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The Structure of C^*-Subalgebras of the Toeplitz Algebra Fixed with Respect to a Finite Group of Automorphisms

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Abstract—In this paper we study the C^*-subalgebras of the Toeplitz algebra $\mathcal{T}$, each element of which is fixed relative to finite subgroup of automorphisms of the algebra $\mathcal{T}$. We prove that such subalgebras have a finite family of unitarily equivalent irreducible representations.

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1. INTRODUCTION

In [1], V. A. Arzumanyan proved that a bicyclic semigroup has, up to unitary equivalence, one finite-dimensional irreducible representation and a series of one-dimensional unitary representations parameterized by a unit circle S^1. A C^*-algebra generated by an infinite-dimensional irreducible representation generates the Toeplitz algebra. L. Coburn [2] showed that nonunitary isometric representations of a semigroup of non-negative integers $\mathbb{Z}_+$ generate C^*-algebras that are canonically isomorphic to the Toeplitz algebra. On the other hand, M. A. Aukhadiev and V. Tepoyan [3] obtained a semigroup criterion under which each non-unitary isometric representation is extended to a representation of an inverse semigroup generated by the given semigroup. From this criterion it follows that each isometric representation of a semigroup $\mathbb{Z}_+$ is extended to a representation of bicyclic semigroup $\mathbb{Z}_+^*$. In this paper we introduce a concept of index of an element of the semigroup $\mathbb{Z}_+^*$ and study subsemigroups $\mathbb{Z}_{+,m}^*$ and $\mathbb{Z}_+^*(m)$, $m \in \mathbb{Z}_+$ generated by the elements whose indices have a common divisor m. We show that non-unitary isometric representations of such semigroups generate subalgebras of the Toeplitz algebra $\mathcal{T}_m$ and $\mathcal{T}(m)$ consisting of the elements that are fixed relative to some finite group of automorphisms of rank m. We give a complete description of irreducible representations of C^*-algebra $\mathcal{T}_m$.

2. BICYCLIC INVERSE SEMIGROUPS

A semigroup S is called *inverse*, if for any $a \in S$ there exists a unique inverse element $a^* \in S$ such that the equalities $a^*aa^* = a^*$ and $aa^*a = a$ hold. It follows from the definition that $a^{**} = a$. An element b of the semigroup S is called an *idempotent*, if $b^2 = b$. Idempotents of an inverse semigroup form a commutative subsemigroup in S, which coincides with the set $P_S = \{c \in S : c = c^*\}$.

An inverse semigroup with a unit e is called a *bicyclic* semigroup, if it is generated by one element a and the relation $a^*a = e$. Note that an inverse semigroup generated by representations of a semigroup of non-negative integers is a bicyclic semigroup. Keeping this in mind, we will further denote a bicyclic semigroup by $\mathbb{Z}_+^*$.

It immediately follows from the equality $a^*a = e$ that each element of a bicyclic semigroup has the form $a^m a^{*n}$, where m and n are non-negative integers.

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Define an *index* of an element $b = a^m a^{*n}$ from $\mathbb{Z}_+^*$ as a number $m - n$ and denote it as $\operatorname{ind}(b)$. Note that $\operatorname{ind}(b \cdot c) = \operatorname{ind}(b) + \operatorname{ind}(c)$ for any elements $b, c \in \mathbb{Z}_+^*$.

Consider a homomorphism $\tau : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$, $\tau(b) = aba^*$, being an embedding of the semigroup $\mathbb{Z}_+^*$ into itself. By τ^n let us denote an nth superposition of the mapping τ, i.e., $\tau^n(b) = a^n b a^{*n}$. Obviously, $\tau^n : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$ is also an embedding. Note that $\operatorname{ind}(\tau^n(b)) = \operatorname{ind}(b)$ for any $n \in \mathbb{Z}_+$ and $b \in \mathbb{Z}_+^*$. Fix the integer m. Let $\mathbb{Z}_{+,m}^* = \{b \in \mathbb{Z}_+^* : \operatorname{ind}(b) = k \cdot m,\, k \in \mathbb{Z}\}$. Note that in case $m = 1$ the semigroup $\mathbb{Z}_{+,1}^* = \mathbb{Z}_+^*$. Let $\mathbb{Z}_+^*(m)$ be an inverse semigroup, generated by the element a^m. Obviously, $\mathbb{Z}_{+,m}^*$ and $\mathbb{Z}_+^*(m)$ are inverse subsemigroups of the bicyclic semigroup $\mathbb{Z}_+^*$. The semigroup $\mathbb{Z}_+^*(m)$ is bicyclic as well. Let us establish a connection between the semigroups $\mathbb{Z}_{+,m}^*$ and $\mathbb{Z}_+^*(m)$.

