thus, studying the question on solvability of the full group of collineations of a non-Desargues semifield plane of even order, we may consider only planes whose autotopism group contains a Baer involution.

2. BAER INVOLUTION IN THE TRANSLATION COMPLEMENT

The structure of the spread set of a semifield plane of even order 2^N admitting a Baer involution in the translation complement was established in [11]. The set of affine points of a semifield plane was considered as a linear space over the field of order two, this made it possible to write down a Baer involution as a linear transformation and use only linear functions in the writing of matrices of the spread set. We will use the same approach in the study of semifield planes of odd order.

Proof of Theorem 1. Let π be a semifield plane of order p^N, $p > 2$ be a prime number, admitting a Baer involution τ in the translation complement. Since τ fixes pointwise a subplane π_0 of maximal order $|\pi_0| = \sqrt{|\pi|}$, therefore $N = 2n$ is an even number.

Consider the set of affine points of π as a $4n$-dimensional linear space over the field $\mathbb{Z}_p$:

$$W \times W = \{(x_1, x_2, \dots, x_{2n}, y_1, y_2, \dots, y_{2n}) | x_i, y_i \in \mathbb{Z}_p\},$$
$$W = \{(x_1, x_2, \dots, x_{2n}) | x_i \in \mathbb{Z}_p\}.$$

Then τ is a linear transformation of the space $W \times W$ which fixes exactly $2n$ one-dimensional subspaces of $W \times W$. Thus, the Jordan normal form of the matrix of τ is formed by $2n$ Jordan cells of the form $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Obviously, one can choose a basis in $W \times W$ with respect to which the Baer involution τ is given by (3).

The matrix of the spread set of a semifield plane π is uniquely determined by any its row, for example, the last one. The other elements of the matrix are additive functions of the elements of this row. Since $\mathbb{Z}_p$ has prime order, these functions are linear:

$$\theta(u_{2n,1}, \dots, u_{2n,2n}) = \begin{pmatrix} u_{11} & \dots & u_{1,2n} \\ u_{21} & \dots & u_{2,2n} \\ \dots & \dots & \dots \\ u_{2n,1} & \dots & u_{2n,2n} \end{pmatrix},$$

where $u_{ij} = q_{ij1}u_{2n,1} + q_{ij2}u_{2n,2} + \dots + q_{ij,2n}u_{2n,2n}$ for $i = 1, 2, \dots, 2n-1$, $j = 1, 2, \dots, 2n$.

Let us rewrite the matrix in the form

$$\theta(V, U) = \begin{pmatrix} m(U) + h(V) & d(U) + f(V) \\ V + s(U) & U + w(V) \end{pmatrix},$$

subdividing it into blocks of order n. The summands V, $h(V)$, $f(V)$, and $w(V)$ contain linear functions depending only on $u_{2n,1}, \dots, u_{2n,n}$. The other summands are defined by the choice of elements $u_{2n,2n+1}, \dots, u_{2n,2n}$. Then, obviously, the last rows of the matrices $s(U)$ and $w(V)$ consist only of zeros.

Let us find out the conditions on the indicated functions under which a plane with spread set R admits a Baer involution τ of the form (3). For brevity, we denote $T = \left(\begin{smallmatrix} -E & 0 \\ 0 & E \end{smallmatrix}\right)$, then $\tau = \left(\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right)$.

Since τ is a collineation, for any matrix $\theta(V, U) \in R$, the product $T^{-1}\theta(V, U)T$ also belongs to R ([12, 13]). We have

$$T^{-1}\theta(V, U)T = \begin{pmatrix} m(U) + h(V) & -d(U) - f(V) \\ -V - s(U) & U + w(V) \end{pmatrix} = \theta(-V, U).$$

Hence $s(U) \equiv 0$, $d(U) \equiv 0$, $h(V) \equiv 0$, and $w(V) \equiv 0$,

$$\theta(V,U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}.$$

Since R is the spread set of a semifield plane, it follows from definition that the sets $Q = \{V\}$ and $K = \{U\}$ are closed with respect to the addition and contain the zero matrix, all nonzero matrices are nondegenerate.

In addition, it is obvious that $E \in K$, $m(E) = E$. What is more, we can assume that the set Q also contains the identity matrix. In fact, let $V_0 \in K$, $V_0 \neq 0$. Let us choose a new basis in W taking the transition matrix $A = \left(\begin{smallmatrix} V_0 & 0 \\ 0 & E \end{smallmatrix}\right)$. Then

$$A\theta(V_0, 0)A^{-1} = \begin{pmatrix} V_0 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & f(V_0) \\ V_0 & 0 \end{pmatrix} \begin{pmatrix} V_0^{-1} & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & V_0 f(V_0) \\ E & 0 \end{pmatrix},$$

and, in addition, $ATA^{-1} = T$ and $A\theta(0,E)A^{-1} = \theta(0,E)$.

