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Characteristic Boundary-Value Problem for a System of First-Order Partial Differential Equations with Shifted Arguments of the Desired Function

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Abstract—For a system of first-order partial differential equations of a form not studied earlier we consider a variant of Goursat problem and prove the existence and uniqueness of the solution to the problem.

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The differential equations, both ordinary and partial, whose arguments have some shift were studied in a few papers (e.g., [1–10]). Various authors call such equations functional differential and with delayed (or deviating) argument. In this paper the variant of shift from [9, 10] is for the first time applied to a system of partial differential equations of the first order.

Let $D = \{0 < x < 1, 0 < y < 1\}$ be a domain. Consider a system of equations

$$\begin{aligned} \alpha_{11}u_{1x} + \alpha_{12}v_{1x} &= a^1_{11}u_1 + a^1_{12}u_2 + a^2_{11}v_1 + a^2_{12}v_2 + f_1; \\ \alpha_{21}u_{2y} + \alpha_{22}v_{2y} &= a^1_{21}u_1 + a^1_{22}u_2 + a^2_{21}v_1 + a^2_{22}v_2 + f_2, \end{aligned} \tag{1}$$

$\alpha_{ij}, a^k_{ij}, f_i \in C(\overline{D})$, $i, j, k = 1, 2$.

Here $v_i(x, y)$ are defined, respectively, with respect to $u_i(x, y)$ by the formula

$$v_i(x, y) \equiv u_i[\lambda(y), \mu(x)], \tag{2}$$

where each of the functions $\lambda, \mu \in C^1[0, 1]$ maps a segment $[0, 1]$ to itself preserving the direction of the motion. In particular,

$$\lambda(0) = \mu(0) = 0, \quad \lambda(1) = \mu(1) = 1,$$

and the second application of the transformation (λ, μ) returns the coordinates (x, y) to their initial position:

$$v_i[\lambda(y), \mu(x)] \equiv u_i\{\lambda[\mu(x)], \mu[\lambda(y)]\} \equiv u_i(x, y). \tag{3}$$

These properties are possessed by, e.g., the pairs $\lambda = y$, $\mu = x$ and $\lambda = (e^y - 1)/(e - 1)$, $\mu = \ln[1 + (e - 1)x]$, where e is the base of the natural logarithm.

Let us define a regular solution of (1) in D as a solution of the class $u_1, u_2, u_{1x}, u_{1y} \in C(D)$.

Goursat Problem. Find a regular in D solution to system (1), continuously extensible to the boundary of D, satisfying the conditions

$$u_1(0, y) = \varphi_1(y), \quad u_2(x, 0) = \psi_2(x), \quad x, y \in [0, 1]. \tag{4}$$

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This set-up encapsulates a well-known [1] Goursat problem with conditions (4) for the system

$$u_{1x} = a_{11}^1 u_1 + a_{12}^1 u_2 + f_1;\ \ u_{2y} = a_{21}^1 u_1 + a_{22}^1 u_2 + f_2.$$

Now back to solution of our problem. In the first Eq. (1) assume $x = t$ and integrate it from 0 to x:

$$(\alpha_{11}u_1)(x,y) + \alpha_{12}(x,y)u_1(\lambda(y),\mu(x)) = \omega_1(x,y) + (\alpha_{12}v_1)(0,y)$$
$$+ \int_0^x (((\alpha_{11t} + a_{11}^1)u_1)(t,y) + (a_{12}^1 u_2)(t,y) + (\alpha_{12t} + a_{11}^2)(t,y)u_1(\lambda(y),\mu(t))$$
$$+ a_{12}^2(t,y)u_2(\lambda(y),\mu(t)))dt, \quad (5)$$

where $\omega_1(x,y) = \alpha_{11}(0,y)\varphi_1(y) + \int\limits_0^x f_1(t,y)dt$. Let us denote the whole right-hand side of (5) as $g_1(x,y)$, then

$$\alpha_{i1}(x,y)u_i(x,y) + \alpha_{i2}(x,y)v_i(x,y) = g_i(x,y), \quad (6)$$

where $i = 1$. In the latter equation assume $x = \lambda(y)$, $y = \mu(x)$, and with (3) we get

$$\alpha_{i2}(\lambda(y),\mu(x))u_i(x,y) + \alpha_{i1}(\lambda(y),\mu(x))v_i(x,y) = g_i(\lambda(y),\mu(x)). \quad (7)$$

