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Boundary Theorems of Uniqueness for Logarithmic-Subharmonic Functions

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Abstract—We investigate the boundary theorems of uniqueness for certain important classes of logarithmic-subharmonic functions defined on the unit disk.

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Numerous important works of famous mathematicians are dealing with uniqueness theorems for meromorphic functions in the unit disk D. A reader can find a detailed description of these results in [1–6]. Certain results of that kind are extended on subharmonic functions; see, e.g., [7–10]. The author continues studies of this subject for logarithmic-subharmonic functions. A non-negative subharmonic functions $u(z)$ is called logarithmic-subharmonic if $\ln u(z)$ is also subharmonic function. We keep notation and definitions of papers [11, 12]. In addition, we say that a subset S_0 of the disk D satisfies condition (C) (see [6]) if

1) the set $E = \{|z|; z \in S_0\}$ is dense in certain interval $[r_0, 1)$ of the real axis,

2) for any $\eta > 0$ there exists a value $\delta > 0$ such that $|\arg z| < \eta$ for all $z \in S_0$ lying inside the ring $1 - \delta < |z| < 1$.

We denote by S_ξ the image of S_0 under rotation $z' = \xi z$, $|\xi| = 1$, then S_ξ has on Γ a unique limit point ξ. A set $N \subset \Gamma$ is called metrically dense on certain arc $\gamma \subset \Gamma$, if linear measure $\mathrm{mes}(\gamma' \cap N)$ is positive for any arc $\gamma' \subset \gamma$. We say (see [13]) that $u(z)$ is subordinated in D to subharmonic function $v(z)$ if $u(z) = v[\omega(z)]$, where the function $\omega(z)$ is analytic in D, and $\omega(0) = 0$, $|\omega(z)| < 1$. It is known (see ([13], P. 109) that the function $u(z)$ is subharmonic in D. The concept of subordination can be introduced analogously in the case of analytic $v(z)$. Then subordinated function $u(z)$ is also analytic in unit disk D. A point $\xi \in \Gamma$ is called uncertainty point if there exist two paths j_1 and j_2 ending at the point ξ such that $C_{j_1}(f, \xi) \cap C_{j_2}(f, \xi) = \emptyset$. We denote by $I(\alpha, \beta)$ the arc on Γ with end points $e^{i\alpha}$ and $e^{i\beta}$, where $0 \le \alpha < \beta \le 2\pi$. Assume that $\sigma(I) = D \cap N_\delta(\xi)$, where $\delta > 0$ and $N_\delta(\xi)$ is δ-neighborhood of the point $\xi = e^{i\theta} \in \Gamma$. The boundary of domain $\sigma(I)$ on Γ is the arc $I(\theta - \delta, \theta + \delta)$. Let $S(\alpha, \beta) = \{z = re^{i\theta} : \alpha < \theta < \beta, 0 \le r < 1\}$ be a sector of the disk D. We call a set $E \subset \Gamma$ (see [2]) the set of the first category if it is a union of countable family of nowhere dense sets. Otherwise it is the set of the second category. A set $E \subset I(\alpha, \beta)$ is called remainder set, if its complement in $I(\alpha, \beta)$ is a set of the first category. We say that a subharmonic function $u(z)$ has a harmonic majorant in the domain G, if there exists a harmonic function $v(z)$ such that $u(z) \le v(z)$ in G. We put $R_+ = [0, +\infty]$.

We consider two theorems, which were obtained by M. Arsove, in a convenient for our consideration form. Theorem A* is a generalization of classic Littlewood theorem for subharmonic in D functions (see [13]), and Theorem B* is a subharmonic analog of the N. N. Luzin and I. I. Privalov uniqueness theorems [1].

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Theorem A* ([7]). *Let a defined in $\sigma(I)$ logarithmic-subharmonic function $u(z)$ have a positive majorant. Then finite radial limit $\lim_{r\to 1} u(re^{i\theta})$ exists everywhere on $I(\theta-\delta,\theta+\delta)$ excluding, maybe, a set E of null measure.*

Theorem B* ([7]). *Let a defined in D logarithmic-subharmonic function $u(z)\not\equiv 0$, and $I(\alpha,\beta)$ be an arc of Γ. If $\limsup_{r\to 1} u(re^{i\theta})<+\infty$ for $\theta\in A$, where A is a remainder set on $I(\alpha,\beta)$, then the set of values θ such that $\liminf_{r\to 1} u(re^{i\theta})=0$ is not metrically dense in $I(\alpha,\beta)$.*

In what follows we need also the theorems proved by E. Collingwood, F. Bagemihl, K. Barth and W. Schneider, J. Meek.