The conditions that $m(U) \in GL_n(p)$ for all $0 \neq U \in K$ and $f(V) \in GL_n(p)$ for all $0 \neq V \in Q$ follow from the fact that the matrix $\theta(V,U) \neq 0$ is nondegenerate because

$$\det\theta(0,U) = \det m(U) \cdot \det U \quad \text{and} \quad \det\theta(V,0) = \det f(V) \cdot \det V.$$

Examining the set of all affine points of π of the form

$$(0,x,0,y) \ \text{ for } \ x = (x_{n+1}, \ldots, x_{2n}), \ \ y = (y_{n+1}, \ldots, y_{2n}), \ \ x_i, y_i \in \mathbb{Z}_p,$$

we see that $(0,x,0,y)^\tau = (0,x,0,y)$. Therefore, such points form a Baer subplane π_0 fixed by the involution τ. Since $(0,x)\theta(0,U) = (0,xU)$, it follows that $K = \{U\}$ is the spread set of π_0. □

Lemma 2. *Let the sets Q and K in the assumptions of Theorem 1 be fields of order p^n in $GL_n(p) \cup \{0\}$. Then one can assume without loss of generality that $Q = K$.*

Proof. Let a matrix D be a generating element of the multiplicative group K^*. Since the fields Q and K are conjugate in $GL_n(p)$, there exists a matrix $P \in GL_n(p)$ such that $C = PDP^{-1}$ is a generating element of the multiplicative group Q^*. Consider the following matrices of the spread set

$$\theta(C,0) = \begin{pmatrix} 0 & f(C) \\ C & 0 \end{pmatrix}, \quad \theta(0,D) = \begin{pmatrix} m(D) & 0 \\ 0 & D \end{pmatrix}$$

and change the basis in W using the transition matrix

$$M = \begin{pmatrix} D^{-1}PDP^{-1} & 0 \\ 0 & E \end{pmatrix},$$

which does not change the form of the Baer involution τ. Then

$$M\theta(0,D)M^{-1} = \begin{pmatrix} D^{-1}PDP^{-1}m(D)PD^{-1}P^{-1}D & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} \overline{m}(D) & 0 \\ 0 & D \end{pmatrix},$$

$$M\theta(C,0)M^{-1} = \begin{pmatrix} 0 & D^{-1}PDP^{-1}f(C) \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{f}(D) \\ D & 0 \end{pmatrix},$$

where $\overline{m}$, $\overline{f}$ are new linear functions defining the spread set (in terms of the other basis). Thus, up to an isomorphism of planes, we can assume that $Q^* = \langle D \rangle$ and $Q = K$. □

Remark. It is obvious that the s-spread set K is defined up to an isomorphism of the corresponding subplane π_0. To prove this fact, it suffices to consider a change of basis of the $4n$-dimensional linear space with block-diagonal transition matrix.

3. AUTOTOPISM GROUP ISOMORPHIC TO A_4

The question on existence of a subgroup in the translation complement of a semifield plane isomorphic to A_4 was considered on repeated occasions on the scientific seminar at the Krasnoyarsk university (run by N. D. Podufalov). In particular, a significant progress in the study of odd order planes admitting a large group of Baer collineations has been achieved by I. V. Busarkina [14], who proved that such planes do not admit A_4.

It should be noted that one can easily give a large number of examples of semifield planes of even order admitting A_4 (see the result of the author in [15]). Thus, of special interest are planes of odd order. The absence of a subgroup isomorphic to A_4 in the autotopism group and, generally, in the linear translation complement was proved in [16] for the case of a semifield plane of rank two over a finite field of odd order.

Let us find out under what conditions on the spread set, in the notation of the preceding Section, a semifield plane of odd order admits a subgroup of autotopisms isomorphic to the alternating group A_4. Let $H < \Lambda$, $H = \langle \tau, \sigma \rangle \leftthreetimes \langle \gamma \rangle$, where $\sigma, \gamma \in \Lambda$, $|\sigma| = 2$, $|\gamma| = 3$, $\sigma\tau = \tau\sigma$, $\tau^\gamma = \sigma$.

Since σ is an involution in Λ, it follows that σ is either a homology or a Baer involution. Since τ and σ are conjugate, σ cannot be a homology.

Let $\pi_0 = \mathfrak{F}(\tau)$ be the Baer subplane fixed by τ. Then the Baer involution σ, permutable with τ, induces a collineation σ_0 on the plane π_0 which can also be either a homology of the plane π_0 or a Baer involution. Consider all possible cases.