From relations (6), (7), being a system of linear algebraic equations, one can calculate u_i, v_i, if the determinant

$$\Delta_i(x,y) = \begin{vmatrix} \alpha_{i1}(x,y) & \alpha_{i2}(x,y) \\ \alpha_{i2}(\lambda(y),\mu(x)) & \alpha_{i1}(\lambda(y),\mu(x)) \end{vmatrix} \quad (8)$$

is nonzero. In this case by Cramer's rule we get

$$u_i(x,y) = \frac{1}{\Delta_i(x,y)}(g_i(x,y)\alpha_{i1}(\lambda(y),\mu(x)) - g_i(\lambda(y),\mu(x))\alpha_{i2}(x,y)), \quad (9)$$

$$v_i(x,y) = \frac{1}{\Delta_i(x,y)}(g_i(\lambda(y),\mu(x))\alpha_{i1}(x,y) - g_i(x,y)\alpha_{i2}(\lambda(y),\mu(x))). \quad (10)$$

From (8), in view of (3), we have $\Delta_i(\lambda(y),\mu(x)) \equiv \Delta_i(x,y)$. Assuming in (10) $y = \mu(x)$, $x = \lambda(y)$, in view of (3) and (2), we get (9). Hence, formulas (9), (10) give the same solutions. Thus, the unique solution to Eq. (6) is (9). Let us now substitute into (9) the value $g_1(x,y)$, being the right-hand side of (5), so we get

$$u_1(x,y) = \frac{1}{\Delta_1(x,y)}(((\alpha_{12}v_1)(0,y) + \int_0^x (((\alpha_{11t} + a_{11}^1)u_1)(t,y)$$
$$+ (a_{12}^1 u_2)(t,y) + (\alpha_{12t} + a_{11}^2)(t,y)u_1(\lambda(y),\mu(t))$$
$$+ a_{12}^2(t,y)u_2(\lambda(y),\mu(t)))dt)\alpha_{11}(\lambda(y),\mu(x))$$
$$- ((\alpha_{12}v_1)(0,\mu(x)) + \int_0^{\lambda(y)} (((\alpha_{11t} + a_{11}^1)u_1)(t,\mu(x)) + (a_{12}^1 u_2)(t,\mu(x)) + (\alpha_{12t}$$
$$+ a_{11}^2)(t,\mu(x))u_1(x,\mu(t)) + a_{12}^2(t,\mu(x))u_2(x,\mu(t)))dt)\alpha_{12}(x,y)) + F_1(x,y), \quad (11)$$

where $F_1(x,y) = \frac{1}{\Delta_1(x,y)}(\omega_1(x,y)\alpha_{11}(\lambda(y),\mu(x)) - \omega_1(\lambda(y),\mu(x))\alpha_{12}(x,y))$.

Let us conduct a similar argument for $u_2(x,y)$. Namely, take the second Eq. (1), assume $y = \tau$ for it and integrate it from 0 to y:

$$(\alpha_{21}u_2)(x,y) + \alpha_{22}(x,y)u_2(\lambda(y),\mu(x)) = \omega_2(x,y) + (\alpha_{22}v_2)(x,0)$$
$$+ \int_0^y (a_{21}^1 u_1(x,\tau) + ((\alpha_{21\tau} + a_{22}^1)u_2)(x,\tau)$$

$$+ a_{21}^2(x,\tau)u_1(\lambda(\tau),\mu(x)) + (\alpha_{22\tau} + a_{22}^2)(x,\tau)u_2(\lambda(\tau),\mu(x)))d\tau, \quad (12)$$

