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On the Maximal Finite-Dimensional Lie Algebras with Given Nilradical

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Abstract—We study the set of finite-dimensional Lie algebras with fixed nilradical (in the capacity of which any nilpotent Lie algebra may serve). We prove an exact estimate for dimensions of Lie algebras from this set. We also show that there may exist several Lie algebras in this set, possessing the maximal dimension. Proofs are based on a concept of algebraic splitting for finite-dimensional Lie algebras.

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The problem of classification of finite-dimensional Lie algebras goes back to S. Lie. For small dimensions, the classification was obtained by S. Lie by himself (and his successors at the initial stage of the development of the theory of the Lie algebras). Then there was obtained a comprehensive classification of semisimple Lie algebras over the fields of characteristic 0, algebraically closed or perfect. By the Levi decomposition, the classification of arbitrary Lie algebras reduces, to a great extent, to the classification of simple Lie algebras (now studied almost exhaustively, with a few intriguing questions left, e.g., on the "origin" of the five simple exceptional Lie algebras) and to the classification of solvable Lie algebras. By the Maltsev splitting, the classification of solvable Lie algebras, according to some papers, is reduced to the classification of nilpotent Lie algebras. That is not exactly the case: It is not sufficient to obtain the classification of nilpotent Lie algebras, it is also necessary to have the description of their differentiation algebras (or at least commutative completely reducible subalgebras in differentiation algebras), which is a very hard problem. Anyhow, approaches to the classification of finite-dimensional Lie algebras require the study and classification of nilpotent Lie algebras. However, great obstacles arise on this way. Arbitrary nilpotent Lie algebras cannot be described to at least some nontrivial degree. Many constructions that perfectly work for semisimple or solvable Lie algebras often "degenerate" into trivialities for nilpotent Lie algebras. For example, the Cartan subalgebra for an arbitrary nilpotent Lie algebra coincides with itself. The classifications of nilpotent Lie algebras have been obtained for the dimensions ≤ 7 (for the fields $\mathbf{C}$, $\mathbf{R}$). For dimensions ≥ 8 we only have partial classifications for some special classes of nilpotent Lie algebras. There is no approach to the classification of an arbitrary finite-dimensional nilpotent Lie algebras. Thus, the reduction of the classification of an arbitrary Lie algebras to the classification of a nilpotent one stubles on (yet?) irresistible obstacles.

A "turning manoeuvre"was proposed for the study of an arbitrary Lie algebras. If we fail to classify all nilpotent Lie algebras, we can consider them as the source object, and to classify more general Lie algebras based on them. Namely, we can study all those Lie algebras, whose nilradical (i.e., the greatest nilpotent ideal) is isomorphic to a given nilpotent of the Lie algebra. This approach has been used in a number of recently published papers, the results of one of these papers [1] will be mentioned below.

Let L be an arbitrary finite-dimensional Lie algebra over a field k. We will normally assume in the paper that $k = \mathbf{C}$ (in fact, it suffices to assume that k is algebraically closed field of zero characteristic). We will also discuss how the statements of the paper should be modified, if the field k is not necessarily algebraically closed, but perfect (e.g., $k = \mathbf{R}$). Let us denote by $\mathcal{L}_n(k)$ the space of n-dimensional Lie

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algebras over k (formed the by components of the structural tensors c_{ij}^k, skew-simmetric with respect to the lower indices and satisfying the consequences from the Jacobi identity). The group $GL_n(k)$ acts on this space in a natural way; the orbits of this action consist of all pairwise isomorphic Lie algebras. Let us denote by $\mathcal{R}_n(k)$ a subset in $\mathcal{L}_n(k)$ formed by a solvable Lie algebras.

Now let N be some nilpotent Lie algebra. Let us denote by $\mathcal{L}(k, N)$ (or, if it does not lead to confusion, briefly, $\mathcal{L}(N)$) the subset in $\mathcal{L}_n(k)$ formed by the Lie algebras (of any finite dimension), whose nilradicals are isomorphic to the nilpotent Lie algebra N. The set $\mathcal{R}(k, N)$ is defined in a similar manner.

We will consider the maximal (in some sense) elements of sets $\mathcal{L}(N)$ and $\mathcal{R}(N)$.