1. Let σ_0 be a homology with axis $[\infty]_0$ and center $(0,0)_0$ (the lower index 0 means that a point or a plane belongs to the subplane π_0). Then σ_0 is given by the $2n \times 2n$-matrix

$$\sigma_0 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}.$$

We have

$$\sigma = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & -E \end{pmatrix}.$$

Since $\tau\sigma = \sigma\tau$, it follows that $A_2 = B_2 = 0$, $A_1^2 = E$, $B_1^2 = E$, $A_1 \neq -E$, $B_1 \neq -E$ (since σ is not a homology).

Let $\gamma \in \Lambda$ be an element of order three such that $\tau^\gamma = \sigma$, $\sigma^\gamma = \tau\sigma$, $(\tau\sigma)^\gamma = \tau$, $\gamma = \left(\begin{smallmatrix} S & 0 \\ 0 & Z \end{smallmatrix}\right)$. Then $S^{-1}TS = A$, $S^{-1}AS = TA$, where

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & -E \end{pmatrix}$$

(the equalities for the matrices Z and B are similar). In more detail, we have:

$$\begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & -E \end{pmatrix},$$

$$\begin{pmatrix} A_1 & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} -A_1 & 0 \\ 0 & -E \end{pmatrix},$$

hence

$$S_4 = 0, \quad S_1 + S_1 A_1 = 0, \quad S_3 - S_3 A_1 = 0, \quad A_1 S_1 + S_1 A_1 = 0, \quad S_2 + A_1 S_2 = 0.$$

Since $S_4 = 0$, we have $|S_2| \neq 0$ and $A_1 = -E$, $|S_3| \neq 0$ and $A_1 = E$. The contradiction obtained shows that σ_0 is not a homology of π_0 with the axis $[\infty]_0$ and the center $(0, 0)_0$.

2. Let σ_0 be a homology with the center $(\infty)_0$ and the axis $[0, 0]_0$. Then

$$\sigma_0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad \sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & -E \end{pmatrix}.$$

For

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & -E \end{pmatrix},$$

a reasoning similar to that in the preceding case implies $B_1 = E$ and $B_1 = -E$, a contradiction.

3. If σ_0 is a homology with the center $(0)_0$ and the axis $[0]_0$, then

$$\sigma_0 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}, \quad \sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & E \end{pmatrix},$$

which also leads to a contradiction.

Lemma 3. *Let $H < \Lambda$, $H = \langle \tau, \sigma \rangle \leftthreetimes \langle \gamma \rangle$, where $\sigma, \gamma \in \Lambda$, $|\sigma| = 2$, $|\gamma| = 3$, $\sigma\tau = \tau\sigma$, $\tau^\gamma = \sigma$. Then the order of π equals p^{4n}, and, without loss of generality, we can represent the autotopisms τ and σ by $8n \times 8n$-matrices of the form* (3) *and* (4)*, respectively,*

$$\gamma = \begin{pmatrix} 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J \end{pmatrix}, \tag{5}$$

where $I^3 = J^3 = E$.

Proof. As was shown above, σ is a Baer involution of π. From the condition $\tau\sigma = \sigma\tau$, we have

$$\sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix},$$

and the restriction of σ to the subplane $\pi_0 = F(\tau)$, $\sigma_0 = \left(\begin{smallmatrix} A_2 & 0 \\ 0 & B_2 \end{smallmatrix}\right)$, is a Baer involution of π_0. Consequently, the number $\sqrt{|\pi_0|}$ is an integer, i.e., $|\pi| = p^{4n}$. We consider then the linear space $W = W_0 \times W_0$ and apply Theorem 1 to σ_0. As a result, we obtain the following $(4n \times 4n)$-matrix:

$$\sigma_0 = \begin{pmatrix} -E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{pmatrix}.$$

Taking into account that the matrices A_2 and B_2 in the expression of σ are reduced to the form $\left(\begin{smallmatrix} -E & 0 \\ 0 & E \end{smallmatrix}\right)$, we can, passing to a new basis, bring the cells A_1 and B_1 to the same form and write down σ in the form (4).

Let

$$\gamma = \begin{pmatrix} S & 0 \\ 0 & Z \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}.$$

Since $\tau^\gamma = \sigma$, $\sigma^\gamma = \tau\sigma$, we have

$$S^{-1} \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} S = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \quad S^{-1} \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} S = \begin{pmatrix} -L & 0 \\ 0 & L \end{pmatrix}.$$