where $\omega_2(x,y) = \alpha_{21}(x,0)\psi_2(x) + \int\limits_0^y f_2(x,\tau)d\tau$. Let us denote the whole right-hand side of (12) as $g_2(x,y)$, then (12) will take the form of (6) for $i=2$. All the arguments between formulas (6), (11), are repeated for $i=2$, hence we do not deploy them here. Now let us substitute into (9), assuming $i=2$, the value $g_2(x,y)$, being the right-hand side of (12), so we get

$$u_2(x,y) = \frac{1}{\Delta_2(x,y)}(((\alpha_{22}v_2)(x,0) + \int_0^y ((a_{21}^1u_1)(x,\tau) + ((\alpha_{21\tau} + a_{22}^1)u_2)(x,\tau)$$
$$+ a_{21}^2(x,\tau)u_1(\lambda(\tau),\mu(x)) + (\alpha_{22\tau} + a_{22}^2)(x,\tau)u_2(\lambda(\tau),\mu(x)))d\tau)\alpha_{21}(\lambda(y),\mu(x))$$
$$- ((\alpha_{22}v_2)(\lambda(y),0) + \int_0^{\mu(x)} ((a_{21}^1u_1)(\lambda(y),\tau) + ((\alpha_{21\tau} + a_{22}^1)u_2)(\lambda(y),\tau)$$
$$+ a_{21}^2(\lambda(y),\tau)u_1(\lambda(\tau),y) + (\alpha_{22\tau} + a_{22}^2)(\lambda(y),\tau)u_2(\lambda(\tau),y))d\tau)\alpha_{22}(x,y)) + F_2(x,y), \quad (13)$$

where $F_2(x,y) = \frac{1}{\Delta_2(x,y)}(\omega_2(x,y)\alpha_{21}(\lambda(y),\mu(x)) - \omega_2(\lambda(y),\mu(x))\alpha_{22}(x,y))$.

Thus, we get a loaded system (11), (13), containing, except the desired unknowns, $v_1(0,y)$ and $v_2(x,0)$ as well. For their determination in (5) assume $y=0$, and in (12) assume $x=0$. We get a system of integral equations

$$(\alpha_{11}u_1)(x,0) - \int_0^x (((\alpha_{11t} + a_{11}^1)u_1)(t,0)dt - \int_0^x (a_{12}^2v_2)(t,0)dt = \omega_1(x,0)$$
$$- (\alpha_{12}v_1)(x,0) + (\alpha_{12}v_1)(0,0) + \int_0^x ((a_{12}^1u_2)(t,0) + (\alpha_{12t} + a_{11}^2)v_1(t,0))dt, \quad (14)$$

$$(\alpha_{21}u_2)(0,y) - \int_0^y ((\alpha_{21\tau} + a_{22}^1)u_2)(0,\tau)d\tau - \int_0^y (a_{21}^2v_1)(0,\tau)d\tau = \omega_2(0,y)$$
$$- (\alpha_{22}v_2)(0,y) + (\alpha_{22}v_2)(0,0) + \int_0^y ((a_{21}^1u_1)(0,\tau) + ((\alpha_{22\tau} + a_{22}^2)v_2)(0,\tau))d\tau.$$

Denote

$$U(x,y) = \begin{bmatrix} u_1(x,0) \\ u_2(0,y) \end{bmatrix}, \quad A_1(x,y) = \begin{bmatrix} \alpha_{11}(x,0) & 0 \\ 0 & \alpha_{21}(0,y) \end{bmatrix},$$
$$A_2(x,y) = \begin{bmatrix} (\alpha_{11x} + a_{11}^1)(x,0) & 0 \\ 0 & (\alpha_{21y} + a_{22}^1)(0,y) \end{bmatrix}, \quad B(x,y) = \begin{bmatrix} 0 & a_{12}^2(x,0) \\ a_{21}^2(0,y) & 0 \end{bmatrix}.$$

We assume $\int\limits_0^{xy} \dots dt\,\tau$ means that we integrate the first equation by t from 0 to x, and the second by τ from 0 to y. Then system (14) takes the form of a vector-matrix equation

$$(A_1U)(x,y) - \int_0^{xy} (A_2U)(t,\tau)dt\tau - \int_0^{xy} (BU)(\lambda(\tau),\mu(t))dt\tau = \Omega(x,y),$$

where $\Omega(x,y)$ is a column vector, whose elements are completely known right-hand sides of the equations of system (14). Next, use the technique that the author applied in [12] for another problem. Namely, let us transform the integral equations to the form solved with respect to the desired function. To do so, we stipulate $\det A_1(x,y) \neq 0$.

