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On Liezation of the Leibniz Algebras and its Applications

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Abstract—We consider some the fundamental properties of the Leibniz algebras. Some results were known before, but in the paper they are proved by a single method of liezation—the transition to a Lie algebra, which gives for a number of cases greatly simplified proof. There are also some new results.

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The main aim of the paper is Leibniz algebras liezation (the procedure of transition to Lie algebras) that allows to give simple proofs of many theorems on Leibniz algebras applying the known results on Lie algebras. We uniformly attain certain results recently proved by a number of authors. The majority of the results of the paer are the direct analogs of the well-known statements on Lie algebras, so some of them appeared in numerous recent publications on Leibniz algebras (we do not give here the explicit references list since the results are scattered through a number of not necessarily printed sources). The really novel part of the paper is the uniform approach and certain fundamental Leibniz algebras statements allocation. We also give new simple proofs of the known results together with certain new results on Leibniz algebras.

We consider finite-dimensional algebras L over some field k of zero characteristic (the only important thing here is that this characteristic differs from 2). A linear mapping $D : L \to L$ is a Lie algebra L derivation if $D(x \cdot y) = Dx \cdot y + x \cdot D(y)$ for all $x, y \in L$ (here "$\cdot$" stands for the multiplication operation in the algebra L). We denote the set of all algebra L derivations by $\mathrm{Der}(L)$. $\mathrm{Der}(L)$ is a Lie algebra with respect to the linear operator commutation operation. We also naturally have the algebra L automorphism group $\mathrm{Aut}(L)$. If k is a field $\mathbf{R}$ or $\mathbf{C}$, then its automorphism group is a Lie group and its Lie algebra coincides with $\mathrm{Der}(L)$. We may consider $\mathrm{Aut}(L)$ as an algebraic group (or, more precisely, as a group of some algebraic group k-points over the field k).

An algebra L over the field k is a left Leibniz algebra if for any $x \in L$ the relative linear left multiplication operator l_x is a derivation of L, i.e., $l_x \in \mathrm{Der}(L)$ (this condition may be called a Leibniz axiom). Similarly we define a right Leibniz algebra—here the algebra L derivations are all the right multiplication operators r_x. Leibniz algebras extend the notion of Lie algebras which are both left and right Leibniz algebras and allow skew-symmetric multiplication operation. So we can say that the Leibniz algebra is a "non-anti-commutative " Lie algebra analog. The term "non-anti-commutative" seems cumbersome (and possesses double negation) so sometimes the authors use the term "non-commutative" Lie algebra analog though this notion is not precise, because it makes to think one that the main Lie algebra operation is commutative.

A. M. Bloch initiated the study of Leibniz algebras in [1, 2]. He named these objects D-algebras showing their direct connection to derivations. Later J.-L. Loday independently re-opened Leibniz algebras [3]. He introduced the term "Leibniz algebra" similarly to the famous "Leibniz function differentiating rule"Ȧlso J.-L. Loday (partly with coauthors) hugely developed Leibniz algebra theory. Later on some authors used to call these algebras "Loday algebras" though J.-L. Loday himself (sometimes under the nom-de-plume "Guillaume William Zinbiel"; here Zinbiel is the inverse of

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Leibniz) in survey [4] noted that this term is false. Still later in [5] attracted the attention to Leibniz algebras.

A lot of the Leibniz algebras properties (including the fundamental ones) is scattered through a number of authors papers and notes. Moreover, since some authors consider left algebras and the other prefer right, the reader does not achieve the uniform view on the matter. We must repeat that here we give certain fundamental definitions and statements on Leibniz algebras. Most of these facts we give with uniform proofs based on the notion of iezation, i.e., on the transition to Lie algebras. It turns out that application of the known Lie algebras properties allows significant simplifications of Leibniz algebras theory printed results proofs.

The notions of left and right Leibniz algebras are mutually parallel. The transfer from the right Leibniz algebra to the left one by the new multiplication $x \circ y = y \cdot x$. It will be convenient to consider the left Leibniz algebras since in this case we have a better view on the connection between operation of multiplication and derivation that puts the differentiation operator on the right to the differentiated object. Note that a number of authors prefer right Leibniz algebras. Then the basic property usually takes the form $x \cdot (y \cdot z) = (x \cdot y) \cdot z - (x \cdot z) \cdot y$. This identity is equivalent to the identity describing the derivation property for the right multiplication r_z by the element z but it seems to be less natural than the one based on the algebra derivation. Moreover, the authors often denote the Leibniz algebras multiplication operation (similarly to the Lie algebra case) by $[x, y]$. Here we use precisely this notation for the Leibniz algebra multiplication operation.