Similar equalities hold for the matrix Z. Consider only the relations connected with the matrix S,

$$\begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix},$$

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} -L & 0 \\ 0 & L \end{pmatrix},$$

$$-S_1 = S_1L, \ \ -S_2 = S_2L, \ \ S_3 = S_3L, \ \ S_4 = S_4L,$$

$$LS_1 = -S_1L, \ \ LS_2 = S_2L, \ \ LS_3 = -S_3L, \ \ LS_4 = S_4L.$$

Let $S_i = \left(\begin{smallmatrix} S_{i1} & S_{i2} \\ S_{i3} & S_{i4} \end{smallmatrix}\right)$. Then

$$S_{12} = S_{14} = S_{11} = 0, \ \ S_{22} = S_{24} = S_{23} = 0, \ \ S_{31} = S_{33} = S_{34} = 0, \ \ S_{41} = S_{43} = S_{42} = 0,$$

and

$$S = \begin{pmatrix} 0 & 0 & S_{21} & 0 \\ S_{13} & 0 & 0 & 0 \\ 0 & S_{32} & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix},$$

where all blocks S_{ij} are nondegenerate matrices. Computing $S^3 = E$, we obtain the equalities

$$S_{21}S_{32}S_{13} = E, \quad S_{13}S_{21}S_{32} = E, \quad S_{32}S_{13}S_{21} = E, \quad S_{44}^3 = E.$$

After the change of basis with the matrix

$$M = \begin{pmatrix} S_{21}^{-1} & 0 & 0 & 0 \\ 0 & S_{21}^{-1}S_{13}^{-1} & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix},$$

which does not change the matrices of τ and σ, we obtain

$$M \begin{pmatrix} 0 & 0 & S_{21} & 0 \\ S_{13} & 0 & 0 & 0 \\ 0 & S_{32} & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & 0 & E & 0 \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix}.$$

Note that, using a block-diagonal matrix M with $n \times n$ diagonal blocks, we cannot, in the general case, bring the block S_{44} to the form E. But if the polynomial $\lambda^3 - 1$ can be factored over $\mathbb{Z}_p$ into linear factors, S_{44} can be reduced to the Jordan normal form. □

Lemma 4. *The matrix of the spread set of a plane π is of the form*

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix} = \begin{pmatrix} m_1(U_2) & m_2(V_2) & f_1(U_1) & f_2(V_1) \\ m_3(V_2) & m_4(U_2) & f_3(V_1) & f_4(U_1) \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix},$$

where $\{V_1\} = Q_1$, $\{U_1\} = K_1$, $\{V_2\} = Q_2$, $\{U_2\} = K_2$, the functions ν, ψ, μ, φ, m_i, and f_i $(i = 1, \dots, 4)$ are linear mappings from the set of $(n \times n)$-matrices to $GL_n(p) \cup \{0\}$, and

$$m_1(E) = m_4(E) = \mu(E) = E.$$

Proof. By Theorem 1, the spread set $K \subset GL_{2n}(p) \cup \{0\}$ of the Baer subplane π_0 is of the form

$$K = \left\{ U = \theta_0(V_2, U_2) = \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix} \;\middle|\; U_2 \in K_2,\ V_2 \in Q_2 \right\},$$

where K_2 and Q_2 are s-spread sets in $GL_n(p) \cup \{0\}$, μ and φ are injective linear mapping from K_2 and Q_2, respectively, to $GL_n(p) \cup \{0\}$, $\mu(E) = E$, $\varphi(E) \neq E$.

Consider the collineation σ and check the condition

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \theta(V,U) \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \in R \ \ \forall U \in K, \ \ \forall V \in Q.$$

Let $V = 0$, then for any $U \in K$ we have

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} m(U) & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} Lm(U)L & 0 \\ 0 & LUL \end{pmatrix} = \theta(0, LUL),$$

therefore $LUL \in K$ and $m(LUL) = Lm(U)L$. Let

$$m(U) = m(V_2, U_2) = \begin{pmatrix} m_1(V_2,U_2) & m_2(V_2,U_2) \\ m_3(V_2,U_2) & m_4(V_2,U_2) \end{pmatrix},$$

where $V_2 \in Q_2$, $U_2 \in K_2$, then

$$Lm(U)L = \begin{pmatrix} m_1(V_2,U_2) & -m_2(V_2,U_2) \\ -m_3(V_2,U_2) & m_4(V_2,U_2) \end{pmatrix}.$$

Taking into account that $LUL = \left(\begin{smallmatrix} \mu(U_2) & -\varphi(V_2) \\ -V_2 & U_2 \end{smallmatrix}\right)$, for any $U_2 \in K_2$ and $V_2 \in Q_2$, we obtain the equalities

$$m_1(V_2,U_2) = m_1(-V_2,U_2), \quad -m_2(V_2,U_2) = m_2(-V_2,U_2),$$
$$-m_3(V_2,U_2) = m_3(-V_2,U_2), \quad m_4(V_2,U_2) = m_4(-V_2,U_2).$$

Since the functions m_i are additive, it follows that m_1 and m_4 do not depend on V_2, m_2 and m_3 do not depend on U_2. Thus, we can write down

$$m(U) = \begin{pmatrix} m_1(U_2) & m_2(V_2) \\ m_3(V_2) & m_4(U_2) \end{pmatrix}.$$