Theorem C* ([2]). *Let either real or complex function $f(z)$ be defined in D, and let $\{S_\xi\}$ be a family of continuums obtained by rotation around the origin of a non-degenerate continuum S_1, which has unique common point $\xi=1$ with Γ. Then $C(f,\xi,S_\xi)=C(f,\xi,D)$ for points ξ of certain remainder set on Γ.*

Theorem D* ([2]). *Let $f(z)$ be any complex-valued function on D. Then the set of its uncertainty points is no more than countable.*

Theorem E* ([6]). *Let $\mu(r)>0$ be any decreasing function on $[0,1)$ such that $\lim_{r\to 1}\mu(r)=0$. Then there exists a holomorphic in D function $h(z)\not\equiv 0$ such that $|h\left(re^{i\theta}\right)|\leq\frac{1}{\mu(r)}$ for $r\to 1$ and $\lim_{r\to 1} h\left(re^{i\theta}\right)=0$ for all values $\theta\in E\subset\Gamma$, where $\operatorname{mes} E=2\pi$.*

Theorem F* ([15]). *Let a normal subharmonic in D function $u(z)$ satisfy the condition $\int\limits_0^{2\pi}\left|u(re^{i\theta})\right|d\theta=O(1)$ for $r\to 1$. Then $u(z)$ has finite angular limits almost everywhere on Γ.*

Let us formulate the main results of the present paper.

Theorem 1. *If we can find for a continuous and logarithmic-subharmonic in D function $f(z)$ a set S_0 with property (C) and a set M of the second category on some arc $\gamma\subset\Gamma$ such that $C(f,\xi,S_\xi)$ is bounded from above at each point $\xi\in M$, and if each point ξ of some metrically dense on the arc γ set N is end point of a curve L_ξ such that $C(f,\xi,L_\xi)=\{0\}$, then $f(z)\equiv 0$.*

Remark 1. If we assume additionally that the paths L_ξ are non-tangent, then we can replace the condition $C(f,\xi,L_\xi)=\{0\}$ by assumption $0\in C(f,\xi,L_\xi)$.

Theorem 2. *Let $\mu(r)>0$ be arbitrary decreasing function on $[0,1)$ such that $\lim_{r\to 1}\mu(r)=0$. Assume that we can find for a logarithmic-subharmonic and continuous in D function $f(z)$ a subset S_0 of the disk D with property (C) and a set M of the second category on some arc $\gamma\subset\Gamma$ such that $f(z)=O(\mu(|z|))$ for $z\to\xi$, $z\in S_\xi$, for any point $\xi\in M$. Then $f(z)\equiv 0$.*

We describe calculating parts of the proof of the main results in the following lemmas.

Lemma 1. *Let $f(z)$ be a continuous logarithmic-subharmonic function, and there exist a subset S_0 with property (C) and a set E of the second Baire category on some arc $\gamma\subset\Gamma$ such that $C(f,\xi,S_\xi)\neq R_+$ at each point $\xi\in E$. If any point ξ of some metrically dense on arc γ set N is end point of a curve L_ξ such that $C(f,\xi,L_\xi)=\{0\}$, then there exists at least one point $\xi_0\in\gamma$ such that the function $f(z)$ has null radial boundary limits at almost all points of certain neighborhood of ξ_0 on Γ.*