It is easy to verify that a left Leibniz algebra left multiplication set constitute a Lie algebra (see the results given below) that we denote by $\mathrm{ad}^l(L)$ (similar result holds true for the right Leibniz algebras).

Consider the simplest examples of the Leibniz algebras in dimensions 1 and 2.

Assume that $\dim_k(L) = 1$ and a be a nonzero element of L. If $[a, a] = 0$, then L is an Abel Lie algebra. If $[a, a] \neq 0$, then $[a, a] = \alpha a$ for some nonzero $\alpha \in k$. Then the Leibniz algebra identity (both in left and right forms) leads us to a contradiction. Hence there exists only one 1-dimensional Leibniz algebra, i.e., a Lie algebra with the trivial (zero) multiplication.

Assume now that $\dim_k(L) = 2$. It is then easy to verify that there exist only four (up to some isomorphism) left Leibniz algebras:

1) two Lie algebras: the Abel a_2 and the solvable r_2 given in some suitable base a, b by the relations $[a, b] = -[b, a] = b$;

2) two Leibniz algebras that are not Lie algebras defined in some suitable base a, b by the following relations (we give only the case of nonzero multiplications):

(i) $[b, b] = a$,

(ii) $[b, a] = a, [b, b] = a$.

Note that algebra (i) is both left and right (since it is commutative) and algebra (ii) is only left; this can be checked by direct identities calculation. It seems easier to apply the Leibniz algebra identities corollaries that we describe. Namely, we have $[[b, b], b] = [a, b] = 0$ for the algebra of (ii) (as it should be for any left Leibniz algebra), but $[b, [b, b]] = [b, a] = a$, and for the right Leibniz algebra we should have zero. So this 2-dimensional algebra is not a left Leibniz algebra.

The left Leibniz algebra identity allows different identities-corollaries. Let us give a useful example. Consider the basic identity together with itself under the argument permutation:

$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$,

$[b, [a, c]] = [[b, a], c] + [a, [b, c]]$ or $[a, [b, c]] = [b, [a, c]] - [[b, a], c]$.

Comparison of this equalities yields the identity $[[a, b], c] = -[[b, a], c]$. In other words, the operation $[-, -]$ is not initially skew-symmetric in a left Leibniz algebra but it turns into skew-symmetric one under multiplication by any element from the right. Similar fact holds true for skew-symmetry on the right Leibniz algebras with natural multiplication by some element from the left.

In particular, we have $[[a, a], b] = 0$ for any a, b. The right Leibniz algebras allow similar identities with permutation of the relative multipliers. So we always have $[a, [b, b]] = 0$ in the case of the right Leibniz algebra.

Note one more useful equality: $l_x l_y - l_y l_x = l_{[x,y]}$. It is the result of the transformation $l_x l_y(a) = [x, [y, a]]$ into $[[x, y], a] + [y, [x, a]]$ applied to the basic identity. Thus, the linear operator l_x set is a

Lie algebra with respect to the commutation operation for a left Leibniz algebra (it is clear that the commutation operation is skew-symmetric). The situation is naturally similar for the right Leibniz algebras (here $r_{yx} = r_x r_y - r_y r_x$).

Let L be an arbitrary left Leibniz algebra. Consider its subset consisting of the elements of the type $[a, a]$ for any $a \in L$. This subset generates the linear subspace $\mathrm{Ker}(L)$. Another possible name of this set is given below. It is not only an ideal of L but also a two-sided, Abel and central from the left one, i.e., the multiplication $[\mathrm{Ker}(L), L]$ of this ideal by L vanishes. It is also a right ideal since $[a, [b, b]] = [a + [b, b], a + [b, b]] - [a, a]$. But the ideal $\mathrm{Ker}(L)$ is not necessarily central from the right. This ideal under certain disguise first appeared in [2].

The ideal $\mathrm{Ker}(L)$ always differs from L because it is Abel, and since any Abel Leibniz algebra is the Lie one, we have $\mathrm{Ker}(L) = \{0\}$.