Let now $U = 0$, then for any $V \in Q$ we have

$$\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} 0 & f(V) \\ V & 0 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} 0 & Lf(V)L \\ LVL & 0 \end{pmatrix} = \theta(LVL, 0).$$

From this condition it follows that the s-spread set Q also defines a semifield plane admitting the Baer involution $\left(\begin{smallmatrix} L & 0 \\ 0 & L \end{smallmatrix}\right)$. Therefore, each matrix V is of the form

$$V = \begin{pmatrix} \nu(U_1) & \psi(V_1) \\ V_1 & U_1 \end{pmatrix},$$

where $\{V_1\} = Q_1$ and $\{U_1\} = K_1$ are s-spread sets in $GL_n(p) \cup \{0\}$, $\nu(E) = E$, $\psi(E) \neq E$. Computing $Lf(V)L = f(LVL)$, we obtain relations for the functions f_i, which are similar to given above. Hence

$$f(V) = \begin{pmatrix} f_1(U_1) & f_2(V_1) \\ f_3(V_1) & f_4(U_1) \end{pmatrix}. \qquad \square$$

Lemma 5. *Let σ_0 be the restriction of σ to π_0, and let $\pi_1 = \mathfrak{F}(\sigma_0)$ be the Baer subplane in π_0 fixed by the involution σ_0. Then $\left(\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right)$ is an autotopism of π_1, and $I = J$.*

Proof. Let us write down the collineation $\gamma = \left(\begin{smallmatrix} S & 0 \\ 0 & Z \end{smallmatrix}\right)$ in the form (5) using the following notation for blocks:

$$E_1 = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}, \quad E_I = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad E_J = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}.$$

Then

$$S = \begin{pmatrix} E_3 & E_1 \\ E_2 & E_I \end{pmatrix}, \qquad Z = \begin{pmatrix} E_3 & E_1 \\ E_2 & E_J \end{pmatrix}, \qquad S^{-1} = S^2 = \begin{pmatrix} E_2 & E_3 \\ E_1 & E_I^2 \end{pmatrix}.$$

Since γ is a collineation, it follows that for each matrix $\theta(V, U)$ from the spread set R of π the product $S^{-1}\theta(V, U)Z$ also belongs to R. In particular, for $V = 0$ and $U = E$, we have

$$S^{-1}\theta(0, E)Z = S^{-1}Z = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & I^2J \end{pmatrix} \in R,$$

whence $S^{-1}Z = E$, $I^2J = I^{-1}J = E$, $I = J$, $Z = S$, $\gamma = \left(\begin{smallmatrix} S & 0 \\ 0 & S \end{smallmatrix}\right)$.

Further, for $V = 0$ and arbitrary $U \in K$, we obtain

$$S^{-1}\theta(0, U)S = \begin{pmatrix} E_2 & E_3 \\ E_1 & E_J^2 \end{pmatrix} \begin{pmatrix} m(U) & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} E_3 & E_1 \\ E_2 & E_J \end{pmatrix}$$
$$= \begin{pmatrix} E_2 m(U) E_3 + E_3 U E_2 & E_2 m(U) E_1 + E_3 U E_J \\ E_1 m(U) E_3 + E_J^2 U E_2 & E_1 m(U) E_1 + E_J^2 U E_J \end{pmatrix} = \theta(\overline{V}, \overline{U})$$

for some $\overline{V} \in Q$, $\overline{U} \in K$. Taking into account the preceding lemma, we can write down the matrices $U \in K$ and $m(U)$ in the form

$$U = \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix}, \qquad m(U) = \begin{pmatrix} m_1(U_2) & m_2(V_2) \\ m_3(V_2) & m_4(U_2) \end{pmatrix}.$$

Then

$$\overline{U} = E_1 m(U) E_1 + E_J^2 U E_J$$
$$= \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_1(U_2) & m_2(V_2) \\ m_3(V_2) & m_4(U_2) \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & J^2 \end{pmatrix} \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$$
$$= \begin{pmatrix} m_1(U_2) & 0 \\ 0 & J^2U_2J \end{pmatrix} \in K.$$

Hence $J^{-1}U_2J \in K_2$ for any $U_2 \in K_2$. Therefore, the matrix $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ defines an autotopism of the semifield plane with spread set K_2, i.e., of π_1. ☐

Lemma 6. *In the notation of Lemma 4, we have* $m_1(U_2) = \mu(J^{-1}U_2J)$, $m_4(U_2) = \mu(JU_2J^{-1})$ *for all* $U_2 \in K_2$, $f_1(U_1) = m_3(JU_1)$, $f_4(U_1) = \varphi(JU_1)J$ *for all* $U_1 \in K_1$, *and in addition* $K_1 = J^{-1}Q_2$.

