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Nonlocal Dezin's Problem for Lavrent'ev–Bitsadze Equation

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Abstract—Using the method of spectral analysis, for the mixed type equation $u_{xx} + (\operatorname{sgn} y)u_{yy} = 0$ in a rectangular domain we establish a criterion of uniqueness of its solution satisfying periodicity conditions by the variable x, a nonlocal condition, and a boundary condition. The solution is constructed as the sum of a series in eigenfunctions for the corresponding one-dimensional spectral problem. At the investigation of convergence of the series, the problem of small denominators occurs. Under certain restrictions on the parameters of the problem and the functions, included in the boundary conditions, we prove uniform convergence of the constructed series and stability of the solution under perturbations of these functions.

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1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let l, α, and β be given positive numbers. We consider the non-homogeneous mixed type equation

$$Lu \equiv u_{xx} + (\operatorname{sgn} y)u_{yy} = F(x, y) \tag{1}$$

in the rectangular domain $D = \{(x, y) \mid 0 < x < l, -\alpha < y < \beta\}$.

Dezin [1, 2] noted that the method of solvable extensions for differential operators can be adopted to the Lavrent'ev–Bitsadze operator L, under conditions of periodicity by x:

$$u(0, y) = u(l, y),\ \ u_x(0, y) = u_x(l, y),\ \ -\alpha \le y \le \beta, \tag{2}$$

and under the gluing conditions

$$u(x, 0+0) = u(x, 0-0) = u(x, 0),\ \ u_y(x, 0+0) = u_y(x, 0-0) = u_y(x, 0),\ \ 0 \le x \le l. \tag{3}$$

In addition, we set the following condition by y:

$$u(x, \beta) = 0,\ \ 0 \le x \le l, \tag{4}$$

$$u_y(x, -\alpha) - \lambda u(x, 0) = 0,\ \ 0 \le x \le l, \tag{5}$$

with a real parameter λ; in [1] we assume $l = 2\pi$, $\alpha = 1$, and $\beta = 1$.

Problem (1)–(5) was investigated by Nakhusheva ([3]; [4], pp. 143–153) for the case $\alpha = l$, $F(x, y) = f(x, y)H(y)$, $H(y)$ being the Heaviside function, and $\lambda \ge 0$. It was proved that for $\lambda < 0$ the homogeneous problem ($f(x, y) \equiv 0$) has non-trivial solutions.

In the paper we assume, for simplicity, that $F(x, y) \equiv 0$, i.e., we consider the homogeneous Lavrent'ev–Bitsadze equation

$$u_{xx} + (\operatorname{sgn} y)u_{yy} = 0, \tag{6}$$

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and we set non-homogeneous conditions by y:

$$u(x,\beta) = \varphi(x), \ \ 0 \le x \le l, \tag{7}$$

$$u_y(x,-\alpha) - \lambda u(x,0) = \psi(x), \ \ 0 \le x \le l. \tag{8}$$

We will investigate solvability of problem (2), (3), (6)–(8) depending on the given parameters l, α, β, λ, and the functions $\varphi(x)$, $\psi(x)$. We find solution

$$u(x,y) \in C^1(\overline{D}) \cap C^2(D_+ \cup D_-) \tag{9}$$

where $D_+ = D \cap \{y > 0\}$, $D_- = D \cap \{y < 0\}$.

We should note that in [5, 6] by methods of spectral analysis there were investigated the Dirichlet problem and the problem with periodicity conditions (2) for the degenerated mixed type equation

$$K(y)u_{xx} + u_{yy} - b^2 K(y)u = 0$$

with $K(y) = (\operatorname{sgn} y)|y|^n$, $n = \text{const} > 0$, $b = \text{const} \ge 0$. Here we apply the method for investigation of the given Dezin's problem.

We will find a criterion of uniqueness of solution to problem (2), (6)–(9). The solution is constructed as the sum of a series in eigenfunctions for the corresponding one-dimensional spectral problem. In reasoning of convergence of the series, the problem of small denominators occurs for the ratio of the sides lengths α/l of D_-. Under some conditions on α/l, λ, β, and functions $\varphi(x)$, $\psi(x)$ we show that the sum $u(x,y)$ of the series satisfies (9). We also prove stability of the solution with respect to perturbation of the given functions $\varphi(x)$ and $\psi(x)$.

We should note that the first nonlocal problems for mixed type equations was investigated in [7–9].

2. UNIQUENESS OF SOLUTION

Separating the variables $u(x,y) = X(x)Y(y)$ in (1), we obtain the following spectral problem for $X(x)$:

$$X''(x) + \widetilde{\lambda} X(x) = 0, \ \ 0 < x < l, \tag{10}$$

$$X(0) = X(l), \ \ X'(0) = X'(l). \tag{11}$$

Problem (10), (11) has a countable set of eigenvalues $\widetilde{\lambda}_k = \mu_k^2 = \left(\frac{2\pi k}{l}\right)^2$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$, all of them are simple and the corresponding system of eigenfunctions is $X_k(x) = \left\{\frac{1}{\sqrt{l}}, \sqrt{\frac{2}{l}}\cos \mu_k x, \sqrt{\frac{2}{l}}\sin \mu_k x\right\}$. The system is orthonormal and complete in $L_2[0,l]$, therefore, it is a orthonormal basis in the space.