Let us call a Lie algebra $L \in \mathcal{L}(N)$ strongly maximal, if it cannot be embedded into some other Lie algebra from $\mathcal{L}(N)$ as a proper subalgebra (in other words, it has a nilradical isomorphic to N). Strongly maximal elements in $\mathcal{R}(N)$ are defined in a similar manner. This definition of maximality is the most natural one, but also it makes sense to consider other notions of maximality, too. One of those is the notion of d-maximality (dimensional maximality). A Lie algebra $L \in \mathcal{L}(N)$ is called d-maximal if there is no Lie algebras in $L \in \mathcal{L}(N)$, whose dimension is greater than that of L. Two notions of maximality are also defined for $L \in \mathcal{R}(k, N)$ in a similar manner. One more (and two similar to the mentioned above for exact Lie algebras) notion of maximality will be proposed below.

It is clear that dimensional maximality implies strong maximality. There is some ground to make a conjecture that the converse is, in general, false.

The goal of this paper is the study of maximal (in one of a few different senses) Lie algebras with a given nilradical. For any nilpotent Lie algebra N we will construct strongly maximal and d-maximal Lie algebras.

We will start our consideration with the case when a Lie algebra L with the given nilradical is solvable. In other words, we will consider spaces $\mathcal{R}(N)$ for an arbitrary nilpotent Lie algebra N. In [1] they give an estimate of dimension $\dim(L)$ for $L \in \mathcal{R}(N)$, where N is an arbitrary finite-dimensional nilpotent Lie algebra (see details below). The same paper, based on a few partial cases (including those previously studied) made a conjecture that the maximal (in our terminology: strongly maximal) Lie algebra in $\mathcal{R}(N)$ is unique up to isomorphism.

Below we will strengthen the estimate for $\dim(L)$ obtained in [1] (and extended it onto arbitrary, not only solvable, like in [1], Lie algebras). Moreover, our estimate will always be exact. The conjecture of uniqueness of the maximal Lie algebra in $\mathcal{R}(N)$ will be turned down using the results given in [2]. Further we will consider the case of arbitrary (and not only solvable) Lie algebras with a given nilradicals. These results can be viewed as a minor contribution into the development of new approaches to classification of arbitrary finite-dimensional Lie algebras.

Let us start with one construction that will play a decisive role in the further deployment. The base of this construction is the notion of splitting of the Lie algebra. The first time a splitting of a (complex) solvable Lie algebra as its embedding into a splittable one, semidirect sum of nilpotent ideal and an Abelian subalgebra (acting semisimply on this ideal), was constructed by A. I. Maltsev. The development of this construction (a construction refinement and extension of the notion of splitting onto the case of solvable real-valued Lie algebras) was given by L. Auslander, as well as G. Mostow and G. Hochschild. Later the idea of splitting, that proved to be so efficient and capable of solving many interesting problems in the theory of Lie algebras and groups, and also their homogeneous spaces, was reinvented several times. A few authors, independently on each other, constructed some analogs of the notion of splitting for their own purposes. However, the question of uniqueness of the notion of splitting has almost never been considered. It is true that in [3] it was proved the statement of that under certian conditions the splitting of the Lie algebra is unique in some natural sense. However, not all variants of the notion of splitting satisfy the conditions required. More detail on the notion of splitting is given in [3]. Below we use a certain modification of the notion of splitting (applicable to arbitrary finite-dimensional Lie algebras over perfect field of characteristic 0), commonly called an algebraic splitting. The idea of this construction dates back to L. Auslander.

Let L be some complex solvable Lie algebra (the case of arbitrary Lie algebras will be considered below). Consider its attached representation $\mathrm{ad} : L \to \mathrm{Der}(L)$, a homomorphism of the Lie algebra into the Lie algebra $\mathrm{Der}(L)$ of its differentiations, defined by action on L of linear operators ad_X, $\mathrm{ad}_X(Y) = [X, Y]$. Let us denote the image of the Lie algebra L under this homomorphism as ad_L. The Lie algebra ad_L is a Lie subalgebra in the Lie algebra $\mathrm{gl}(L)$ of linear transformations of the vector space L. Consider the algebraic closure $\langle \mathrm{ad}_L \rangle$ of this Lie subalgebra. It is contained in $\mathrm{Der}(L)$ because the Lie algebra $\mathrm{Der}(L)$ is, obviously, algebraic. The Lie algebra $\langle \mathrm{ad}_L \rangle$ is algebraic and hence it is splittable: it allows the Chevalley decomposition into semidirect sum of a nilpotent ideal U, consisting of nilpotent linear transformations of Lie algebra, and an Abelian Lie subalgebra F, consisting of semisimple (i.e., acting on the space L well reducibly, or, which is the same, reductively) elements.