Lemma 1. *The semigroup $\mathbb{Z}_{+,m}^*$ can be represented as*

$$\mathbb{Z}_{+,m}^* = \bigcup_{k=0}^{m-1} \tau^k(\mathbb{Z}_+^*(m)).$$

Proof. Let us show that for any element $b \in \mathbb{Z}_{+,m}^*$ $b \in \tau^l(\mathbb{Z}_+^*(m))$ is true for some $0 \le l \le m-1$. Indeed, if $b \in \mathbb{Z}_{+,m}^*$, then $b = a^{mk+l} a^{*mr+l}$, where $0 \le l \le m-1$, $k, r \in \mathbb{Z}_+$. Then

$$b = a^l a^{mk} a^{*mr} a^{*l} \in \alpha^l(\mathbb{Z}_+^*(m)).$$

The inclusion $\bigcup\limits_{k=0}^{m-1} \tau^k(\mathbb{Z}_+^*(m)) \subset \mathbb{Z}_{+,m}^*$ is obvious. □

3. TOEPLITZ ALGEBRA'S SUBSLAGEBRAS, FIXED RELATIVE TO FINITE GROUP OF AUTOMORPHISMS

Consider a Hilbert space $l^2(\mathbb{Z}_+)$ with a natural orthonormal basis $\{\, e_k\}_{k \in \mathbb{Z}_+}$. Let T be a shift operator on $l^2(\mathbb{Z}_+)$, i.e., acting on the basis in the following way:

$$Te_k = e_{k+1}.$$

Obviously, $T^*T = I$, where T^* is the adjoint operator to the operator T, I is the identity operator and $TT^* = P$ is the projector on $l^2(\mathbb{Z}_+ \setminus \{0\})$. Thus, a semigroup generated by the operators T and T^*, forms an inverse bicyclic semigroup. Each element of this semigroup has a form $T^n T^{*m}$, $n, m \in \mathbb{Z}_+$. We will further call these elements monomials [4], and the number $n - m$ an index of the monomial $T^n T^{*m}$ and denote $\operatorname{ind}(T^n T^{*m})$. Finite linear combinations of monomials form an involutive subalgebra of the algebra $B(l^2(\mathbb{Z}_+))$ of all linear bounded operators of the Hilbert space $l^2(\mathbb{Z}_+)$. A uniform closure of this subalgebra in $B(l^2(\mathbb{Z}_+))$ is called the *Toeplitz algebra* and is denoted as $\mathcal{T}$.

Let $C(S^1; \mathcal{T}) = C(S^1) \otimes \mathcal{T}$ be a C^*-algebra of all continuous mappings from a unit circle S^1 to the algebra $\mathcal{T}$ with the norm

$$\|A\| = \sup_{S^1} \|A(e^{i\theta})\|, A \in C(S^1; \mathcal{T}).$$

Let $A_{\theta_0} \in C(S^1; \mathcal{T})$, $A_{\theta_0}(e^{i\theta}) = A(e^{i(\theta+\theta_0)})$ be a shift operator on $e^{i\theta_0}$. Since $\|A_{\theta_0}\| = \|A\|$, the shift operator A_θ generates a representation

$$\sigma : S^1 \to \operatorname{Aut}(C(S^1; \mathcal{T})), \sigma(e^{i\theta})(A) = A_\theta.$$

Each element A from $C(S^1; \mathcal{T})$ can be represented as a formal series

$$A(e^{i\theta}) \simeq \bigoplus_{k=-\infty}^{\infty} A_k e^{ik\theta},$$

where

$$A_k = \frac{1}{2\pi} \int_0^{2\pi} \sigma(e^{i\theta})(A) e^{-ik\theta} d\theta.$$