Let there exist a solution to problem (2), (6)–(9). According to [5, 6], we introduce the functions

$$u_0(y) = \frac{1}{\sqrt{l}} \int_0^l u(x,y)\,dx, \ \ u_k(y) = \sqrt{\frac{2}{l}} \int_0^l u(x,y) \cos \mu_k x\,dx, \tag{12}$$

$$v_k(y) = \sqrt{\frac{2}{l}} \int_0^l u(x,y) \sin \mu_k x\,dx, k \in \mathbb{N}, \tag{13}$$

and show that they satisfy differential equations

$$u_k''(y) - (\operatorname{sgn} y)\mu_k^2 u_k(y) = 0, \ \ y \in (-\alpha, 0) \cup (0, \beta), \tag{14}$$

$$u_0''(y) = 0, \ \ y \in (-\alpha, 0) \cup (0, \beta), \tag{15}$$

$$v_k''(y) - (\operatorname{sgn} y)\mu_k^2 v_k(y) = 0, \ \ y \in (-\alpha, 0) \cup (0, \beta), k \in \mathbb{N}. \tag{16}$$

Then we find the general solution to (14)

$$u_k(y) = \begin{cases} c_k e^{\mu_k y} + d_k e^{-\mu_k y}, & y > 0, \\ a_k \cos \mu_k y + b_k \sin \mu_k y, & y < 0. \end{cases} \tag{17}$$

Here a_k, b_k, c_k, and d_k are arbitrary constants. We choose the constants so that the conjugation conditions

$$u_k(0+0) = u_k(0-0), \;\; u'_k(0+0) = u'_k(0-0) \tag{18}$$

hold. Since (17) satisfy (18), we find

$$c_k = \frac{a_k + b_k}{2}, \;\; d_k = \frac{a_k - b_k}{2}.$$

Then, taking into account the found values of c_k and d_k, we see that (17) has the form

$$u_k(y) = \begin{cases} a_k \cosh \mu_k y + b_k \sinh \mu_k y, & y > 0; \\ a_k \cos \mu_k y + b_k \sin \mu_k y, & y < 0. \end{cases} \tag{19}$$

To find a_k and b_k we use (7), (8), and (12). We have

$$u_k(\beta) = \sqrt{\frac{2}{l}} \int_0^l u(x,\beta) \cos \mu_k x \, dx = \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \cos \mu_k x \, dx = \varphi_k, \tag{20}$$

$$u'_k(-\alpha) - \lambda u_k(0) = \sqrt{\frac{2}{l}} \int_0^l [u_y(x,-\alpha) - \lambda u(x,0)] \cos \mu_k x \, dx = \sqrt{\frac{2}{l}} \int_0^l \psi(x) \cos \mu_k x \, dx = \psi_k. \tag{21}$$

Now, based on (19)–(21), we obtain the system

$$\begin{gathered} a_k \cosh \mu_k \beta + b_k \sinh \mu_k \beta = \varphi_k, \\ a_k \left(\sin \mu_k \alpha - \frac{\lambda}{\mu_k} \right) + b_k \cos \mu_k \alpha = \frac{\psi_k}{\mu_k}. \end{gathered} \tag{22}$$

If the determinant of (22)

$$\Delta(k) = \cos \mu_k \alpha \cosh \mu_k \beta - \sinh \mu_k \beta \sin \mu_k \alpha + \frac{\lambda}{\mu_k} \sinh \mu_k \beta \neq 0 \tag{23}$$

for all $k \in \mathbb{N}$, then the system has a unique solution

$$a_k = \frac{1}{\Delta(k)} \left[\varphi_k \cos \mu_k \alpha - \psi_k \frac{\sinh \mu_k \beta}{\mu_k} \right], \tag{24}$$

$$b_k = \frac{1}{\Delta(k)} \left[\varphi_k \left(\frac{\lambda}{\mu_k} - \sin \mu_k \alpha \right) + \psi_k \frac{\cosh \mu_k \beta}{\mu_k} \right]. \tag{25}$$

We should note that, in addition to k, $\Delta(k)$ depends on the parameters α, β, l, and λ.

Substituting (24) and (25) into (22), we find the final form of the functions

$$u_k(y) = \begin{cases} \varphi_k \frac{A_k(\alpha,y)}{\Delta(k)} - \psi_k \frac{\sinh \mu_k(\beta - y)}{\Delta(k)}, & y > 0; \\ \varphi_k \frac{B_k(\alpha,y)}{\Delta(k)} + \psi_k \frac{C_k(y,\beta)}{\Delta(k)}, & y < 0, \end{cases} \tag{26}$$

where

$$A_k(\alpha, y) = \cos \mu_k \alpha \cosh \mu_k y - \sin \mu_k \alpha \sinh \mu_k y + \frac{\lambda}{\mu_k} \sinh \mu_k y, \tag{27}$$

$$B_k(\alpha, y) = \cos \mu_k (\alpha + y) + \frac{\lambda}{\mu_k} \sin \mu_k y, \tag{28}$$

$$C_k(y, \beta) = \cosh \mu_k \beta \sin \mu_k y - \sinh \mu_k \beta \cos \mu_k y. \tag{29}$$

By the same way, beginning from (15), based on the conjugation conditions and the boundary conditions

(7), (8), we find

$$u_0(y) = \varphi_0 \frac{1+\lambda y}{1+\lambda\beta} + \psi_0 \frac{y-\beta}{1+\lambda\beta}, \ \ -\alpha \le y \le \beta. \tag{30}$$

Here $1+\lambda\beta \ne 0$, $\varphi_0 = \frac{1}{\sqrt{l}} \int\limits_0^l \varphi(x)\,dx$, $\psi_0 = \frac{1}{\sqrt{l}} \int\limits_0^l \psi(x)\,dx$. (We deduce (30) and show that $1+\lambda\beta \ne 0$, using (26)–(29) and (23) for the case $k=0$.)