Proof. We apply the method from [6]. If a mapping $f(z)$ and a set S_0 in the disk D satisfies condition (C), then by virtue of Theorem C* the set of points $\xi \in \Gamma$ such that $C(f,\xi,S_\xi) \neq C(f,\xi,D)$ is the set of the first category with respect to Γ. This result under assumptions of the lemma means that the arc $\gamma \subset \Gamma$ contains at least one point $\xi_0 \in \gamma$ such that $C(f,\xi_0,S_{\xi_0}) = C(f,\xi_0,D)$, and, consequently, $C(f,\xi_0,D) \neq R_+$. Since the function $f(z)$ is continuous and D is connected, the limit set $C(f,\xi_0,D)$ is also connected. Hence, this limit set is some segment $[a,b]$, where $a \geq 0$, $b \leq +\infty$ and $a < b$. The set N is metrically dense on γ and $C(f,\xi,L_\xi) = \{0\}$ for $\xi \in N$, whence, limit set $C(f,\xi_0,D)$ is bounded from above. Consequently, the function $f(z)$ is bounded from above subharmonic function in some neighborhood of the arc $\gamma_0 \subset \gamma$ ($\xi_0 \in \gamma_0$). Therefore, the function $f(z)$ has positive harmonic majorant in the mentioned neighborhood, and Theorem A* implies that the function $f(z)$ has finite radial limits at each point of boundary of the neighborhood of point ξ_0 on Γ excluding, maybe, some set F, $\operatorname{mes} F = 0$. Since any countable set has null measure, we obtain for $\xi \in N$ by virtue of Theorem D* and condition $C(f,\xi,L_\xi) = \{0\}$ that the function $f(z)$ has null limits at any point of the boundary of the mentioned neighborhood of point ξ_0 on Γ excluding, maybe, some set E, $\operatorname{mes} E = 0$. ☐

Remark 2. If we assume additionally that the curves L_ξ are non-tangential, then we can replace the assumption $C(f,\xi,L_\xi) = \{0\}$ in the formulation of Lemma by weaker condition $0 \in C(f,\xi,L_\xi)$ for $\xi \in N$.

To prove Theorem 2 we need

Lemma 2. *Let $f(z)$ be continuous subharmonic function in the disk D. If there exist a subset S_0 of the disk D with property (C) and a set M of the second category on $\gamma \subset \Gamma$ such that $+\infty \notin \bigcup_{\xi\in M} C(f,\xi,S_\xi)$, then we can find an arc $\gamma_0 \subset \gamma$ such that*

1) *the set $M \cap \gamma_0$ is dense on γ_0,*

2) *$M \cap \gamma_0$ is the set of the second category on γ_0,*

3) *the function $f(z)$ is uniformly bounded from above in an appropriate neighborhood of the arc γ_0 in D.*

Proof. We apply the scheme, which was offered for meromorphic functions by E. Collingwood (see, e.g., [6]) for chords instead of S_ξ. We denote by $E(\theta,N)$ the set of all points $\zeta = e^{i\theta} \in M$ where $f(z) < N$ for any $z \in S_\zeta$, N is an integer number. Let us consider a sequence of numbers $N_1 < N_2 < \cdots < N_\nu < \cdots$. Clearly, $E(\theta,N_\nu) \subset E(\theta,N_{\nu+1})$ for $\nu = 1,2,\ldots$ Then $M = \bigcup_{\nu=1}^{\infty} E(\theta,\nu)$. But M is a set of the second category on $\gamma \subset \Gamma$. Therefore, at least one of sets $E(\theta,\nu)$ (for example, $E(\theta,\nu_0)$), is a set of the second category on $\gamma \subset \Gamma$. Hence, there exists an arc $\gamma_0 \subset \gamma$, where the set $E(\theta,\nu_0)$ is also dense. Since $E(\theta,\nu_0) \subset M$, it follows that the M is dense on γ_0, what proves Proposition 1. Since the $E(\theta,\nu_0) \subset M$ is of the second category on γ_0, therefore Proposition 2 is valid. At any point $\zeta = e^{i\theta} \in E(\theta,\nu_0) \cap \gamma_0$ we have for all $z \in S_\zeta$

$$f(z) < \nu_0. \tag{1}$$

Let us prove that the inequality

$$f(z) \leq \nu_0 \tag{2}$$

is satisfied inside curvilinear quadrangle G, which does not contain the origin, and is bounded by arc γ_0, two curves S_ζ drawn at endpoints of the arc γ_0, and the circle $|z| = r$ intersecting these curves S_ζ. Indeed, by virtue of inequality (1) and since $E(\theta,\nu_0)$ is dense on γ_0, every point of the domain G is condensation point for the set of points z where $f(z) < \nu_0$. By virtue of the continuity of the function $f(z)$ we have inequality (2), and Proposition 3 is proved. ☐