Thus, we will solve the system

$$U(x,y) - A_1^{-1}(x,y)\int_0^{xy}(A_2U)(t,\tau)dt\tau - A_1^{-1}(x,y)\int_0^{xy}(BU)(\lambda(\tau),\mu(t))dt\tau = A_1^{-1}(x,y)\Omega(x,y).$$

In the operator form (introducing an understandable notation K), it looks in the following way:

$$U - KU = A_1^{-1}\Omega. \tag{15}$$

For matrices, let us use the norm ([13], P. 410) $\|A\| = \max\limits_{1\le i\le 2}\sum\limits_{k=1}^{2}|a_{ik}|$. Let the estimate $\|A_1^{-1}A_2\| < M$, $\|A_1^{-1}B\| < M$ be fulfilled, where $M > 0$ is a certain constant. Let us check that the operator K is continuous on the set of defined on D continuous vector functions. Let φ_1, φ_2 be continuous vector functions. Then

$$\|K\varphi_1 - K\varphi_2\| < 2M(x+y)\|\varphi_1 - \varphi_2\| < 4M\|\varphi_1 - \varphi_2\|.$$

Obviously, for any $\varepsilon > 0$ there exists $\delta = \varepsilon/(4M)$ such that from the condition $\|\omega_1 - \omega_2\| < \delta$ it follows $\|K\omega_1 - K\omega_2\| < \varepsilon$. The continuity of the operator K is proved.

Let us show that a certain power of K is a shrinking mapping. Indeed,

$$\|K^2\varphi_1 - K^2\varphi_2\| < \frac{(4M)^2}{2!}\|\varphi_1 - \varphi_2\|, \dots,$$
$$\|K^n\varphi_1 - K^n\varphi_2\| < \frac{(4M)^n}{n!}\|\varphi_1 - \varphi_2\|.$$

For some n we have $\frac{(4M)^n}{n!} < 1$, thus, K^n is shrinking. It is known ([14], P. 84), if K is a continuous mapping of a dense metric space to itself such that its certain power is shrinking, then the equation $\omega - K\omega = 0$ has a unique zero solution. Then (15) has a unique solution in the class of continuous vector functions. Thus, we proved that the elements $u_1(x,0)$, $u_2(0,y)$, and so, $v_1(0,y)$, $v_2(x,0)$, as well, are defined in a unique way.

Now consider system (11), (13) for finding $u_1(x,y)$, $u_2(x,y)$. Introduce the vector

$$U_1(x,y) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}.$$

Then

$$U_1(\lambda(y),\mu(x)) = \begin{bmatrix} u_1(\lambda(y),\mu(x)) \\ u_2(\lambda(y),\mu(x)) \end{bmatrix} = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix}.$$

Denote

$$C_1(t,y) = \begin{bmatrix} (\alpha_{11t}(t,y) + a_{11}^1(t,y)) & a_{12}^1(t,y) \\ 0 & 0 \end{bmatrix},$$

$$C_2(x,\tau) = \begin{bmatrix} 0 & 0 \\ a_{21}^1(x,\tau) & (\alpha_{21\tau}(x,\tau) + a_{22}^1(x,\tau)) \end{bmatrix},$$

$$C_3(t,\mu(x)) = \begin{bmatrix} (\alpha_{11t}(t,\mu(x)) + a_{11}^1(t,\mu(x))) & a_{12}^1(t,\mu(x)) \\ 0 & 0 \end{bmatrix},$$

$$C_4(\lambda(y),\tau) = \begin{bmatrix} 0 & 0 \\ a_{21}^1(\lambda(y),\tau) & (\alpha_{21\tau}(\lambda(y),\tau) + a_{22}^1(\lambda(y),\tau)) \end{bmatrix},$$