Repeating the arguments, similar to those used in constructing (26), based on the general solution of (16) we find

$$v_k(y) = \begin{cases} \widetilde{\varphi}_k \frac{A_k(\alpha,y)}{\Delta(k)} - \widetilde{\psi}_k \frac{\sinh \mu_k(\beta-y)}{\Delta(k)}, & y>0; \\ \widetilde{\varphi}_k \frac{B_k(\alpha,y)}{\Delta(k)} + \widetilde{\psi}_k \frac{C_k(y,\beta)}{\Delta(k)}, & y<0, \end{cases} \tag{31}$$

where

$$\widetilde{\varphi}_k = \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \sin \mu_k x\,dx, \ \ \widetilde{\psi}_k = \sqrt{\frac{2}{l}} \int_0^l \psi(x) \sin \mu_k x\,dx.$$

Now we can prove the uniqueness theorem for solutions to problem (2), (6)–(9). Let $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$ and (23) hold for all $k \in \mathbb{N}_0$. Then $\varphi_k = \psi_k = \widetilde{\varphi}_k = \widetilde{\psi}_k = 0$ for all $k \in \mathbb{N}_0$ and $\varphi_0 = \psi_0 = 0$. From (26), (30), (31), (12), and (13) it follows that

$$\int_0^l u(x,y)\,dx = 0, \ \ \int_0^l u(x,y) \cos \mu_k x\,dx = 0, \ \ \int_0^l u(x,y) \sin \mu_k x\,dx = 0, \ \ k = 1,2,\dots$$

Completeness of the system $\left\{\frac{1}{\sqrt{l}}, \sqrt{\frac{2}{l}} \cos \mu_k x, \sqrt{\frac{2}{l}} \sin \mu_k x\right\}$ in $L_2[0,l]$ implies that $u(x,y) = 0$ a. e. on $[0,l]$ for every $y \in [-\alpha,\beta]$. Since $u(x,y)$ is continuous in $\overline{D}$, we have $u(x,y) \equiv 0$ in $\overline{D}$.

Let for some α, β, l, λ, and $k = p \in \mathbb{N}_0$ condition (23) be violated, i.e., $\Delta_{\alpha\beta}(p) = 0$. Then the homogeneous problem (2), (6)–(9) with $\varphi(x) = \psi(x) \equiv 0$ has non-trivial solutions

$$u_p(x,y) = u_p(y) \left(A_1 + A_2 \cos \mu_p x + A_3 \sin \mu_p x\right), \tag{32}$$

$$u_p(y) = \begin{cases} \frac{\sinh \mu_p(\beta-y)}{\sinh \mu_p \beta}, & y>0, \ \ p \in \mathbb{N}; \\ \frac{\mu_p \cos \mu_p(\alpha+y) + \lambda \sin \mu_p y}{\mu_p \cos \mu_p \alpha}, & y<0, \ \ p \in \mathbb{N}; \\ u_0(y) = a_0(y-\beta), \ \ a_0 = \text{const} \ne 0, & -\alpha \le y \le \beta, \ \ p = 0. \end{cases} \tag{33}$$

Here A_1, A_2, and A_3 are arbitrary constants.

Now a natural question arises about existence of roots of the equation $\Delta(k) = 0$. We represent $\Delta(p)$ it in the form

$$\Delta(p) = \sqrt{\cosh 2\mu_p\beta} \sin(\theta_p - 2\pi p \widetilde{\alpha}) + \frac{\lambda}{\mu_p} \sinh \mu_p \beta = 0 \tag{34}$$

where

$$\theta_p = \arcsin \frac{\cosh \mu_p \beta}{\sqrt{\cosh 2\mu_p \beta}}, \ \ \widetilde{\alpha} = \frac{\alpha}{l}.$$

We see from this that (34) has a countable set of zeroes

$$\widetilde{\alpha} = \frac{(-1)^n}{2\pi p} \arcsin \frac{\lambda \sinh \mu_p \beta}{\mu_p \sqrt{\cosh 2\mu_p \beta}} + \frac{\theta_p}{2\pi p} + \frac{\pi n}{2\pi p}, \ \ n \in \mathbb{N}_0, \tag{35}$$

if the condition

$$\frac{|\lambda|}{\mu_p} \sinh \mu_p \beta \Big/ \sqrt{\sinh^2 \mu_p \beta + \cosh^2 \mu_p \beta} \le 1 \tag{36}$$

holds. When $\mu_p \ge |\lambda|$, i.e., $p \ge |\lambda| l/2\pi$, inequality (36) is valid. Therefore, the uniqueness criterion is proved.

Theorem 1. *If there exists a solution to problem* (2), (6)–(9)*, then it is unique if and only if* (23) *holds for all* $k \in \mathbb{N}_0$.