Proof. Based on the proof of the preceding Lemma, consider $\theta(\overline{V},\overline{U}) = S^{-1}\theta(0,U)S$. Since

$$\overline{U} = U(0, J^{-1}U_2J) = \begin{pmatrix} \mu(J^{-1}U_2J) & 0 \\ 0 & J^{-1}U_2J \end{pmatrix},$$

we have $m_1(U_2) = \mu(J^{-1}U_2J)$. Further, we have

$$m(\overline{U}) = E_2m(U)E_3 + E_3UE_2 = \begin{pmatrix} m_4(U_2) & 0 \\ 0 & \mu(U_2) \end{pmatrix} = \begin{pmatrix} m_1(J^{-1}U_2J) & m_2(0) \\ m_3(0) & m_4(J^{-1}U_2J) \end{pmatrix}.$$

Equating the blocks, we obtain for all $U_2 \in K_2$ and $V_2 \in Q_2$

$$m_1(J^{-1}U_2J) = m_4(U_2), \quad m_4(J^{-1}U_2J) = \mu(U_2).$$

Hence $m_4(U_2) = m_1(J^{-1}U_2J) = \mu(J^{-2}U_2J^2) = \mu(JU_2J^{-1})$. Further,

$$\overline{V} = E_1m(U)E_3 + E_J^2UE_2 = \begin{pmatrix} m_2(V_2) & 0 \\ 0 & J^2V_2 \end{pmatrix},$$

$$f(\overline{V}) = E_2m(U)E_1 + E_3UE_J = \begin{pmatrix} m_3(V_2) & 0 \\ 0 & \varphi(V_2)J \end{pmatrix} = \begin{pmatrix} f_1(J^2V_2) & f_2(0) \\ f_3(0) & f_4(J^2V_2) \end{pmatrix}.$$

Since $\overline{V} \in Q$, we have $J^2V_2 \in K_1$ for all $V_2 \in Q_2$, i.e., $K_1 = J^{-1}Q_2$. Since Q contains the identity matrix, the set Q_2 contains the matrix J. Comparing V and $\overline{V}$, we obtain

$$\begin{pmatrix} \nu(J^{-1}V_2) & \psi(0) \\ 0 & J^{-1}V_2 \end{pmatrix} = \begin{pmatrix} m_2(V_2) & 0 \\ 0 & J^{-1}V_2 \end{pmatrix},$$

hence $m_2(V_2) = \nu(J^{-1}V_2) \ \ \forall V_2 \in Q_2$.

Writing down $f(\overline{V})$, we have $\left(\begin{smallmatrix} f_1(J^{-1}V_2) & f_2(0) \\ f_3(0) & f_4(J^{-1}V_2) \end{smallmatrix}\right) = \left(\begin{smallmatrix} m_3(V_2) & 0 \\ 0 & \varphi(V_2)J \end{smallmatrix}\right)$, which implies $f_1(U_1) = m_3(JU_1)$, $f_4(U_1) = \varphi(JU_1)J \ \ \forall U_1 \in K_1$. □

Taking into account Lemma 6, we change the blocks $m_i(U_2)$ and $f_i(V_1)$, $i = 1,2,3,4$, in the statement of Lemma 4 and obtain the following expression for a matrix from the spread set of π:

$$\theta(V,U) = \begin{pmatrix} \mu(J^{-1}U_2J) & \nu(J^{-1}V_2) & m_3(JU_1) & f_2(V_1) \\ m_3(V_2) & \mu(JU_2J^{-1}) & f_3(V_1) & \varphi(JU_1)J \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}.$$

Lemma 7. *$Q_2 = J^{-1}Q_1$, $K_1 = JQ_1$, and, in addition, for all $V_1 \in Q_1$, $U_1 \in K_1$, $V_2 \in Q_2$, we have $f_1(U_1) = \psi(J^{-1}U_1)$, $f_2(V_1) = \varphi(J^{-1}V_1)J^{-1}$, $f_3(V_1) = \nu(JV_1)$, $m_3(V_2) = \psi(JV_2)$.*

Proof. Consider an arbitrary element

$$V = \begin{pmatrix} \nu(U_1) & \psi(V_1) \\ V_1 & U_1 \end{pmatrix}, \ \ V \in Q,$$

and compute the product

$$S^{-1}\theta(V,0)S = \begin{pmatrix} E_3VE_3 + E_2f(V)E_2 & E_3VE_1 + E_2f(V)E_J \\ E_J^2VE_3 + E_1f(V)E_2 & E_J^2VE_1 + E_1f(V)E_J \end{pmatrix} = \theta(\overline{V},\overline{U}).$$

Here

$$\overline{U} = E_J^2VE_1 + E_1f(V)E_J = \begin{pmatrix} 0 & f_2(V_1)J \\ J^2V_1 & 0 \end{pmatrix} \in Q,$$

then $J^2V_1 \in Q_2$ and $\varphi(J^2V_1) = f_2(V_1)J$. Since $V_1 \in Q_1$ is arbitrary, we have $Q_2 = J^{-1}Q_1$, $J \in Q_1$, $f_2(V_1) = \varphi(J^{-1}V_1)J^{-1}$.