$$D_1(t,y) = \begin{bmatrix} (\alpha_{12t}(t,y) + a_{11}^2(t,y)) & a_{12}^2(t,y) \\ 0 & 0 \end{bmatrix},$$

$$D_2(x,\tau) = \begin{bmatrix} 0 & 0 \\ a_{21}^2(x,\tau) & (\alpha_{22\tau}(x,\tau) + a_{22}^2(x,\tau)) \end{bmatrix},$$

$$D_3(t,\mu(x)) = \begin{bmatrix} (\alpha_{12t}(t,\mu(x)) + a_{11}^2(t,\mu(x))) & a_{12}^2(t,\mu(x)) \\ 0 & 0 \end{bmatrix},$$

$$D_4(\lambda(y),\tau) = \begin{bmatrix} 0 & 0 \\ a_{21}^2(\lambda(y),\tau) & (\alpha_{22\tau}^1(\lambda(y),\tau) + a_{22}^2(\lambda(y),\tau)) \end{bmatrix}, \quad \Phi(x,y) = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}.$$

Then (11), (13) can be represented in the vector-matrix form

$$\begin{aligned} U_1(x,y) &= \frac{\alpha_{11}^1(\lambda(y),\mu(x))}{\Delta_1(x,y)} \int_0^x C_1(t,y)U_1(t,y)dt + \frac{\alpha_{21}^1(\lambda(y),\mu(x))}{\Delta_2(x,y)} \int_0^y C_2(x,\tau)U_1(x,\tau)d\tau \\ &- \frac{\alpha_{12}^1(x,y)}{\Delta_1(x,y)} \int_0^{\lambda(y)} C_3(t,\mu(x))U_1(t,\mu(x))dt - \frac{\alpha_{22}^1(x,y)}{\Delta_2(x,y)} \int_0^{\mu(x)} C_4(\lambda(y),\tau)U_1(\lambda(y),\tau)d\tau \\ &+ \frac{\alpha_{11}^1(\lambda(y),\mu(x))}{\Delta_1(x,y)} \int_0^x D_1(t,y)U_1(\lambda(y),\mu(t))dt + \frac{\alpha_{21}^1(\lambda(y),\mu(x))}{\Delta_2(x,y)} \int_0^y D_2(x,\tau)U_1(\lambda(\tau),\mu(x))d\tau \\ &- \frac{\alpha_{12}^1(x,y)}{\Delta_1(x,y)} \int_0^{\lambda(y)} D_3(t,\mu(x))U_1(x,\mu(t))dt - \frac{\alpha_{22}^1(x,y)}{\Delta_2(x,y)} \int_0^{\mu(x)} D_4(\lambda(y),\tau)U_1(\lambda(\tau),y)d\tau + \Phi(x,y), \end{aligned}$$

or in the operator form (introducing the notation K_1)

$$U_1 - K_1U_1 = \Phi(x,y). \tag{16}$$

Then we can prove that the operator K_1 is continuous on the set of the defined on D continuous vector functions, and that for some n K_1^n is shrinking. We skip the detailed proof of this fact to keep the paper small enough. From [14] (P. 84) it follows that (16) has a unique solution. If we substitute this solution into (5), (12), then they will also become identities due to the equivalence of the transformations carried out. Equations themselves (5), (12) were obtained by direct integration of (1) with conditions (4) in mind. So, we can one time differentiate (5) and (12) with respect to x and y, correspondingly, that will lead to identity in (1). Thus, we have

Theorem. *If* $(\Delta_1 \cdot \Delta_2)(x,y) \cdot \alpha_{11}(x,0) \cdot \alpha_{21}(0,y) \neq 0$ *for* $(x,y) \in \overline{D}$ (Δ_1, Δ_2 *is defined by* (8), $i = 1,2$)*, then the solution to problem* Γ *exists and is unique.*

All the the conducted arguments are equally good for the case when the domain D is a rectangle with sides parallel to coordinate axes. A square domain is chosen solely for the purpose of some simplification of the formulas.

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