3. CONSTRUCTION OF SOLUTION TO THE PROBLEM

Under fulfillment of (23), we find the solution to problem (2), (6)–(9) formally as the sum of the series

$$u(x,y) = \frac{1}{\sqrt{l}} u_0(y) + \sqrt{\frac{2}{l}} \sum_{k=1}^{+\infty} u_k(y) \cos \mu_k x + v_k(y) \sin \mu_k x. \tag{37}$$

Here the coefficients of $u_0(y)$, $u_k(y)$, and $v_k(y)$ are defined by (30), (26), and (31). Since $\Delta(k)$ is in denominators of expressions giving the coefficients of (37) and, as we demonstrated above, $\Delta(k)$, as a functions of α, has a countable set of zeroes (35), we see that for α close to a root of (34), the value $\Delta(k)$ can be sufficiently small. Therefore, the problem of small denominators arises (see, e.g., [5, 6, 10]. Consequently, to prove the existence of solution to the problem we need to show that there exist positive α, β, l, and λ such that for these values of the parameter $\Delta(k)$ is separated from zero for sufficiently large k with an appropriate asymptotics.

Lemma 1. *If* $\widetilde{\alpha} = \alpha/l$ *is natural and* $\lambda > -2\pi/l$*, then there exists a positive* C_0*, depending on* λ *and* l*, such that for all* $k \in \mathbb{N}$

$$\Delta(k) \ge e^{\mu_k \beta} C_0 > 0. \tag{38}$$

Proof. Let $\widetilde{\alpha} = p \in \mathbb{N}$. Then in view of (34)

$$\Delta(k) = \cosh \mu_k \beta + \frac{\lambda}{\mu_k} \sinh \mu_k \beta. \tag{39}$$

For $\lambda \ge 0$ it follows

$$\Delta(k) \ge \cosh \mu_k \beta > \frac{1}{2} e^{\mu_k \beta}.$$

If $\lambda < 0$, then from (39) we have

$$\Delta(k) > e^{\mu_k \beta} \left(\frac{1}{2} - \frac{|\lambda|}{2\mu_k} \right) \ge e^{\mu_k \beta} \left(\frac{1}{2} - \frac{|\lambda| l}{4\pi} \right).$$

This implies (38). □

Lemma 2. *If* $\widetilde{\alpha} = p/q \notin \mathbb{N}$, $p, q \in \mathbb{N}$, $(q,4) = 1$, $(p,q) = 1$*, then there exist positive* C_0 *and* k_0 $(k_0 \in \mathbb{N})$*, which, generally speaking, depend on* α*,* l*, and* λ*, such that for all* $k > k_0$

$$|\Delta(k)| \ge C_0 e^{\mu_k \beta} > 0. \tag{40}$$

Proof. Let $\widetilde{\alpha} = p/q \in \mathbb{Q}$ where $p, q \in \mathbb{N}$, $(p,q) = 1$, $(q,4) = 1$, and $\frac{p}{q} \notin \mathbb{N}$. In the case, let we divide $2kp$ by q with remainder. We have $2kp = sq + r$ where $s, r \in \mathbb{N}_0$ and $0 \le r < q$. Then

$$\Delta(k) = \sqrt{\cosh 2\mu_k \beta} (-1)^{s+1} \sin \left(\frac{\pi r}{q} - \theta_k \right) + \frac{\lambda}{\mu_k} \sinh \mu_k \beta. \tag{41}$$

The case $r = 0$ is reduced to the situation $\widetilde{\alpha} \in \mathbb{N}$ considered above.

Let $0 < r < q$. Then $1 \le r \le q-1$, $q \ge 2$. Since $\theta_k \to \frac{\pi}{4}$ as $k \to +\infty$, we have $\theta_k = \frac{\pi}{4} + \varepsilon_k$ where $\varepsilon_k \to 0$ as $k \to +\infty$. Taking this into account, there exists a natural k_1 such that for all $k > k_1$

$$\left| \sin \left(\frac{\pi r}{q} - \theta_k \right) \right| = \left| \sin \left(\frac{\pi r}{q} - \frac{\pi}{4} - \varepsilon_k \right) \right| \ge \frac{1}{2} \left| \sin \left(\frac{\pi r}{q} - \frac{\pi}{4} \right) \right| = C_1 > 0. \tag{42}$$

Then, taking into account estimate (42), we have from (41)

$$|\Delta(k)| = \sqrt{\cosh 2\mu_k\beta}\left|(-1)^{s+1}\sin\left(\frac{\pi r}{q}-\theta_k\right)+\frac{\lambda}{\mu_k}\frac{\sinh\mu_k\beta}{\sqrt{\cosh 2\mu_k\beta}}\right|$$
$$> \frac{e^{\mu_k\beta}}{\sqrt{2}}\left(\left|\sin\left(\frac{\pi r}{q}-\theta_k\right)\right|-\frac{|\lambda|}{\mu_k}\frac{1}{\sqrt{2}}\right) \geq \frac{e^{\mu_k\beta}}{\sqrt{2}}\left(C_1-\frac{|\lambda| l}{\pi\sqrt{2}k}\right) \quad (43)$$

for $k > k_1$. From this we see that there exists $k_2 \in \mathbb{N}$ such that for all $k > k_2$

$$\frac{|\lambda| l}{\pi\sqrt{2}k} < \frac{|\lambda| l}{\pi\sqrt{2}k_2} \leq \frac{C_1}{2}.$$

Then (43) implies the desired estimate (40) for all $k > k_0 = \max\{k_1, k_2\}$. □

Lemma 3. *If $\widetilde{\alpha}$ is an irrational algebraic number of degree 2, then there exist positive constants β_0, λ_0, and C_0, such that for all $\beta > \beta_0$, $|\lambda| < \lambda_0$, and $k \in \mathbb{N}$*

$$|\Delta(k)| \geq \frac{C_0}{k}e^{\mu_k\beta}. \quad (44)$$

Proof. Beforehand, we will estimate the values of

$$\Delta_1(k) = \sin(2k\pi\widetilde{\alpha}-\theta_k) = (-1)^n \sin\left[\pi k\left(2\widetilde{\alpha}-n/k\right)-\theta_k\right], \ \ n \in \mathbb{N}. \quad (45)$$