Further, $m(\overline{U}) = E_3VE_3 + E_2f(V)E_2$, hence

$$\begin{pmatrix} m_1(0) & m_2(J^2V_1) \\ m_3(J^2V_1) & m_4(0) \end{pmatrix} = \begin{pmatrix} 0 & f_3(V_1) \\ \psi(V_1) & 0 \end{pmatrix},$$

then $m_2(J^2V_1) = f_3(V_1)$ and $m_3(J^2V_1) = \psi(V_1)$, i.e., $m_3(V_2) = \psi(JV_2)$ for any $V_2 \in Q_2$. Consider $\overline{V} = E_J^2VE_3 + E_1f(V)E_2 \in Q$. We obtain

$$\begin{pmatrix} 0 & f_1(U_1) \\ J^2U_1 & 0 \end{pmatrix} = \begin{pmatrix} \nu(0) & \psi(J^2U_1) \\ J^2U_1 & 0 \end{pmatrix}.$$

Then $J^2U_1 \in Q_1$, $J^{-1}K_1 = Q_1$, $f_1(U_1) = \psi(J^2U_1)$. For $f(\overline{V}) = E_3VE_1 + E_2f(V)E_J$, we obtain

$$\begin{pmatrix} 0 & f_4(U_1)J \\ \nu(U_1) & 0 \end{pmatrix} = \begin{pmatrix} f_1(0) & f_2(J^2U_1) \\ f_3(J^2U_1) & f_4(0) \end{pmatrix}.$$

Equating the corresponding elements and using the equalities obtained above, we arrive at the desired result. □

Lemma 7 completes the proof of Theorem 2.

4. EXAMPLES OF SEMIFIELD PLANES OF ORDER 81

Consider a semifield plane π of order 3^4 whose autotopism group contains a Baer involution. Using Theorem 1, we define a plane π by an 8-dimensional linear space over the field $\mathbb{Z}_3$ and a spread set $R \subset GL_4(3) \cup \{0\}$ of the form (1). Then the spread set $K \subset GL_2(3) \cup \{0\}$ of the Baer subplane π_0 is a 2-dimensional linear space

$$K = \{U = u_1D + u_2E \mid u_1, u_2 \in \mathbb{Z}_3\}, \tag{6}$$

$\{D, E\}$ is a basis of K. Obviously, the subplane π_0 is Desargues, K is a field of order 9. By Lemma 2, $Q = K$. Without loss of generality, we can take $D = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The linear functions m and f can be represented in the form

$$m(u_1D + u_2E) = u_1M + u_2E, \qquad f(u_1D + u_2E) = u_1F_1 + u_2F_2$$

for each $U = u_1D + u_2E \in K$. Here $M, F_1, F_2 \in GL_2(3)$, $m(E) = E$, $F_2 = f(E) \neq E$.

If the matrices M, F_1, and F_2 are chosen in such a way that for all $x, y, z, t \in \mathbb{Z}_3$ the matrix

$$\theta(xD + yE, zD + tE) = \begin{pmatrix} zM + tE & xF_1 + yF_2 \\ xD + yE & zD + tE \end{pmatrix}$$

$$= x \begin{pmatrix} 0 & F_1 \\ D & 0 \end{pmatrix} + y \begin{pmatrix} 0 & F_2 \\ E & 0 \end{pmatrix} + z \begin{pmatrix} M & 0 \\ 0 & D \end{pmatrix} + t \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

is either zero (for $x = y = z = t = 0$) or nondegenerate, then the triple of matrices M, F_1, F_2 defines a semifield plane π satisfying the indicated conditions.

With the use of computer, we have obtained 106 collections of matrices M, F_1, F_2, i.e., constructed 106 semifield planes of order 81 admitting a Baer involution in the translational complement. Considering basis transformations in the 8-dimensional linear space preserving the Baer involution τ on the form (3), one can reduce this list.

In the table below we present the collections of matrices M, F_1, F_2 defining six pairwise nonisomorphic planes and the orders of the left, the middle, and the right nuclei W_l, W_m, W_r of the corresponding semifields. The plane π_6 is coordinatized by a field and therefore is Desargues.