Lemma 3. *Let a defined in D function $u(z)$ be subordinated to a logarithmic-subharmonic function $v(z)$. Then $u(z)$ is logarithmic-subharmonic function in D.*

Proof. Lemma 3 implies that $u_1(z) = \ln v(z)$ is logarithmic-subharmonic function. Hence, the function $u_2(z) = u_1(\omega(z)) = \ln v(\omega(z))$ is subordinated in D to subharmonic function $u_1(z)$. I. I. Privalov proved ([13], P. 109) that in this case the function $u_2(z)$ is subharmonic in D. Then the function $\exp\{u_2(z)\}$ is subharmonic by virtue of the properties of subharmonic functions (ibid., P. 59). Whence, $\exp\{u_2(z)\}$ is logarithmic-subharmonic function, and there is valid the relation

$$\exp\{u_2(z)\} = \exp\{u_1(\omega(z))\} = v(\omega(z)) = u(z). \tag{3}$$

We conclude from (3) that the function $u(z)$ is logarithmic-subharmonic. □

Proof of Theorem 1 is based on Lemma 1. According to the lemma, there exists at least one point $\xi_0 \in \gamma$ such that the function $f(z)$ has null radial boundary limits at almost all points of certain neighborhood of ξ_0 in Γ. Since the function $f(z)$ is bounded from above in the mentioned neighborhood, and, consequently, has positive harmonic majorant, by virtue of Theorem B* we have $f(z) \equiv 0$. □

Remark 3. In a special case, where S_ξ, L_ξ are radii and M is remainder set on γ, Theorem 1 for subharmonic functions was proved by M. Arsove [7].

Proof of Theorem 2. According to Lemma 2, the function $f(z)$ is uniformly bounded from above in a neighborhood of certain arc $\gamma_0 \subset \gamma$ and $M_0 = M \cap \gamma_0$ is set of the second category on γ_0. By virtue of Theorem E* there exists a holomorphic in D function $h(re^{i\theta})$ such that $|h(re^{i\theta})| < \frac{1}{\mu(r)}$ as $r \to 1$ and $\lim\limits_{r\to1} h(re^{i\theta}) = 0$ for all values $\theta \in E \subset \Gamma$, $\operatorname{mes} E = 2\pi$. Let $N = E \cap \gamma_0$. Then the set N is metrically dense on γ_0. Clearly, $Q = N \cup M_0$ is the set of the second category and positive measure. Since $f(z)$ and $|h(z)|$ are logarithmic-subharmonic functions, the function $F(z) = f(z) \cdot |h(z)|$ is also logarithmic-subharmonic, and it satisfies assumptions of Theorem 1 on the arc γ_0, if we consider the radii with end points in corresponding points ξ in the capacity of S_ξ and L_ξ. Hence, $F(z) \equiv 0$, and, consequently, $f(z) \equiv 0$. □

Let us study certain applications of the obtained results.

Theorem 3. *Let a defined in D function $u(z)$ be subordinated to a logarithmic-subharmonic function $v(z)$ satisfying assumptions of Theorem 1. Then $u(z) \equiv 0$.*

Proof. By virtue of Lemma 3 the function $u(z)$ is logarithmic-subharmonic in D. Its subordination means validity of the representation

$$u(z) = v(\omega(z)). \tag{4}$$

Relations (4), $\omega(0) = 0$ and $|\omega(z)| < 1$ imply that analytic function $\omega(z)$ maps any closed domain $\overline{G} \subset D$ containing point $z = 0$ onto closed domain $\overline{G}_1 < |\omega| < 1$ containing the point $\omega = 0$. Theorem 1 enables to conclude that $v(z) \equiv 0$. It follows from relation (4) that $u(z) \equiv 0$ on $\overline{G}$. Thus, $u(z) \equiv 0$ in D by virtue of the uniqueness theorem for logarithmic-subharmonic functions. □

Corollary 1. Let a defined in D function $u(z)$ be subordinated to a normal logarithmic-subharmonic function $v(z)$ such that $\lim\limits_{z\to\xi,\, z\in L_\xi} v(z) = 0$ at every point $\xi \in E \subset \Gamma$, $\operatorname{mes} E > 0$, where $L_\xi \subset D$ is a non-tangential to Γ path ending at the point ξ. Then $u(z) \equiv 0$.