For every $k \in \mathbb{N}$ we can find $n \in \mathbb{N}$ such that

$$\left|\alpha_1 - \frac{n}{k}\right| < \frac{1}{2k}, \ \ \alpha_1 = 2\widetilde{\alpha}, \quad (46)$$

holds [11]. From the Liouville theorem ([12], P. 60) it follows that for every irrational algebraic number α_1 of degree 2 there exists $\delta > 0$ such that for every integer p, q $(q > 0)$

$$|\alpha_1 - p/q| > \delta/q^2. \quad (47)$$

Let $n \in \mathbb{N}$ be such that (46) holds. Then the inequality

$$|\pi k\left(\alpha_1 - n/k\right)| < \pi/2, \quad (48)$$

equivalent to (46), is valid. We will take into consideration that $\theta_k \to \pi/4$ as $k \to \infty$. Using decreasing of $u = \frac{\cosh x}{\sqrt{\cosh 2x}}$ and increasing of $\arcsin u$, we deduce that

$$\pi/4 < \theta_k \leq \theta_1 < \pi/2. \quad (49)$$

Applying (48) and (49), we have

$$0 \leq |\pi k\left(\alpha_1 - n/k\right)-\theta_k| \leq \pi k\left|\alpha_1 - n/k\right| + \theta_k < \pi/2+\theta_1 < \pi.$$

We see that one of the following two cases is possible:

$$0 \leq |\pi k\left(\alpha_1 - n/k\right)-\theta_k| < \pi/2, \quad (50)$$
$$\pi/2 \leq |\pi k\left(\alpha_1 - n/k\right)-\theta_k| < \pi/2+\theta_1. \quad (51)$$

If (51) holds, then we obtain

$$|\Delta_1(k)| = \left|\sin\left[\pi k\left(\alpha_1 - n/k\right)-\theta_k\right]\right|$$
$$> \sin\left(\pi/2+\theta_1\right) = \cos\theta_1 = (\sinh\mu_1\beta)/\sqrt{\cosh 2\mu_1\beta} = C_2 \geq C_2/k. \quad (52)$$

If (50) is valid, then, taking into account the well-known inequality

$$\sin x \geq \frac{2}{\pi}x, \ \ 0 \leq x < \frac{\pi}{2},$$

we have

$$|\Delta_1(k)| = \left|\sin\left[\pi k\left(\alpha_1 - \frac{n}{k}\right) - \theta_k\right]\right| \geq \frac{2}{\pi}\left|\pi k\left(\alpha_1 - \frac{n}{k}\right) - \theta_k\right|. \quad (53)$$

Now we will estimate the expression in the right-hand side of (53). We have

$$|\pi k\alpha_1 - \pi n - \theta_k| = \left|\pi k\alpha_1 - \pi n - \frac{\pi}{4} - \theta_k + \frac{\pi}{4}\right| = \left|\pi k\alpha_1 - \pi\frac{4n+1}{4}\right| - \left|\theta_k - \frac{\pi}{4}\right|. \quad (54)$$

Taking into consideration (46), we estimate the first summand in the right-hand side of (54):

$$\pi k\left|\alpha_1 - \pi\frac{4n+1}{4k}\right| > \frac{\pi\delta}{16k}. \quad (55)$$

Using the equality

$$\arcsin x - \arcsin y = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), \text{ if } x\cdot y > 0,$$

we estimate the second summand:

$$\left|\theta_k - \frac{\pi}{4}\right| = \left|\arcsin\frac{\cosh\mu_k\beta}{\sqrt{\cosh 2\mu_k\beta}} - \arcsin\frac{1}{\sqrt{2}}\right| = \left|\arcsin\left(\frac{1}{\sqrt{2}}\frac{\cosh\mu_k\beta - \sinh\mu_k\beta}{\sqrt{\cosh 2\mu_k\beta}}\right)\right|$$
$$= \arcsin\frac{1}{e^{2\mu_k\beta}\sqrt{1+e^{-4\mu_k\beta}}} < \frac{\pi}{2e^{2\mu_k\beta}}, \quad (56)$$

since

$$|\arcsin x| < \frac{\pi}{2}|x|, \ \ 0 < |x| < 1.$$

Now from (55) and (56) we deduce that

$$|\Delta_1(k)| > \frac{2}{\pi}\left(\frac{\pi\delta}{16k} - \frac{\pi}{2e^{2\mu_k\beta}}\right) = \frac{\delta}{8k} - \frac{1}{e^{2\mu_k\beta}} > \frac{\delta}{8k} - \frac{1}{2\mu_k\beta} = \frac{1}{k}\left(\frac{\delta}{8} - \frac{l}{4\pi\beta}\right) = \frac{C_3}{k} \quad (57)$$

where $C_3 = (\pi\delta\beta - 2l)/8\pi\beta > 0$ for $\beta > \beta_0 = 2l/\pi\delta$.