Consider a Lie algebra $A(L) = F + L$, a semidirect sum of ideals isomorphic to L and an Abelian Lie subalgebra isomorphic to F whose action on L is the natural action of linear transformations (since $F \subset \mathrm{gl}(L)$). The obtained Lie algebra $A(L)$ is often called an algebraic splitting of the Lie algebra L. More precisely, the notion of splitting also includes the embedding of the source Lie algebra L, but in this case this embedding is obvious. There are other methods of construction of an "algebraic splitting", but the method given above seems to be most natural (other methods are based on some exact linear transformation of the Lie algebra L).

Denote by W the nilradical of the Lie algebra $A(L)$. Then it is easy to show that $A(L) = F + W$ and $\dim(W) = \dim(L)$.

Consider an arbitrary nilpotent Lie algebra N, and some maximal torus T in $\mathrm{Der}(N)$ (a maximal Abelian Lie subalgebra consisting of semisimple elements). We will denote the dimension of the torus T by $r_s(\mathrm{Der}(N))$ and call it a semisimple rank of the Lie algebra $\mathrm{Der}(N)$ and the corresponding Lie algebra N.

Assume that $H(N) = T + N$ is a semidirect sum, corresponding to the natural action on the Lie algebra T by differentiation on N. We may call this Lie algebra a pseudoholomorph (a holomorph of Lie algebra N, similar to that in the abstract group theory, is defined as the Lie algebra $\mathrm{Der}(N) + N$). It is clear that $H(N) \in \mathcal{R}(N)$ for the source nilpotent Lie algebra N, and also that

$$\dim H(N) = r_s(\mathrm{Der}(L)) + \dim(N).$$

Now we can formulate the first result of this paper.

Theorem 1. *Let N be an arbitrary nilpotent complex finite-dimensional Lie algebra, and $R \in \mathcal{R}(N)$. Then $\dim(R) \leq \dim H(N)$.*

In particular, the Lie algebra $H(N)$ is d-maximal in $\mathcal{R}(N)$ and is strongly maximal in $\mathcal{R}(N)$.

Proof. Consider a Lie algebra $R \in \mathcal{R}(N)$ and its algebraic splitting $A(R)$ with a natural embedding $i : R \hookrightarrow A(R)$. We have $A(R) = F + R = F + W$ (we use the notation introduced above in the description of the algebraic splitting of Lie algebras). An Abelian Lie algebra R/N is embedded into $A(R)/N = F \oplus W/N$ in a natural way, where the Lie algebra W/N (denote it as A) is Abelian.

The Abelian Lie algebra F consists of semisimple differentiations of the Lie algebra R. Since N is a characteristic ideal in R, any differentiation of the Lie algebra R keeps the ideal N invariant. Hence, there is a natural homomorphism $\lambda : F \to F|_N$ (differentiation of the Lie algebra R is mapped onto its construction on the nilradical N).