By $\widetilde{\mathcal{T}}$ let us denote a C^*-subalgebra of the algebra $C(S^1) \otimes \mathcal{T}$, generated by the monomials $\widetilde{T}^n\widetilde{T}^{*m}$, $\widetilde{T}^n\widetilde{T}^{*m}(e^{i\theta}) = e^{ik\theta}T^nT^{*m}$, $n, m \in \mathbb{Z}_+$, where $k = n - m$. It is obvious that the algebra $\widetilde{\mathcal{T}}$ is also invariant with respect to the shifts by the elements of the group S^1, i.e., $\sigma(e^{i\theta})(A) \in \widetilde{\mathcal{T}}$ for any A from $\widetilde{\mathcal{T}}$. In [5, 6] it is shown that the mapping $\widetilde{A} \mapsto A$, $A = \widetilde{A}(1)$, generates an isomorphism between the C^*-algebras $\widetilde{\mathcal{T}}$ and $\mathcal{T}$. Analogous results for a more general case were obtained in [7]. Thus, the representation $\sigma : S^1 \to \mathrm{Aut}(\widetilde{\mathcal{T}})$ generates a representation $\sigma_0 : S^1 \to \mathrm{Aut}(\mathcal{T})$:

$$\sigma_0(e^{i\theta})(A) = \widetilde{A}(e^{i\theta}),$$

where $A = \widetilde{A}(1)$. Note that $\sigma_0(e^{i\theta})(T^nT^{*m}) = e^{ik\theta}T^nT^{*m}$, $n, m \in \mathbb{Z}_+$, $k = n - m$. The notion of index of monomial can be extended to the elements $\widetilde{T}^n\widetilde{T}^{*m}$ of the algebra $\widetilde{\mathcal{T}}$: $\mathrm{ind}(\widetilde{T}^n\widetilde{T}^{*m}) = n - m$. By construction of the algebra $\widetilde{\mathcal{T}}$ one can see that if $\widetilde{A} = \widetilde{T}^n\widetilde{T}^{*m}$, $\widetilde{B} = \widetilde{T}^k\widetilde{T}^{*l}$, and $n - m \neq k - l$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{A}(e^{i\theta})\widetilde{B}^*(e^{i\theta})d\theta = 0.$$

Hence, the algebra $\widetilde{\mathcal{T}}$ can be written as

$$\widetilde{\mathcal{T}} = \overline{\bigoplus_{k=-\infty}^{\infty} \widetilde{\mathcal{L}}_k},$$

where $\widetilde{\mathcal{L}}_k$ is a closed subspace in $\widetilde{\mathcal{T}}$, generated by monomials of index k, i.e., consisting of those $\widetilde{A} \in \widetilde{\mathcal{T}}$ such that

$$\sigma(e^{i\theta})(A) = e^{ik\theta}A.$$

Thus,

$$\mathcal{T} = \overline{\bigoplus_{k=-\infty}^{\infty} \mathcal{L}_k}, \tag{3.1}$$

where $\mathcal{L}_k$ is a closed subspace in $\mathcal{T}$, generated by monomials of index k. Hence

$$\mathcal{L}_k = \{A \in \mathcal{T};\ \sigma_0(e^{i\theta})(A) = e^{ik\theta}A\}.$$

Each element A from $\mathcal{T}$ can be represented as a formal series

$$A \simeq \sum_{k=-\infty}^{\infty} A_k, \ \text{ where } A_k \in \mathcal{L}_k.$$

Let B be a C^*-subalgebra of a unital C^*-algebra A. A positive linear mapping $P : A \to B$ is called a *conditional expectation*, if it preserves the unit, $P(b) = b$ for any $b \in B$ and $P(abc) = aP(b)c$ for any $a, c \in B$, $b \in A$.

Let $\mathcal{T}_m$ be a C^*-subalgebra of the Toeplitz algebra $\mathcal{T}$, generated by monomials of index m.

Theorem 1. *For a C^*-algebra $\mathcal{T}_m$ the following relations are fulfilled*:

a) $\mathcal{T}_m = \overline{\bigoplus_{k=-\infty}^{\infty} \mathcal{L}_{km}}$,

b) *there exists a conditional expectation* $P_m : \mathcal{T} \to \mathcal{T}_m$.

Proof. a) Obvious.

b) Let G_m be a finite subgroup of the group S^1 of order m. Then

$$G_m = \{z \in S^1 : z^m = 1\} = \{e^{i\frac{2\pi k}{m}};\ k = 0, \dots, m-1\}.$$

If $\mathrm{ind}(T^nT^{*l}) = jm$, $j \in \mathbb{Z}$, then

$$\sigma_0(e^{i\frac{2\pi k}{m}})(T^nT^{*l}) = e^{i\frac{2\pi kjm}{m}}T^nT^{*l} = T^nT^{*l}.$$

Hence

$$\mathcal{T}_m = \{A \in \mathcal{T} : \sigma_0(e^{i\frac{2\pi k}{m}})(A) = A,\ k = 0, \dots, m-1\}.$$