Let us return to the estimate

$$|\Delta(k)| \geq \frac{e^{\mu_k\beta}}{\sqrt{2}}\left|\Delta_1(k) + \frac{\lambda}{\mu_k}\sinh\mu_k\beta/\sqrt{\cosh 2\mu_k\beta}\right|.$$

Using (52) and (57), we obtain the desired estimate (44):

$$|\Delta(k)| \geq \frac{e^{\mu_k\beta}}{\sqrt{2}}\left(|\Delta_1(k)| - \frac{|\lambda|}{\mu_k}\frac{1}{\sqrt{2}}\right) > \frac{e^{\mu_k\beta}}{\sqrt{2}}\left(\frac{C_4}{k} - \frac{|\lambda|l}{2\pi\sqrt{2}k}\right) = \frac{e^{\mu_k\beta}}{k\sqrt{2}}\left(C_4 - \frac{|\lambda|l}{2\pi\sqrt{2}}\right) = \frac{e^{\mu_k\beta}}{k}C_0$$

where $C_0 = \left(C_4 - \frac{|\lambda|l}{2\pi\sqrt{2}}\right)\frac{1}{\sqrt{2}} > 0$ for $|\lambda| < \lambda_0 = \frac{C_4 2\pi\sqrt{2}}{l}$ and $C_4 = \min\{C_2, C_3\}$. □

Lemma 4. *Let the assumptions of Lemma 1 be valid. Then for every $k \in \mathbb{N}$ and $y \in [-\alpha, \beta]$*

$$|u_k(y)| \leq M_1(|\varphi_k| + |\psi_k|), \quad |u_k'(y)| \leq M_2 k(|\varphi_k| + |\psi_k|), \quad |u_k''(y)| \leq M_3 k^2(|\varphi_k| + |\psi_k|);$$
$$|v_k(y)| \leq M_1\left(|\widetilde{\varphi}_k| + |\widetilde{\psi}_k|\right), \ |v_k'(y)| \leq M_2 k\left(|\widetilde{\varphi}_k| + |\widetilde{\psi}_k|\right), \ |v_k''(y)| \leq M_3 k^2\left(|\widetilde{\varphi}_k| + |\widetilde{\psi}_k|\right);$$

here and below M_i are positive constants.

By (38), the above estimates follow immediately from (26)–(29) and (31).

By Lemma 4, the terms of series (37), their first order derivatives in $\overline{D}$ and second order ones in $\overline{D}_+$ and $\overline{D}_-$ are majorized, by modulus, by the terms of the number series

$$M_4\sum_{k=1}^{+\infty}k^2\left(|\varphi_k| + |\psi_k| + |\widetilde{\varphi}_k| + |\widetilde{\psi}_k|\right). \quad (58)$$

From the theory of Fourier series it is known that if $\varphi(x), \psi(x) \in C^3[0,l]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = 0, 1, 2$, then (58) is majorized by the convergent number series

$$M_5 \sum_{k=1}^{+\infty} \frac{1}{k} \left(|\varphi_k^{(3)}| + |\psi_k^{(3)}| + |\widetilde{\varphi}_k^{(3)}| + |\widetilde{\psi}_k^{(3)}| \right) \tag{59}$$

where

$$\varphi_k^{(3)} = \sqrt{\frac{2}{l}} \int_0^l \varphi^{(3)}(x) \sin \mu_k x \, dx, \;\; \psi_k^{(3)} = \sqrt{\frac{2}{l}} \int_0^l \psi^{(3)}(x) \sin \mu_k x \, dx,$$
$$\widetilde{\varphi}_k^{(3)} = \sqrt{\frac{2}{l}} \int_0^l \widetilde{\varphi}^{(3)}(x) \cos \mu_k x \, dx, \;\; \widetilde{\psi}_k^{(3)} = \sqrt{\frac{2}{l}} \int_0^l \widetilde{\psi}^{(3)}(x) \cos \mu_k x \, dx,$$

and

$$\sum_{k=1}^{+\infty} |\varphi_k^{(3)}|^2 \le \|\varphi^{(3)}(x)\|_{L_2}^2, \;\; \sum_{k=1}^{+\infty} |\psi_k^{(3)}|^2 \le \|\psi^{(3)}(x)\|_{L_2}^2,$$
$$\sum_{k=1}^{+\infty} |\widetilde{\varphi}_k^{(3)}|^2 \le \|\varphi^{(3)}(x)\|_{L_2}^2, \;\; \sum_{k=1}^{+\infty} |\widetilde{\psi}_k^{(3)}|^2 \le \|\psi^{(3)}(x)\|_{L_2}^2.$$

Consequently, the sum of (37) satisfies (9).

If the assumptions of Lemma 3 hold, then (37) and the series of its terms' derivatives up to the second order are majorized by the terms of the series

$$M_5 \sum_{k=1}^{+\infty} k^3 \left(|\varphi_k| + |\psi_k| + |\widetilde{\varphi}_k| + |\widetilde{\psi}_k| \right). \tag{60}$$

For convergence of (60) it suffices to claim that $\varphi(x), \psi(x) \in C^4[0,l]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = \overline{0,3}$.

Now let the assumptions of Lemma 2 be valid. Then, by (40), series (37) and the series of their terms' derivatives up to the second order are majorized by

$$M_5 \sum_{k=k_0+1}^{+\infty} \frac{1}{k} \left(|\varphi_k^{(3)}| + |\psi_k^{(3)}| + |\widetilde{\varphi}_k^{(3)}| + |\widetilde{\psi}_k^{(3)}| \right). \tag{61}$$

If for the numbers $\widetilde{\alpha}$, specified in Lemma 2, $\Delta(k) \neq 0$ for all $k = \overline{1, k_0}$, then, by the Weierstrass theorem, convergence of (61) implies that the sum of (37) satisfy (9) and (6).