Lemma 1. *The homomorphism $\lambda : F \to \mathrm{Der}(N)$ is monomorphic.*

Proof. Consider some element $f \in \ker \lambda$. We have $f(N) = \{0\}$. The linear mapping f is semisimple. Hence, there exists a basis $\{e_i\}$, $\{e_\alpha\}$ such that the elements e_α form a basis in N, and the elements e_i generate a linear subspace in R, complementary to N, and all these vectors are eigenvectors for the linear transformation f. In particular, let $f(e_i) = \mu_i e_i$, $\mu_i \in \mathbf{C}$. Since f is a differentiation, therefore $f([e_i, e_\alpha]) = [f(e_i), e_\alpha] + [e_i, f(e_\alpha)]$. But $[e_i, e_\alpha] \in N$ and, hence, $f([e_i, e_\alpha]) = 0$. Since $f(e_\alpha) = 0$ (as long as $e_\alpha \in N$), we get $[f(e_i), e_\alpha] = 0$, and since $f(e_i) = \mu_i e_i$, we get $\mu_i[e_i, e_\alpha] = 0$ for all admissible (as indices of basis elements) values of the indices i and α. If all $\mu_i = 0$, then $f = 0$, i.e., the element of the kernel is zero. Consider the case when some of μ_i (let it be μ_1) is nonzero. Then we have $[e_1, e_\alpha] = 0$ for all basis elements e_α of the Lie subalgebra N and the element e_1 belongs to the centralizer $Z_R(N)$ of the Lie subalgebra N in R. Let us prove that this is impossible. Below we will prove $Z_R(N) = Z(N)$. In particular, then $Z_R(N) \subset N$ and, hence, $e_1 \in N$, which is impossible because e_1 is a basis element of a subspace in R, complementary to N. The contradiction leaves only one possibility for f: $f = 0$, that yields $\ker \lambda = \{0\}$.

It only remains to prove the equality $Z_R(N) = Z(N)$. Let $X \in Z_R(N)$ be some element. Let us show that $(\mathrm{ad}_X)^2 = 0$.

Since the nilradical N contains a commutant $[R, R]$ (here it is essential that the Lie algebra R is solvable), we have $\mathrm{ad}_X(R) \subset N$. From the fact that X centralizes N it follows $\mathrm{ad}_X(N) = \{0\}$. As a result we get $(\mathrm{ad}_X)^2 = 0$. In particular, the element X is nilpotent and, hence, must belong to N. Thus, we get $Z_R(N) \subset N$. This, in turn, implies $Z_R(N) = Z(N)$. The required equality is proved, and the proof of Lemma 1 is complete.

A simple equality $Z_R(N) = Z(N)$ from the proof of Lemma 1 does not seem to be new, but the author failed to find its proof in the literature. Let us emphasize once again that it is true only for solvable Lie algebras. For example, for a reductive Lie algebra $L = \mathrm{gl}_2(\mathbf{C}) = \mathrm{sl}_2(\mathbf{C}) \oplus \mathbf{C}$ we have $N = \mathbf{C}$, and $Z_L(N) = L \neq Z(N)$.

Now let us go back to the proof of Theorem 1.

Consider the image $\lambda(F)$ of the monomorphism λ (using Lemma 1). This Abelian subalgebra in $\mathrm{Der}(N)$ consists of semisimple differentiations. Let us recall that we introduced a subalgebra $T \subset \mathrm{Der}(N)$ above, a maximal Abelian subalgebra, consisting of semisimple differentiations. Since such subalgebra up to conjugation in $\mathrm{Der}(N)$ is unique, it yields $\dim(\lambda(F)) \leq \dim T$. But then due to monomorphy property of λ we have $\dim(F) \leq \dim(T)$ as well.

Let us recall the embedding $R/N \hookrightarrow F \oplus A$, described above. Due to the construction of the algebraic splitting, the image of R/N under this embedding has a zero intersection with the Abelian subalgebra A. Hence it follows $\dim(R/N) \leq \dim F$. Combining this inequality with the one above, we get the inequality $\dim(R/N) \leq \dim T$. But $\dim R = \dim R/N + \dim(N)$ and thus $\dim(R) \leq \dim T + \dim(N)$. By definition $H(N) = T + N$. As a result we get $\dim(R) \leq \dim(H(N))$. Thus, the Lie algebra $H(N)$ has the greatest dimension among all the Lie algebras from $\mathcal{R}(N)$, i.e., $H(N)$ is d-maximal in $\mathcal{R}(N)$. But then, evidently, it is strongly maximal in $\mathcal{R}(N)$ as well. □

The inequality $\dim(R) \leq \dim H(N)$ or its equivalent $\dim(R) \leq \dim(N) + r_s(\mathrm{Der}(N))$ is stronger than the inequality $\dim(R) \leq \dim N + \dim N/[N, N]$ proved in [1]. The reason is that it is always true that $r_s(\mathrm{Der}(N)) \leq \dim N/[N, N]$ because semisimple differentiations of nilpotent Lie algebra are unambiguously defined by their action on invariant subspace complementary to $[N, N]$, generating N and having dimension that exactly equals $\dim(N/[N, N])$. Moreover, our estimate is exact: For any nilpotent Lie algebra N we explicitly constructed a Lie algebra $H(N)$, for which the inequality turns into equality.