Proof. Indeed, by virtue of the well-known Rung theorem [14] the function $v(z)$ has null angular boundary limit at any point $\xi \in E$. On the other hand, by the uniqueness theorem for logarithmic-subharmonic functions from [8] we have $v(z) \equiv 0$ in D. Then we repeat the considerations from the proof of Theorem 3, and by means of relation (4) obtain the identity $u(z) \equiv 0$ in D. □

Corollary 2. Let a defined in D function $u(z)$ be subordinated to a normal logarithmic-subharmonic function $v(z)$ satisfying the condition

$$\int_0^{2\pi} v^+(re^{i\theta})d\theta = O(1) \text{ for } r \to 1. \tag{5}$$

If any point ξ of certain set $E \subset \Gamma$, $\operatorname{mes} E > 0$, is end point of a path $L_\xi \subset D$ such that $\lim\limits_{z\to\xi, z\in L_\xi} u(z) = 0$, then $u(z) \equiv 0$.

Proof. By virtue of Theorem F* under condition (5) a normal subharmonic function $u(z)$ has finite angular limits everywhere on Γ excluding, maybe, a set E_1, $\operatorname{mes} E_1 = 0$. By virtue of Theorem D* function $v(z)$ has null angular limits everywhere on set E excluding, maybe, a set $F = (E \setminus E_1) \cup E_2$, where E_2 is countable set, $\operatorname{mes} E_2 = 0$, $\operatorname{mes} F > 0$. Hence, $v(z) \equiv 0$ and $u(z) \equiv 0$ in D by the uniqueness theorem for logarithmic-subharmonic functions [8]. □

Let us consider an application of Theorem 2.

Theorem 4. *Let $\mu(r) > 0$ be a decreasing function on $[0, 1)$ such that $\lim\limits_{r\to1} \mu(r) = 0$. Let $u(z)$ be subordinated in D to a logarithmic-subharmonic function $v(z)$ satisfying assumptions of Theorem 2. Then $u(z) \equiv 0$.*

The proof is analogous to the proof of Theorem 3, but instead of Theorem 1 we use Theorem 2.

Let us consider one more general assertion.

Theorem 5. *Let a defined in the sector $S(\alpha, \beta)$ subharmonic function $u(z)$ satisfy the condition*

$$\int_\alpha^\beta \left|u(re^{i\theta})\right| d\theta = O(1) \text{ as } r \to 1,$$

where $0 \le \alpha < \beta \le 2\pi$. If any point $\xi \in I(\alpha, \beta)$ is end point of a path L_ξ such that

$$C(u, \xi, L_\xi) = \{0\}, \tag{6}$$

then $u(z) \equiv 0$.

Proof. By means of I. I. Privalov considerations (see [13], pp. 194–196) we represent the function $u(z)$ in the sector $S(\alpha, \beta)$ as a sum of negative subharmonic function $u_1(z)$ and positive harmonic function $h(z)$, i.e., $u(z) = u_1(z) + h(z)$. As known, a positive harmonic function $h(z)$ (see [13]) has finite angular limits everywhere on $I(\alpha, \beta)$ excluding, maybe, a set E_1, $\operatorname{mes} E_1 = 0$. On the other hand, a negative subharmonic function $u_1(z)$ has positive harmonic majorant in $S(\alpha, \beta)$. Therefore, by virtue of Theorem A* the function $u_1(z)$ has finite radial limits everywhere on $I(\alpha, \beta)$ excluding, maybe, a set E_2, $\operatorname{mes} E_2 = 0$. By virtue of Theorem D* and condition (6) we obtain equality $\lim\limits_{r\to1} u(re^{i\theta}) = \lim\limits_{r\to1} u(r\xi) = 0$ for any $\xi \in I(\alpha, \beta) \setminus E$, where $E = E_1 \cup E_2 \cup E_3$, E_3 is countable set, and $\operatorname{mes} E = 0$. We conclude due to Theorem B* that $u(z) \equiv 0$. □

Remark 4. If we assume additionally that the paths L_ξ are non-tangential, then condition (6) can be replaced by weaker one $0 \in C(u, \xi, L_\xi)$.

Thus, we prove here new boundary theorems of uniqueness not only for logarithmic-subharmonic functions, but for subordinated subharmonic functions as well. The author does not know studies of the boundary uniqueness for this class of subharmonic functions.

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