If for the numbers $\widetilde{\alpha}$, specified in Lemma 2, $\Delta(k) = 0$ for some $k = k_1, k_2, \dots, k_p \le k_0$, $1 \le k_1 \le k_2 < \dots < k_p$, k_i, then problem (2), (3), (6)–(9) is solvable if and only if

$$\begin{aligned} &\varphi_k \mu_k \cos \mu_k \alpha - \psi_k \sinh \mu_k \beta = 0, \\ &\widetilde{\varphi}_k \mu_k \cos \mu_k \alpha - \widetilde{\psi}_k \sinh \mu_k \beta = 0, \;\; k = k_1, k_2, \dots, k_p. \end{aligned} \tag{62}$$

Then the solution is defined as

$$u(x,y) = u_0(y) + \sum_{k=1}^{\infty} (k \neq k_1, k_2, \dots, k_p) \left[u_k(y) \cos \mu_k x + v_k(y) \sin \mu_k x \right] + \sum_m \widetilde{u}_m(x,y); \tag{63}$$

in the latter sum m takes the values $k_1, k_2, \dots, k_p$, and

$$\widetilde{u}_p(x,y) = \widetilde{u}_p(y) \cos \mu_p x + \widetilde{v}_p(y) \sin \mu_p x \tag{64}$$

where

$$\widetilde{u}_p(y) = \begin{cases} \varphi_p \frac{\sinh \mu_p y}{\sinh \mu_p \beta} + C_p \frac{\sinh \mu_p(\beta - y)}{\sinh \mu_p \beta}, & y > 0; \\ \psi_p \frac{\sin \mu_p y}{\mu_p \cos \mu_p \alpha} + C_p \frac{\mu_p \cos \mu_p (y+\alpha) + \lambda \sin \mu_p y}{\mu_p \cos \mu_p \alpha}, & y < 0; \end{cases} \tag{65}$$

$$\widetilde{v}_p(y) = \begin{cases} \widetilde{\varphi}_p \frac{\sinh \mu_p y}{\sinh \mu_p \beta} + C_p \frac{\sinh \mu_p(\beta - y)}{\sinh \mu_p \beta}, & y > 0; \\ \widetilde{\psi}_p \frac{\sin \mu_p y}{\mu_p \cos \mu_p \alpha} + C_p \frac{\mu_p \cos \mu_p (y+\alpha) + \lambda \sin \mu_p y}{\mu_p \cos \mu_p \alpha}, & y < 0, \end{cases} \tag{66}$$

and C_p is an arbitrary constant.

We should note that (64)–(66) are written, taking into account the non-zero solutions (32) and (33) of the homogeneous problem.

Therefore, the following statements are valid.

Theorem 2. *If the assumptions of Lemma* 1 *hold,* $1 + \lambda\beta \neq 0$, $\varphi(x), \psi(x) \in C^3[0, l]$, $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = 0, 1, 2$, *then there exists a unique solution to problem* (2), (6)–(9)*, and this solution is given by* (37).

Theorem 3. *If the assumptions of Lemma* 3 *are valid,* $1 + \lambda\beta \neq 0$, $\varphi(x), \psi(x) \in C^4[0, l]$, $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = \overline{0, 3}$, *then there exists a unique solution to problem* (2), (6)–(9)*, and this solution is given by* (37).

Theorem 4. *Let the assumptions of Lemma* 2 *be valid* (*therefore,* (39) *holds for all* $k > k_0$), $1 + \lambda\beta \neq 0$, $\varphi(x), \psi(x) \in C^3[0, l]$, $\varphi^{(i)}(0) = \varphi^{(i)}(l)$, $\psi^{(i)}(0) = \psi^{(i)}(l)$, $i = 0, 1, 2$. *If* $\Delta(k) \neq 0$ *for all* $k = \overline{1, k_0}$, *then there exists a unique solution to problem* (2), (6)–(9)*, and this solution is given by* (37). *If* $\Delta(k) = 0$ *for some* $k = k_1, k_2, \ldots, k_p \leq k_0$, *then problem* (2), (6)–(9) *is solvable if and only if* (62) *are valid, and in the case this solution is given by* (63).

4. STABILITY OF SOLUTION UNDER PERTURBATIONS OF $\varphi(x)$ AND $\psi(x)$

Consider the well-known norms

$$\|u\|_{L_2[0,1]} = \|u\|_{L_2} = \left(\int_0^l |u(x, y)|^2 \, dx \right)^{1/2},$$

$$\|u(x, y)\|_{C(\overline{D})} = \max_{\overline{D}} |u(x, y)|.$$

Theorem 5. *Let assumptions of either Theorem* 2 *or Theorem* 3 *be valid. Then for solution* (37) *to problem* (2), (6)–(9) *we have the estimates*

$$\|u(x, y)\|_{L_2} \leq M_6 \left(\|\varphi(x)\|_{L_2[0,l]} + \|\psi(x)\|_{L_2[0,l]} \right), \tag{67}$$

$$\|u(x, y)\|_{C(\overline{D})} \leq M_7 \left(\|\varphi(x)\|_{C[0,l]} + \|\psi(x)\|_{C[0,l]} + \|\varphi'(x)\|_{C[0,l]} + \|\psi'(x)\|_{C[0,l]} \right) \tag{68}$$

where M_6 *and* M_7 *do not depend on* $\varphi(x)$ *and* $\psi(x)$.