In [1], for a few partial cases it was noted that for an arbitrary nilpotent Lie algebra N the maximal Lie algebra in $\mathcal{R}(N)$ is always unique up to an isomorphism. We will show below that this statement is false in the general case. In [1] it was proposed as a conjecture that a d-maximal Lie algebra in $\mathcal{R}(N)$ is unique. Apart from a concrete counterexample, that requires the use of nilpotent Lie algebras of sufficiently large dimension (because for small dimension the conjecture is true) and is explicitly given below, let us first make some guiding considerations.

Let N be a nilpotent Lie algebra for which $r_s(\mathrm{Der}(N)) = 1$. There are many such Lie algebras, for example, among the Lie algebras, for which $\dim(N/[N, N]) = 2$ (in particular, among philoformic). Then any non-nilpotent (i.e., different from N) Lie algebra from $\mathcal{R}(N)$ has a dimension $\dim(N) + 1$. Moreover, in the algebraic splitting $A(R)$ subalgebra F is one-dimensional, and a nilpotent ideal W (see construction of the splitting above) contains N as an ideal of codimension 1 (this simple remark is a partial case of a more general statement proved in Proposition below). Thus R/N is a one-dimensional subalgebra in the two-dimensional Abelian Lie algebra $F \oplus W/N$ (the embedding is described above). Deforming this one-dimensional Lie subalgebra inside containing it two-dimensional Lie algebra we can get the deformations of the Lie algebra R into unsplittable ones. Since $H(N)$ is splittable, we have all reasons to assume that in the general case there exist unsplittable Lie algebras from $\mathcal{R}(N)$ of the same dimension (and hence d-maximal), too.

Let us get to a more detailed description of the example of non-uniqueness of the maximal (and d-maximal) Lie algebra in $\mathcal{R}(N)$. One maximal Lie algebra in $\mathcal{R}(N)$ is well-known, it is $H(N)$. It is splittable: it can be decomposed into a semidirect sum of a nilradical and an Abelian subalgebra acting on the nilradical in a semisimple way. For our goal we need, as stated above, to find at least one unsplittable maximal Lie algebra in $\mathcal{R}(N)$. Then we will get an example of non-uniqueness of a maximal Lie algebra in $\mathcal{R}(N)$. It turns out that for such an example for N we should take a philoformic Lie algebra (or, which is the same, a nilpotent Lie algebra, whose nilpotence class is maximally large; it is less for unity than the dimension of this Lie algebra). Such Lie algebras are described, e.g., in [2]. For such Lie algebras it is always true that $r_s(\mathrm{Der}(N)) \leq 2$. Consider [2] only those philoformic Lie algebras, for which $r_s(\mathrm{Der}(N)) = 1$. If N has the dimension of n, then in the selected case any non-nilpotent Lie algebra R has the dimension of $n+1$. One of these Lie algebras R is splittable (it is isomorphic to the Lie algebra $H(N)$ constructed above). In [2] it is stated that for some series of the philoformic Lie algebras we consider there are also unsplittable ones among their corresponding Lie algebras of dimension $n+1$ from $\mathcal{R}(N)$.