Define $P_m : \mathcal{T} \to \mathcal{T}_m$, assuming

$$P_m(A) = \frac{1}{m} \sum_{k=0}^{m-1} \sigma_0(e^{i\frac{2\pi k}{m}})(A).$$

One can verify that

$$\sigma_0(e^{i\frac{2\pi k}{m}})(P_m(A)) = P_m(A)$$

for all $k = 0, 1, \dots, m-1$ and A from $\mathcal{T}$, and $P_m(A) = A$, if A is from $\mathcal{T}_m$. □

4. PROPERTIES OF THE ALGEBRA $\mathcal{T}_m$

Let $\pi : \mathbb{Z}_+^* \to \mathcal{T}$ be an exact representation of the inverse semigroup $\mathbb{Z}_+^*$ as the Toeplitz algebra:

$$\pi(a^n a^{*m}) = T^n T^{*m}.$$

The following theorem is obvious.

Theorem 2. *The restriction of the representation π to a subsemigroup $\mathbb{Z}_{+,m}^*$ generates the algebra $\mathcal{T}_m$.*

Let us define an endomorphism $\alpha : \mathcal{T} \to \mathcal{T}$:

$$\alpha(A) = TAT^*,\ \ A \in \mathcal{T}.$$

The following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}_+^* & \xrightarrow{\tau} & \mathbb{Z}_+^* \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathcal{T} & \xrightarrow{\alpha} & \mathcal{T}. \end{array}$$

Indeed, $\pi(\tau(b)) = T\pi(b)T^* = \alpha(\pi(b))$, $b \in \mathbb{Z}_+$, that means that τ extends to the endomorphism α of the Toeplitz algebra.

Denote by $\mathcal{T}(m)$ a C^*-subalgebra of the Toeplitz algebra generated by the semigroup $\pi(\mathbb{Z}_+^*(m))$. Obviously, $\mathcal{T}(m) \subset \mathcal{T}_m$.

Theorem 3. *The C^*-algebra $\mathcal{T}_m$ as a vector space can be represented as a direct sum*

$$\mathcal{T}_m = \mathcal{T}(m) \oplus \alpha(\mathcal{T}(m)) \oplus \cdots \oplus \alpha^{m-1}(\mathcal{T}(m)).$$

Proof. From Lemma 1 and the fact that the algebra $\mathcal{T}_m$ is generated by the semigroup $\pi(\mathbb{Z}_{+,m}^*)$, it follows that for any generating element $V \in \mathcal{T}_m$ it is true that $V \in \alpha^l(\mathcal{T}(m))$ for some $0 \le l \le m-1$.

Let us show that $\alpha^k(\mathcal{T}(m)) \cap \alpha^j(\mathcal{T}(m)) = 0$ for $k \ne j$. Let $V \in \alpha^k(\mathcal{T}(m)) \cap \alpha^j(\mathcal{T}(m))$, then $V = \alpha^k(T^{mn}T^{*ml}) = \alpha^j(T^{mr}T^{*ms})$. Hence, $V = T^{mn+k}T^{*ml+k} = T^{mr+j}T^{*ms+j}$, i.e., $mn + k = mr + j$ and $ml + k = ms + j$. Since $0 \le k \le m-1$ and $0 \le j \le m-1$, these equalities are possible only for $k = j$, $n = r$, $l = s$. Hence the statement of the theorem follows from equality (3.1). □

Corollary 1. The algebra $\mathcal{T}_m$ is a C^*-algebra generated by the operators T^m, T^{*m} and the projectors T^lT^{*l}, where $0 \le l \le m-1$.

Proof. By definition, the algebra $\mathcal{T}_m$ is generated by the elements V, whose indices are divisible by m, i.e., $V = T^{mk+l}T^{*mn+l}$, where $k, n, l \in \mathbb{Z}_+$, $0 \le l \le m-1$. Then $V = T^{mk}T^lT^{*l}T^{*mn} = (T^m)^k(T^lT^{*l})(T^{*m})^n$. This yields the statement of the corollary. □

Let us denote by P_l the projector T^lT^{*l}, $1 \le l \le m-1$. Obviously, $P_iP_j = P_jP_i = P_j$ for $1 \le i \le j \le m-1$. This means that $P_i > P_j$, $i < j$. Thus, it is true that $I = P_0 > P_1 > \cdots > P_{m-1}$.

Lemma 2. *The following inclusion is true*:

$$\alpha^k(\mathcal{T}(m))\alpha^j(\mathcal{T}(m)) \subset \alpha^k(\mathcal{T}(m)) \oplus \alpha^j(\mathcal{T}(m)),$$

where $0 \le k, j \le m-1$.