The factor-algebra $L/\mathrm{Ker}(L)$ is clearly a nonzero Lie algebra, moreover, $\mathrm{Ker}(L)$ is the smallest of all the L ideals whose factor-algebras are Lie algebras. Hence we consider $\mathrm{Ker}(L)$ as some "non-Lie kernel" and call it a "liezator" of L. We can also describe the ideal $\mathrm{Ker}(L)$ as a span of all elements of the type $[x, y] + [y, x]$. We say that a factor-algebra $L/\mathrm{Ker}(L)$ is a liezation of L an denote it by $L^\star$. There exists a natural Lie algebra $L^\star$ action on the vector space $\mathrm{Ker}(L)$ (the multiplication on which is trivial). The factorization by $\mathrm{Ker}(L)$ then erases the difference between right and left Leibniz algebras, i.e., they define a Lie algebra structure on $L/\mathrm{Ker}(L)$.

If a Leibniz algebra L is commutative (i.e., $[x, y] = [y, x]$), then the subset $[x, y] + [y, x]$ coincides with the commutant $[L, L] = \{[x, y] \mid x, y \in L\}$. Then the algebra L liezation is simply the Abel Lie algebra $L/[L, L]$. Commutative Lie algebras are nilpotent Leibniz algebras of nilpotence class 2.

Consider the left center $Z^l(L) = \{x \in L \mid [x, L] = 0\}$. Similarly we can introduce the right center $Z^k(L)$, moreover, both centers can be constructed for both left and right Leibniz algebras. Since $[[x, y], L] = -[[y, x], L]$, $Z^l(L)$ is a two-sided ideal for any left Leibniz algebra. The right center $Z^k(L)$ is a subalgebra since $[x, [u, v]] = [[x, u], v] + [u, [x, v]] = 0$ for all $x \in L$, $u, v \in Z^r(L)$. In the general case the right and left ideals do not coincide, they can even be of different dimensions. Due to the given above facts $\mathrm{Ker}(L) \subset Z^l(L)$. Hence $L/Z^l(L)$ is a Lie algebra isomorphic to the Lie algebra $\mathrm{ad}^l(L)$ constructed above.

Leibniz algebras naturally allow a lot of notions from Lie algebras theory. For example, we may consider the commutant series $D^n(L)$: $D^1(L) = [L, L]$, $D^{k+1}(L) = [D^k(L), D^k(L)]$. Note that the members of this series are two-sided ideals for left and right Leibniz algebras: this is an easy consequence of the Leibniz identity. A Leibniz algebra is solvable if the commutant series goes all the way down to $\{0\}$ for some finite step. It seems easy to verify that the Leibniz algebra solvable ideals sum is also a solvable ideal. Hence, a Leibniz algebra L possesses a maximal solvable ideal R comprising all the other solvable ideals and called a Leibniz algebra L radical. Since $\mathrm{Ker}(L)$ is an ideal with trivial multiplication, this ideal belongs to the radical R of the Leibniz algebra L.

Consider now the decreasing series $C^n(L)$: $C^1(L) = [L, L]$, $C^{k+1}(L) = [L, C^k(L)]$. Despite certain asymmetry of this definition (we perform the multiplication by L from the left) the elements of the series are two-sided ideals, since by no matter what left or right Leibniz identity we have $[C^k(L), L] = [L, C^k(L)]$. So $C^k(L)$ consists of linear k-element products combinations with an arbitrary brackets positioning. In particular, it follows that $[C^p(L), C^q(L)] \subset C^{p+q}(L)$. We can similarly define the upper central series also consisting of two-sided ideals.

A Leibniz algebra is nilpotent if its decreasing central series attains zero with finite number of steps. Note that by the nilpotent Leibniz algebra definition its left and right centers are nonzero.

Proposition 1. *Any Leibniz algebra L possesses a maximal nilpotent ideal comprising all the algebra L nilpotent ideals.*

Proof. Let I be some nilpotent ideal of L. Then the sum $I + \mathrm{Ker}(L)$ is also a nilpotent ideal since the kernel $\mathrm{Ker}(L)$ is left central. So it suffices to consider only nilpotent ideals containing $\mathrm{Ker}(L)$.

Consider a liezation epimorphism $\mathcal{L} : L \to L/\mathrm{Ker}(L) = L^\star$. It is well-known that the Lie algebra $L/\mathrm{Ker}(L)$ possesses a maximal nilpotent ideal. Its inverse image under the mapping $\mathcal{L}$ is also nilpotent due to $\mathrm{Ker}(L)$ centrality and is also maximal due to the given above facts on the maximal nilpotent ideals of L. $\square$

A Leibniz algebra L nilradical is a maximal nilpotent ideal of L. It exists by Proposition 1. By definition a nilradical is a characteristic ideal, i.e., it does not change under all the Leibniz algebra L automorphisms. Clearly this ideal belongs to the Leibniz algebra radical and is a nilradical of this solvable radical. A nilradical comprise the right center together with the ideal $\mathrm{Ker}(L)$.