Proof. We will prove the theorem following paper [13]. Since the system $X_k(x)$ is orthonormal in $L_2[0, l]$, by Lemma 4, we obtain from (37) that

$$\|u(x, y)\|_{L_2}^2 = u_0^2(y) + \sum_{k=1}^{+\infty} (u_k^2(y) + v_k^2(y)) \leq 2\widetilde{M}_1^2 \left(\varphi_0^2 + \psi_0^2 \right)$$

$$+ 2M_1^2 \sum_{k=1}^{+\infty} \left(|\varphi_k|^2 + |\psi_k|^2 + |\widetilde{\varphi}_k|^2 + |\widetilde{\psi}_k|^2 \right)$$

$$\le M_6^2\left(\varphi_0^2+\sum_{k=1}^{+\infty}(\varphi_k^2+\widetilde{\varphi}_k^2)+\psi_0^2+\sum_{k=1}^{+\infty}(\psi_k^2+\widetilde{\psi}_k^2)\right)\le M_6^2\left(\|\varphi(x)\|_{L_2}^2+\|\psi(x)\|_{L_2}^2\right).$$

Thus follows (67). Let (x,y) be an arbitrary point from $\overline{D}$. By Lemma 4,

$$|u(x,y)|\le|u_0(y)|+M_1\sum_{k=1}^{+\infty}(|u_k(y)|+|v_k(y)|)$$

$$\le\widetilde{M}_1\left(|\varphi_0|+|\psi_0|\right)+M_1\sum_{k=1}^{+\infty}\left(|\varphi_k|+|\psi_k|+|\widetilde{\varphi}_k|+|\widetilde{\psi}_k|\right). \quad (69)$$

We have

$$\varphi_k=-\frac{\varphi_k^{(1)}}{\mu_k},\ \ \widetilde{\varphi}_k=\frac{\widetilde{\varphi}_k^{(1)}}{\mu_k},\ \ \psi_k=-\frac{\psi_k^{(1)}}{\mu_k},\ \ \widetilde{\psi}_k=\frac{\widetilde{\psi}_k^{(1)}}{\mu_k}, \quad (70)$$

where

$$\varphi_k^{(1)}=\sqrt{\frac{2}{l}}\int_0^l\varphi'(x)\sin\mu_kx\,dx,\ \ \widetilde{\varphi}_k^{(1)}=\sqrt{\frac{2}{l}}\int_0^l\varphi'(x)\cos\mu_kx\,dx,$$

$$\psi_k^{(1)}=\sqrt{\frac{2}{l}}\int_0^l\psi'(x)\sin\mu_kx\,dx,\ \ \widetilde{\psi}_k^{(1)}=\sqrt{\frac{2}{l}}\int_0^l\psi'(x)\cos\mu_kx\,dx.$$

Taking into account (70), based on the Cauchy–Schwarz inequality we obtain from (69) that

$$|u(x,y)|\le\widetilde{M}_1\left(|\varphi_0|+|\psi_0|\right)+M_1\sum_{k=1}^{+\infty}\frac{1}{\mu_k}\left(|\varphi_k^{(1)}|+|\widetilde{\varphi}_k^{(1)}|+|\psi_k^{(1)}|+|\widetilde{\psi}_k^{(1)}|\right)\le\widetilde{M}_1\left(|\varphi_0|+|\psi_0|\right)+$$

$$+M_1\frac{l}{\pi}\left(\sum_{k=1}^{+\infty}\frac{1}{k^2}\right)^{1/2}\left[\left(\sum_{k=1}^{+\infty}|\varphi_k^{(1)}|^2\right)^{1/2}+\left(\sum_{k=1}^{+\infty}|\widetilde{\varphi}_k^{(1)}|^2\right)^{1/2}+\left(\sum_{k=1}^{+\infty}|\psi_k^{(1)}|^2\right)^{1/2}+\left(\sum_{k=1}^{+\infty}|\widetilde{\psi}_k^{(1)}|^2\right)^{1/2}\right]\le$$

$$\le\widetilde{M}_2\left[\|\varphi(x)\|_{L_2}+\|\psi(x)\|_{L_2}+\left(\sum_{k=1}^{+\infty}(|\varphi_k^{(1)}|^2+|\widetilde{\varphi}_k^{(1)}|^2)\right)^{1/2}+\left(\sum_{k=1}^{+\infty}(|\psi_k^{(1)}|^2+|\widetilde{\psi}_k^{(1)}|^2)\right)^{1/2}\right]\le$$

$$\le\widetilde{M}_2\left[\|\varphi(x)\|_{L_2}+\|\psi(x)\|_{L_2}+\left(|\varphi_0^{(1)}|^2+\sum_{k=1}^{+\infty}(|\varphi_k^{(1)}|^2+|\widetilde{\varphi}_k^{(1)}|^2)\right)^{1/2}+\right.$$

$$\left.+\left(|\psi_0^{(1)}|^2+\sum_{k=1}^{+\infty}(|\psi_k^{(1)}|^2+|\widetilde{\psi}_k^{(1)}|^2)\right)^{1/2}\right]\le$$

$$\le M_7\left(\|\varphi(x)\|_{L_2}+\|\psi(x)\|_{L_2}+\|\varphi'(x)\|_{L_2}+\|\psi'(x)\|_{L_2}\right),$$

where $\varphi_0^{(1)}=\sqrt{\frac{1}{l}}\int\limits_0^l\varphi'(x)\,dx=0$, $\psi_0^{(1)}=\sqrt{\frac{1}{l}}\int\limits_0^l\psi'(x)\,dx=0$. Thus follows (68). □

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