Proof. Let $V_1 \in \alpha^k(\mathcal{T}(m))$ be an element of the form $V_1 = T^{nm+k}T^{*ml+k}$, and $V_2 \in \alpha^j\mathcal{T}(m)$ be an element of the form $V_2 = T^{cm+j}T^{*am+j}$. Then if $cm+j > ml+k$, then

$$\begin{aligned} V_1V_2 = T^{nm+k}T^{*ml+k}T^{cm+j}T^{*am+j} &= T^{nm+k+cm+j-(ml+k)}T^{*am+j} \\ &= T^jT^{m(n+c-l)}T^{*am}T^{*j} \in \alpha^j(\mathcal{T}(m)), \end{aligned} \tag{4.1}$$

and if $cm+j < ml+k$, then

$$\begin{aligned} V_1V_2 &= T^{nm+k}T^{*(ml+k)}T^{cm+j}T^{*(am+j)} \\ &= T^{nm+k}T^{*(-cm)-j+(ml+k)}T^{*(am+j)} = T^kT^{mn}T^{*m(l+a-c)}T^{*k} \in \alpha^k(\mathcal{T}(m)). \end{aligned} \tag{4.2}$$

If $cm+j = ml+k$, then, in view of $0 \le k, j \le m-1$, we have $j = k$, $c = l$. Hence, $V_1, V_2, V_1V_2 \in \alpha^k(\mathcal{T}(m))$. Thus, the statement of the lemma is true for monomials. In order to complete the proof, we note that any element $C \in \alpha^k(\mathcal{T}(m))\alpha^j(\mathcal{T}(m))$ has the form of $C = AB$, where A is a linear combination of the elements of the form V_1, and B is a linear combination of the elements of the form V_2. Thus, AB is a linear combination of the elements of type (4.1) or (4.2). Thus, $AB \in \alpha^k(\mathcal{T}(m)) \oplus \alpha^j(\mathcal{T}(m))$. $\square$

Thus, the following theorem is true.

Theorem 4. *If* $0 \le i_1 < i_2 < \cdots < i_k \le m-1$, *where* $i_k \in \mathbb{Z}_+$, *then a direct sum of vector spaces*

$$\alpha^{i_1}(\mathcal{T}(m)) \oplus \cdots \oplus \alpha^{i_k}(\mathcal{T}(m))$$

is a C^**-subalgebra in the algebra* $\mathcal{T}_m$.

5. REPRESENTATIONS OF THE ALGEBRA $\mathcal{T}_m$

Let us represent a Hilbert space $l^2(\mathbb{Z}_+)$ with the basis $\{e_k\}_{k\in\mathbb{Z}_+}$ as a direct sum

$$l^2(\mathbb{Z}_+) = H_0 \oplus H_1 \oplus \cdots \oplus H_{m-1},$$

where the basis of the subspace H_i consists of $\{e_{i+km}\}_{k\in\mathbb{Z}_+}$, $0 \le i \le m-1$.

Lemma 3. *The subspaces* H_i, $0 \le i \le m-1$, *are invariant with respect to the algebra* $\mathcal{T}_m$.

Proof. Let $A \in \mathcal{T}_m$, then A is a linear combination of the elements of the form $V = T^kT^{*l}$, where $\operatorname{ind}(V) = k-l$ is divisible by m, i.e., $k-l = dm$, $d \in \mathbb{Z}$. Then for any H_i and $e_j \in H_i$, if $Ve_j \ne 0$, then $Ve_j = e_{j+ind(V)} = e_{j+dm} \in H_i$. Thus, $Ae_j \in H_i$. $\square$

Theorem 5. *The restriction of the* C^**-algebra* $\mathcal{T}_m$ *to* H_i, $0 \le i \le m-1$, *generates* m *unitarily nonequivalent irreducible infinite-dimensional representations.*

Proof. Consider the representation $\pi : \mathcal{T}_m \to B\Big(\bigoplus\limits_{i=0}^{m-1} H_i\Big)$, $\pi(A) = A$, $A \in \mathcal{T}_m$. By Lemma 3 all the spaces H_i, $0 \le i \le m-1$, are invariant with respect to the algebra $\mathcal{T}_m$, hence $\pi(A)$ can be represented as a direct sum

$$\pi(A) = \pi_0(A) \oplus \pi_1(A) \oplus \cdots \oplus \pi_{m-1}(A),$$

where $\pi_i(A) = A|_{H_i}$ for any $A \in \mathcal{T}_m$, $0 \le i \le m-1$.