Consider also a normalizer concept. The left normalizer $N_L^l(U)$ of the Leibniz algebra L subset $U \subset L$ is a set of elements $a \in L$ such that $[a, U] \subset U$. The right normalizer $N_L^r(U)$ requires $[U, a] \subset U$. Any left Leibniz algebra left normalizer is a subalgebra but not any right normalizer has this property.

Proposition 2 (Engel's theorem for Leibniz algebras). *If all left multiplication operators l_x in the left Leibniz algebra L are nilpotent, then the algebra L is itself nilpotent. In particular, all the left multiplication operators possess the common eigenvector with zero eigenvalue and in some base all the operators l_x matrices are nilpotent upper triangular.*

Proof. The Lie algebra $L/\,\mathrm{Ker}(L)$ has the classical Engel's theorem. So the statement of Proposition 2 holds for any such algebra. But $\mathrm{Ker}(L)$ is a central ideal so its action by left multiplications on L is trivial. Thus the Leibniz algebra L is nilpotent as a central extension of a nilpotent Lie algebra.

Left multiplications matrices upper nilpotent forms also follow from the above by induction on the algebra dimension. □

First the proof of this statement was given in [5]. We have a stronger result [6]: If all the left multiplications are nilpotent in a left Leibniz algebra, then so are all the right multiplications and in the appropriate bases the right and left multiplication operators matrices are upper triangular nilpotent. The proof of this statement is based itself on the identity $(r_x)^n = (-1)^n r_x (l_x)^{n-1}$ proved by induction. In the case of $n = 2$ this relation follows from the identity given above: $[[x, a], a] = -[[a, x], a]$. Then we consider $(r_x)^3(a) = [[[a, x], x], x]$, transform this relation in a similar way and apply the induction on n. So the nilpotences "from the left" and "from the right" are mutually equivalent. It also follows from the fact on the similarity of $[C^k(L), L]$ and $[L, C^k(L)]$.

Corollary 1. The normalizers (both left and right) of the Leibniz subalgebra M in a nilpotent Leibniz algebra L do not coincide with the subalgebra M. They strictly contain this subalgebra.

Proof. Let us start with the left centralizer. The considerations here coincide with those for the Lie algebra case, i.e., we consider the action on the factor-space L/M induced by the left multiplications. The right multiplications operator set constitute a Lie algebra nilpotent in our case. By the classical Engel's theorem for Lie algebras this action has a common eigenvector with zero eigenvalue. A representative of this vector in L belongs to $N_L(M)$ but not to M. We give an analogous consideration for the right centralizer but with the stronger Engel's theorem variant of [6]. □

The following statement is also a well-known result for Lie algebras (see, e.g., [7]).

Proposition 3. *Let L be a nilpotent Leibniz algebra. Then a subspace $V \subset L$ generates the Leibniz algebra L if and only if $V + [L, L] = L$.*

Proof. Assume that $V + [L, L] = L$. Denote by M the subalgebra of L generated by the subspace V. We prove that $M = L$ by induction on $\mathrm{codim}(M)$ and $\dim(L)$. This statement is obvious for $\mathrm{codim}(M) = 0$ and $\dim(L) = 0$.

Assume that $\mathrm{codim}(M) > 0$. Consider the left normalizer $M' = N_L(M)$. It is strictly greater than M since L is nilpotent by assumption. Hence $\mathrm{codim}(M') < \mathrm{codim}(M)$ and by the induction hypothesis $M' = L$. So M is a left ideal of L complementary to $[L, L]$. Next we prove that M is a right and thus a two-sided ideal. In order to do this we apply the following

Lemma. *Let I be a left Leibniz algebra L left ideal. Then the right normalizer $N_L^r(I)$ is a subalgebra of L.*

Proof. Consider two elements $n, n' \in N_L^r(I)$. We have $[I,[n,n']] = [[I,n],n'] + [n,[I,n']]$. Both the subspaces $[I,n]$ and $[I,n']$ belong to I by the normalizer definition and $[n,I] \subset I$ since I is a left ideal of L. We then have the inclusion $[I,[n,n']] \subset I$, i.e., $[n,n'] \in N_L^r(I)$ and $N_L^r(I)$ is a subalgebra of L. □

In our case $I = M$ is a left ideal of L. So by Lemma $N_L^r(M)$ is a subalgebra of L strictly containing M by Corollary 1. We then proceed with induction by $\operatorname{codim}_L(M)$. Since $\operatorname{codim} L(N_L^r(M)) < \operatorname{codim}_L(M)$, the induction hypothesis yields $N_L^r(M) = L$, i.e., M is a right ideal of L.