Plane	Orders of nuclei W_l, W_m, W_r	M	F_1	F_2
π_1	3, 3, 9	$\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$
π_2	3, 9, 3	$\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$
π_3	9, 3, 3	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$
π_4	9, 9, 9	$\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
π_5	9, 9, 9	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$
π_6	81, 81, 81	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Constructed examples are also presented in author's paper [5]. Let now a semifield plane π of order 3^4 admit a subgroup of autotopisms isomorphic to the alternating group A_4. By Theorem 2, π can be given by an 8-dimensional linear space over $\mathbb{Z}_3$ and a spread set of the form (2). Hence it follows that $Q_1 = K_1 = Q_2 = K_2 = \mathbb{Z}_3$, $Q = K$ is a field of order nine, and matrices from $R \subset GL_4(3) \cup \{0\}$ are of the form

$$\theta(x, y, z, t) = \begin{pmatrix} t & j^{-1}z & aj^{-1}y & bjx \\ ajz & t & jx & by \\ y & ax & t & bz \\ x & y & z & t \end{pmatrix}, \quad x, y, z, t \in \mathbb{Z}_3.$$

Here the coefficients $a, b, j \in \mathbb{Z}_3^*$, $j^3 = 1$, define the plane π in the case when all non-zero matrices $\theta(x, y, z, t)$ are nondegenerate. An immediate verification of possible values of a, b, and j and computation of the determinants shows that the following statement holds.

Lemma 8. *The autotopism group of a semifield plane of order 3^4 does not contain a subgroup isomorphic to the alternating group A_4.*

Consider next a spread set of the form (2) for the case $|\pi| = 3^8$. In this case, $R \subset GL_8(3) \cup \{0\}$, U_1, V_1, U_2, and V_2 are (2×2)-matrices over $\mathbb{Z}_3$. Without loss of generality, we can assume that $Q_1 = K_1 = Q_2 = K_2 = K$ is field (6) of order 9. In addition, one can easily check that in this case $J = E$. Since the cells

$$\begin{pmatrix} \nu(U_1) & \psi(V_1) \\ V_1 & U_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix}$$

form spread sets of semifield planes of order 3^4 admitting a Baer involution, one should consider examples of planes given above. It is clear that (see Remark to Theorem 1), checking possible spread sets K, it suffices to consider only pairwise nonisomorphic planes, and, for the set Q, all 106 constructed examples must be considered.

Consider linear mappings

$$\mu(xD + yE) = xM + yE, \ \ \varphi(xD + yE) = xF_1 + yF_2,$$
$$\nu(xD + yE) = xN + yE, \ \ \psi(xD + yE) = xP_1 + yP_2,$$

where the triples (M, F_1, F_2) and (N, P_1, P_2) define semifield planes of order 3^4 admitting Baer involution. Then a matrix from the spread set of a plane of order 3^8 admitting a subgroup of autotopisms isomorphic to A_4 can be written in the form

$$\theta(x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2) = x_1 \begin{pmatrix} 0 & 0 & 0 & F_1 \\ 0 & 0 & N & 0 \\ 0 & P_1 & 0 & 0 \\ D & 0 & 0 & 0 \end{pmatrix} + y_1 \begin{pmatrix} 0 & 0 & 0 & F_2 \\ 0 & 0 & E & 0 \\ 0 & P_2 & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix}$$
$$+ z_1 \begin{pmatrix} 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & F_1 \\ N & 0 & 0 & 0 \\ 0 & D & 0 & 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & F_2 \\ E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & N & 0 & 0 \\ P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_1 \\ 0 & 0 & D & 0 \end{pmatrix}$$
$$+ y_2 \begin{pmatrix} 0 & E & 0 & 0 \\ P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_2 \\ 0 & 0 & E & 0 \end{pmatrix} + z_2 \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix} + t_2 \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}.$$

In order for a semifield plane given by matrices M, F_1, F_2, N, P_1, P_2 to exist, it is necessary that the matrices

$$\theta(x_1, y_1, z_1, t_1, 0, 0, 0, 0) \ \text{and} \ \theta(0, 0, 0, 0, x_2, y_2, z_2, t_2)$$

be nondegenerate for all nonzero vectors (x_i, y_i, z_i, z_i). The checking of the 1696 possibilities for the matrices $(M, F_1, F_2, N, P_1, P_2)$ with the computation of the determinants of order 4 leads to the restriction of the list to 508 and 674 collections, respectively, in which only 386 collections of matrices are common. Then, the computation of the determinants of order 8 for the obtained collections and all 8-dimensional vectors $(x_1, \ldots, t_2)$ gives the negative result.

Lemma 9. *The group of autotopisms of a semifield plane of order 3^8 does not contain a subgroup isomorphic to the alternating group A_4.*

The absence of a subgroup of autotopisms isomorphic to the alternating group A_4 in the group of collineations of a semifield plane of odd order, in the case of rank two [16], in the case of orders 3^4 and p^4 for an arbitrary prime number $3 < p \leq 73$ (the fact has been checked by direct computation), in the case of order 3^8 allows us to suggest the conjecture that such planes do not exist in the general case. Note that this result would certainly impose a significant restriction on the structure of the autotopism group of a semifield plane and give possibilities to make next steps in the verification of its solvability.

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