From Corollary 1 we have that the algebra $\mathcal{T}_m$ acts on each of the spaces H_i as the Toeplitz algebra. Indeed, since $\mathcal{T}_m$ is generated by the operators T^m, T^{*m}, $T^l T^{*l}$, $0 \le l \le m-1$, and $T^m e_{i+mk} = e_{i+m(k+1)}$ for any k, and $T^{*m} e_{i+mk} = e_{i+m(k-1)}$ for $k \ne 0$ and $T^{*m} e_i = 0$, the operator T^m is a shift operator on the basis $\{e_{i+km}\}_{k \in \mathbb{Z}_+}$ in the space H_i. This means that the representations π_i, $0 \le i \le m-1$, are irreducible and the algebra $\mathcal{T}_m$ has m irreducible representations.

Let us show that the representations π_i are unitarily nonequivalent. Suppose the contrary, among π_i, $0 \le i \le m-1$, there are unitarily equivalent, i.e., there exist π_i, π_j, $0 \le i, j \le m-1$, and a unitary operator $U : H_j \to H_i$ such that

$$U\pi_j(A) = \pi_i(A)U$$

for any $A \in \mathcal{T}_m$. Suppose $i > j$ for definiteness. Suppose $A = T^i T^{*i}$. Then $\pi_i(A)U = (T^i T^{*i})|_{H_i} = I|_{H_i}$, and $\pi_j(A) = (T^i T^{*i})|_{H_j} \ne I|_{H_j}$, since $T^i T^{*i} e_j = 0$. Hence, $U\pi_j(A)e_j = 0$, and $\pi_i(A)Ue_j \ne 0$. We get a contradiction. Thus, the algebra $\mathcal{T}_m$ has m irreducible unitarily nonequivalent infinite-dimensional representations. □

Corollary 2. Let $0 \le i_1 < i_2 < \cdots < i_k \le m-1$, where $i_k \in \mathbb{Z}_+$, then a C^*-subalgebra $\alpha^{i_1}(\mathcal{T}(m)) \oplus \cdots \oplus \alpha^{i_k}(\mathcal{T}(m))$ has k unitarily nonequivalent, irreducible, infinite-dimensional representations.

Lemma 4. *Let $\pi : \mathcal{T}_m \to B(H)$ be an irreducible representation of $\mathcal{T}_m$ on the Hilbert space H, then the restriction $\pi|_{\mathcal{T}(m)} : \mathcal{T}(m) \to B(H)$ is irreducible, too.*

Proof. Suppose the contrary, let $\pi|_{\mathcal{T}(m)}$ on H act reducibly, i.e., $H = \bigoplus\limits_{i \in I} H_i \oplus H_0$, where $\pi|_{\mathcal{T}(m)}(H_i) \subset H_i$, $i \in I$ or $i = 0$. The operator $\pi(T^m)$ is an isometric operator, hence, by Wold–von Neumann theorem [8], is a sum of shift operators and a unitary operator. Let $\pi(T^m)$ on H_i, $i \in I$, act as a shift operator, and on H_0 as a unitary operator. Let us show that $H_0 = 0$. Note that $T^m T^{*m} P_k = T^m T^{*m}$, $k < m$, where $P_k = T^k T^{*k}$ is a projector. This yields $\pi(T^m T^{*m})\pi(P_k) = \pi(T^m T^{*m})$. Since $\pi(T^m)$ on H_0 acts as a unitary operator, $\pi(T^m T^{*m}) = \pi(I) = 1$ and, hence, $\pi(P_k) = 1$, $k = 0, \ldots, m-1$. The latter and $\pi|_{\mathcal{T}(m)}(H_0) \subset H_0$ yield that H_0 is a proper invariant subspace for π, which contradicts the fact that π is an irreducible representation of $\mathcal{T}_m$, i.e., $H_0 = 0$. Thus, $H = \bigoplus\limits_{i \in I} H_i$ and the operator $\pi(T^m)$ is a shift operator on each H_i, $i \in I$. This means that H contains a nonempty set of initial elements.

Denote by $L_0 = \{h \in H : \pi((T^m)^*)h = 0\}$ a set of initial elements in H. Let $k = \dim(L_0)$. If $k = 1$, then $\pi|_{\mathcal{T}(m)}$ is irreducible and the theorem is proved. Suppose the contrary, i.e., $k > 1$.