So we may assume that M is a two-sided ideal of L. We now apply the induction by $\dim(L)$.

Consider the natural liezation epimorphism $\mathcal{L} : L \to L/\operatorname{Ker}(L) = L^\star$. It is clear that $\mathcal{L}([L,L]) = [L^\star, L^\star]$. So we may apply one of Proposition 3 statements to the subspace $\mathcal{L}(V)$ in the case of Lie algebras. But then $V^\star$ generates the Lie algebra $L^\star$, hence $M + \operatorname{Ker}(L) = L$.

Now for the Lie algebra L commutant we have $[L,L] = [M + \operatorname{Ker}(L), M + \operatorname{Ker}(L)] = [M,M] + [M,\operatorname{Ker}(L)] + [\operatorname{Ker}(L), M] + [\operatorname{Ker}(L),\operatorname{Ker}(L)]$. But $[\operatorname{Ker}(L),\operatorname{Ker}(L)]$ and $[\operatorname{Ker}(L), M]$ vanish due to left centrality of Z. We also have the inclusion $[M,\operatorname{Ker}(L)] \subset M$ since M is a right ideal of L. Thus, $[L,L] = [M,M] + M \subset M$. Hence, M contains $[L,L]$. Since $M + [L,L] = L$, we obtain $M = L$.

Conversely if a subspace V generates a Leibniz algebra L, then its image under the natural epimorphism $L \to L/[L,L]$ generates the Abel Leibniz algebra $L/[L,L]$. Since the multiplication of $L/[L,L]$ is trivial, the subspace of this algebra generates the algebra itself only if this subspace coincides with the whole algeba. But this is equivalent to the equality $V + [L,L] = L$. □

It seems interesting to note that not all even simple and well-known statements on nilpotent Lie algebras can be carried over the Leibniz algebras. For example, we have the following simple stetement on dimension 2 and higher Lie algebras: *the commutant codimension is* ≥ 2. This fact is false for Leibniz algebras (though not only for nilpotent but also for solvable Leibniz algebras we have $\operatorname{codim}_L[L,L] > 0$). For example, the 2-dimensional Leibniz algebra $\langle a,b\rangle$ with $[a,a] = b$ (see above) is nilpotent but its commutant codimension equals 1. This is true since its liezation is 1-dimensional and the statement is false for 1-dimensional Lie algebras. Hence we obtain the following

Corollary 2. If a Leibniz algebra L is nilpotent and $\operatorname{codim}_L([L,L]) = 1$, then the algebra L is generated by 1 element.

Thus in the case of $\operatorname{codim}_L([L,L]) = 1$ a nilpotent algebra is in some sense "cyclic". These algebras investigation is a special feature of the Leibniz algebras theory that does not have a Lie algebra analog. Such "cyclic" nilpotent algebras can be completely and explicitly described.

Note one more useful corollary of Proposition 3.

Corollary 3. The minimal number of a nilpotent Leibniz algebra L generators equals $\dim L/[L,L]$.

Proposition 4. *Let R be a Leibniz algebra L radical and N be a nilradical of L. Then* $[L,R] \subset N$.

Proof. This statement is well-known for Lie algebras L. Assume now that L is an arbitrary Leibniz algebra. Then $\operatorname{Ker}(L) \subset N \subset R$. Put $L^\star = L/\operatorname{Ker}(L)$ (the algebra L liezation), $R^\star = R/\operatorname{Ker}(L)$, $N^\star = N/\operatorname{Ker}(L)$. The radical and nilradical definitions imply that $R^\star$ and $N^\star$ are radical and nilradical of the Leibniz algebra $L^\star$, respectively (here we note that $\operatorname{Ker}(L) \subset Z^l(L)$). Since $L^\star$ is a Lie algebra, we have $[L^\star, R^\star] \subset N^\star$. And since $\operatorname{Ker}(L) \subset N \subset R$, we conclude that $[L,R] \subset N$. □

Corollaries 4 and 5 first appeared in [5]. We give a simpler proof of Corollary 5.