Let us show that the projectors $P_0 = I$, $P_i = T^i T^{*i}$ $(i = 1, \ldots, m-1)$ translate L_0 into itself, i.e., $\pi(P_i)(L_0) \subset L_0$, $i = 0, \ldots, m-1$. Indeed, let $h \in L_0$, then $\pi(T^m T^{*m})\pi(P_i)h = \pi(T^m T^{*m})h = 0$ and $P_i h \in L_0$, $i = 0, 1, \ldots, m-1$. From a family of commuting projectors $I = P_0 > P_1 > \cdots > P_{m-1}$, let us construct a family of orthoprojectors $Q_0, Q_1, \ldots, Q_{m-1}$ in the following way: $Q_i = P_i - P_{i+1}$, $i = 0, 1, \ldots, m-2$, and $Q_{m-1} = P_{m-1}$. Then the space L_0 can be represented as a direct sum of subspaces

$$L_0 = E_0 \oplus E_1 \oplus \cdots \oplus E_{m-1},$$

where $E_j = \pi(Q_j)(L_0)$, $j = 0, \ldots, m-1$. Note that at least one of these subspaces is nonzero. Since $\dim(L_0) > 1$, there exist $x_1, x_2 \in L_0$ such that $x_1 \perp x_2$ x_1, x_2 belong to the same E_j or different E_i, E_j, $i \ne j$. Obviously, $\pi(Q_j)x_l$ may be equal to either zero or x_l, where $l = 1, 2$, $j = 0, \ldots, m-1$. Since $P_i = \sum\limits_{j=i}^{m-1} Q_j$, $\pi(P_i)x_l$ may also be equal to either zero or x_l, where $l = 1, 2$, $i = 0, \ldots, m-1$.

Consider the subspaces $K_1 = \pi(\mathcal{T}(m))x_1$, $K_2 = \pi(\mathcal{T}(m))x_2$. Obviously, $K_1 \cap K_2 = \emptyset$. By the above argument $\pi(P_i)(K_1) \subset K_1$, $\pi(P_i)(K_2) \subset K_2$ for $i = 0, \ldots, m-1$. Thus by Corollary 1 we get $\pi(\mathcal{T}_m)(K_1) \subset K_1, \pi(\mathcal{T}_m)(K_2) \subset K_2$, i.e., π is a reducible representation. A contradiction. $\square$

Since the algebra $\mathcal{T}(m)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$ and is generated by a bicyclic semigroup, it has one infinite-dimensional irreducible representation. A question, in a sense converse to Lemma 4, arises: How can an irreducible representation of the algebra $\mathcal{T}(m)$ be extended to irreducible representations of the algebra $\mathcal{T}_m$ and how many such representations exist? It turns out that the following theorem is true.

Theorem 6. *The C^*-algebra $\mathcal{T}_m$ has exactly m irreducible unitarily nonequivalent infinite-dimensional representations.*

Proof. Let $\pi : \mathcal{T}(m) \to B(H)$ be an irreducible infinite-dimensional representation of the algebra $\mathcal{T}(m)$. Let us extend the definition of the representation π to the representation of the algebra $\mathcal{T}_m$. By Corollary 1, we get that we need to extend the definition of π onto the projectors P_i, $i = 1, \ldots, m-1$. Then $T^mT^{*m}P_i = T^mT^{*m}$, $i = 1, \ldots, m-1$. This yields $\pi(T^mT^{*m})\pi(P_i) = \pi(T^mT^{*m})$, i.e., $\pi(P_i) \geq \pi(T^mT^{*m})$. Thus, either $\pi(P_i){=}\pi(T^mT^{*m})$, or $\pi(P_i){=}I$. Thus, due to the inequalities $I > P_1 > P_2 > \cdots > P_{m-1}$ we get m different representations of the algebra $\mathcal{T}_m$:

1) $\pi_0(T^m) = \pi(T^m), \pi_0(P_1) = \cdots = \pi_0(P_{m-1}) = \pi(T^mT^{*m})$,

2) $\pi_1(T^m) = \pi(T^m), \pi_1(P_1) = I, \pi_1(P_2) = \cdots = \pi_1(P_{m-1}) = \pi(T^mT^{*m})$,

3) $\pi_2(T^m) = \pi(T^m), \pi_2(P_1) = \pi_2(P_2) = I, \pi_2(P_3) = \cdots = \pi_2(P_{m-1}) = \pi(T^mT^{*m})$,

......

m) $\pi_{m-1}(T^m) = \pi(T^m), \pi_{m-1}(P_1) = \cdots = \pi_{m-1}(P_{m-1}) = I$.

All possible extensions of the representation π are exhausted by the representations constructed.