Corollary 4. $[R,R] \subset N$. In particular, $[R,R]$ is nilpotent.

Corollary 5. A Leibniz algebra L is solvable if an only if $[L,L]$ is nilpotent.

Proof. One way the statement of Corollary 5 is contained in Corollary 4. Consider the converse.

Assume that $[L, L]$ is nilpotent. Consider the radical R of the algebra L. The Lie algebra L/R is a semisimple Lie algebra since it does not possess any solvable ideals. A semisimple Lie algebra coincides with its commutant. Hence in the case of a nilpotent $[L, L]$ this semisimple algebra is trivial. Thus L is solvable. □

The following statement of [8] was proved for the right multiplication operators in the right Lie algebra. But the proof given there is cumbersome so we give here a simple one.

Proposition 5. *A Leibniz algebra (right and left) multiplications are degenerate.*

Proof. The statement is trivial in the case of Lie algebras because then $[x, x] = 0$ for any $x \in L$, that is the vector x itself is an eigenvector for the 0 eigenvalue.

Now in arbitrary Leibniz algebra $\mathrm{Ker}(L)$ is a two-sided ideal different from L (see above) and $L/\mathrm{Ker}(L)$ is a Lie algebra of a positive dimension. Let x be some nonzero element of L. The subspace $\mathrm{Ker}(L)$ is invariant for the linear operators l_x and r_x. The linear operators induced by these multiplications on the factor-space $L/\mathrm{Ker}(L)$ are degenerate (their determinants vanish). Then they are also degenerate on the whole space L. □

The Leibniz algebras structure can be described by an analog [9] of the Lie algebras Levi theorem for which we give a somewhat modified proof.

Proposition 6 (Levi theorem for Leibniz algebras)**.** *Any Leibniz algebra L allows existence of a subalgebra S (being a semisimple Lie algebra) such that we have the decomposition $L = S + R$, here R is the radical and $S \cap R = \{0\}$.*

Proof. Consider the Lie algebra $L^\star = L/\mathrm{Ker}(L)$. By the classical Levi theorem we have a semisimple subalgebra $S^\star \subset L^\star$ that defines a decomposition into a semidirect sum $L^\star = S^\star + R^\star$. Here $R^\star$ is the Lie algebra $L^\star$ radical. If R is the Leibniz algebra L radical and $\mathcal{L} : L \to L^\star$ is the natural liezation morphism, then it is easy to conclude that $\mathcal{L}(R) = R^\star$. Put $F = \mathcal{L}^{-1}(S^\star)$. Then F is a Leibniz subalgebra of L containing $\mathrm{Ker}(L)$. Moreover, the factor-algebra $F/\mathrm{Ker}(L)$ is isomorphic to the semisimple Lie algebra $S^\star$.

Now the Abel Leibniz algebra $\mathrm{Ker}(L)$ is an $S^\star$-module for the Lie algebra $S^\star$. Semisimplicity of the Lie algebra $S^\star$ implies that the subalgebra F (which we consider as the extension of $S^\star$ with $\mathrm{Ker}(L)$) splits due to the Whitehead lemma for the semisimple Lie algebras. Thus F contains a subalgebra (a semisimple Lie algebra) complementary to the Abel ideal $\mathrm{Ker}(L)$. It is clear that $S + R = L$ and we arrive at the desired decomposition. □

The examples of [9] show that the subalgebra S is not unique. The minimal examples' Leibniz algebras dimension equals 6. In the case of the Lie algebras the semisimple factor is unique up to conjugation by the Maltsev theorem.

Proposition 7. *Assume that L is a Leibniz algebra such that its left center is central and right, i.e., we have $[L, Z^l(L)] = \{0\}$. If the Lie algebra $L/Z^l(L)$ is semisimple, then L is a reductive Lie algebra (its radical is central).*

Proof. Consider the Levi decomposition $L = S + R$. Since by assumption the Lie algebra $L/Z^r(L)$ is semisimple, we have $R = Z^l(L)$. The left action of the subalgebra S on $Z^l(L)$ is trivial. Since $[L, Z^l(L)] = \{0\}$, we have also a trivial right action of S on $Z^l(L)$. Hence S is permutable with the radical and the Levi decomposition for L is direct. The radical $R = Z^l(L)$ is Abel one so L is a reductive Lie algebra. □

This statement of Proposition also can be found in [5] for the special case of the simple classical Lie algebra S.

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