Note that the representations $\pi_0, \pi_1, \ldots, \pi_{m-1}$ are unitarily equivalent to the representations obtained in Theorem 5. Indeed, consider a representation $\pi_k = \pi|_{H_k}$ for some k from Theorem 5. The basis of the subspace H_k coincides with the set $\{e_{k+nm}\}_{n\in\mathbb{Z}_+}$. Obviously, all the projectors $\pi_k(P_i)$, $1 \leq i \leq m-1$, preserve their basis elements, except for the initial element e_k, $\pi_k(P_i)e_k = e_k$ for $i \leq k$ and $\pi_k(P_i)e_k = 0$ for $i > k$. Thus, the following identities are true:

$$\pi_k(P_1) = \cdots = \pi_k(P_k) = I \text{ and } \pi_k(P_{k+1}) = \cdots = \pi_k(P_{m-1}) = \pi(T^mT^{*m}).$$

It follows from Theorem 5 that the representations $\pi_0, \pi_1, \ldots, \pi_{m-1}$ are irreducible and unitarily nonequivalent. $\square$

6. REPRESENTATIONS OF THE SEMIGROUP $\mathbb{Z}^*_{+,m}$

In [1] it is proved that for the bicyclic semigroup $\mathbb{Z}^*_+$ there exist, up to unitary equivalence, one infinite-dimensional irreducible representations and a series of one-dimensional unitary representations, parameterized by a unit circle S^1. Naturally, a question arises, how many irreducible representations exist for its subsemigroup $\mathbb{Z}^*_{+,m}$? The following Theorem is true.

Theorem 7. *The inverse semigroup $\mathbb{Z}^*_{+,m}$ has, up to unitary equivalence, exactly m infinite-dimensional irreducible representations and a series of one-dimensional unitarily non-equivalent representations, parameterized by a unit circle S^1.*

In the proof of the first part of this Theorem we use the methods developed in the previous Sections for the description of representations of the algebra $\mathcal{T}_m$. In particular, the irreducible representation $\mathbb{Z}^*_{+,m}$, restricted on $\mathbb{Z}^*_+(m)$, turns out to be irreducible as well. Conversely, extending the definition of the irreducible representation $\mathbb{Z}^*_+(m)$ to the projectors $P_1, P_2, \ldots, P_{m-1}$, one can construct exactly m different irreducible representations of the semigroup $\mathbb{Z}^*_{+,m}$, like in the proof of Theorem 6. The second part of the theorem follows from the next lemma.

Lemma 5. *Let J be a subsemigroup in $\mathbb{Z}^*_{+,m}$ generated by the elements of zero index. The homomorphism* $\mathrm{ind} : \mathbb{Z}^*_{+,m} \to \mathbb{Z}$ *generates a short exact sequence of semigroups*

$$0 \to J \to \mathbb{Z}^*_{+,m} \to \mathbb{Z} \to 0,$$

where $\mathrm{id} : J \to \mathbb{Z}^*_{+,m}$ *is an embedding.*

The following theorem is analogous to a well-known theorem of L. Coburn [2] that any two C^*-algebras generated by nonunitary isometric representations of the semigroup $\mathbb{Z}^*_+$ are canonically isomorphic.

Theorem 8. *Any two C^*-algebras generated by exact representations of the semigroup $\mathbb{Z}^*_{+,m}$ are canonically isomorphic.*

Proof. Let $\pi : \mathbb{Z}^*_{+,m} \to B(H)$ be an exact representation of the semigroup $\mathbb{Z}^*_{+,m}$. Then by Theorem 7 π can be represented as

$$\pi = \bigoplus_{k=0}^{m-1} \pi_k \oplus \Big(\bigoplus_{t\in\Gamma} \tau_t \Big),$$

where $\pi_k,\ k = 0, 1, \dots, m-1$, are all irreducible representations of the semigroup $\mathbb{Z}^*_{+,m}$, similar to those described in Theorem 5 for the algebra $\mathcal{T}_m$, and $\tau_t,\ t \in \Gamma$, are one-dimensional representations parameterized by the subset Γ of the unit circle S^1. By Theorem 5 it follows that the algebras generated by $\pi_k(\mathbb{Z}^*_{+,m})$, $k = 0, 1, \dots, m-1$, are isomorphic to the Toeplitz algebra.

Consider the representation $\pi_{m-1} \oplus \Big(\bigoplus_{t\in S^1} \tau_t \Big)$. Then by L. Coburn's theorem [2] the algebra generated by this representations is canonically isomorphic to the algebra generated by the representation π_{m-1}. Thus the algebra, generated by the representation π is canonically isomorphic to the algebra generated by the representation $\bigoplus_{k=0}^{m-1} \pi_k$. This yields the statement of the theorem. $\square$

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