Namely, this takes place for the Lie algebras denoted in [2] as $C_n(\alpha_1, \ldots, \alpha_t)$, where $n = 2m+2$, $t = m-1$ ([2] considers a Lie algebra of this class C, of dimension $n+1$; for us it is more convenient to consider a Lie algebra of dimension n, so we make some renotation of parameters compared to [2]). The parameters α_i, considered up to proportion, define a Lie algebra $C_n(\alpha_1, \ldots, \alpha_t)$ uniquely up to an isomorphism. For our goal we need $t \geq 2$. Then there must be $m \geq 3$ and thus the dimension n of the Lie algebra will be ≥ 8 (let us note that for dimension of up to 6 inclusively the question of uniqueness is solved in the affirmative way). For $N = C_n(\alpha_1, \ldots, \alpha_t)$ in [2] all possible solvable Lie algebras R are given, with this N as their nilradical. It turns out that for $n \geq 8$ (or, which is equivalent, for $t \geq 2$) there are infinitely many such Lie algebras R, and all of them, except for one, are unsplittable. All these Lie algebras R from $\mathcal{R}(N)$ have the dimension $n+1$, thus, they are maximal and d-maximal. Hence, there is no uniqueness of maximal and d-maximal Lie algebras in $\mathcal{R}(N)$ for $n \geq 8$. The case of $n = 7$ remains unconsidered (for $n \leq 6$ the uniqueness holds, see above).

The following statement describes the structure of d-maximal Lie algebras in $\mathcal{R}(N)$.

Proposition. *Let R be some d-maximal Lie algebra in $\mathcal{R}(N)$ for an arbitrary nilpotent Lie algebra N. Then for the algebraic splitting of this Lie algebra the following statements are true*:

(i) $\dim F = r_S(\mathrm{Der}(N)), \dim W/N = \dim R/N$,

(ii) *a Lie subalgebra R/N in the direct sum $F \oplus W/N$ has zero intersection with W/N and is isomorphically projected onto F by the projection of the direct sum onto the first direct summand.*

Proof. We will use a few elements of the proof of Theorem 1. Let R be some d-maximal Lie algebra in $\mathcal{R}(N)$. Its dimension must be equal the dimension of the Lie algebra $H(N)$, which in turn equals $\dim(N) + r$, where $r = r_s(\mathrm{Der}(N))$.

Consider the monomorphism λ for the Lie algebra R, used in the proof of Theorem 1. Since R has the same dimension, as $H(N)$, this monomorphism will be an isomorphism. In particular, $\dim(F) = \dim(T) = r$. It is always true that $\dim(W/N) = \dim(R/N)$. In our case we also have $\dim(F) = r$. For the embedding of R/N as a subalgebra into the direct sum $F \oplus W/N$ the intersection of this subalgebra with W/N is always trivial (due to the construction of the splitting). □

The Proposition gives the range for the Lie algebras of maximal possible dimension in $\mathcal{R}(N)$.

Now consider an arbitrary (and not only solvable) finite-dimensional Lie algebras. We will prove the statement by analogy to Theorem 1. This will require some additional considerations.

Let L be some finite-dimensional Lie algebra, $L = S + R$ be its Levi decomposition (R is its radical, S is the semisimple part), N is a nilradical in L. Let us consider a set $\mathcal{L}(k, N)$ of Lie algebras L over a field k (for now assuming that k is algebraically closed and is of characteristic 0; below we will discuss the case when k is perfect), whose nilradicals are isomorphic to some given nilpotent Lie algebra N. Unlike the case of solvable Lie algebras considered above, in this general case there are no maximal elements in $\mathcal{L}(N)$ (neither strongly maximal, nor d-maximal). Since $L \in \mathcal{L}(N)$ and S_1 is an arbitrary semisimple Lie algebra, $L \oplus S_1$ has the same nilradical, as L, and thus belongs to $\mathcal{L}(N)$. This is the reason for which it is necessary to restrict the class of considered Lie algebras in consideration of maximal elements.

A Lie algebra L is called exact, if the action (induced by the adjoint action L onto itself) of its semisimple part S on its radical R is exact, i.e., has a zero kernel. This is equivalent to the condition of exactness of the natural representation of $S \to \mathrm{Der}(R)$. A solvable Lie algebra is exact by a trivial reason (it has no semisimple part). It is worthy to note that this definition does not depend (as it is easy to verify) on the choice of maximal semisimple Lie subalgebra S in L. The description of exact Lie algebras was given in [4].

Note that the adjoint action of the Lie algebra L onto itself induces an action L on $Z(N)$.

Theorem 2. *A Lie algebra L is exact if and only if at least one of the following two equivalent conditions holds*:

(i) *the kernel of action L on N equals $Z(N)$,*

(ii) *the action of the semisimple part S of the Lie algebra L on $Z(N)$ is exact* (*i.e., it has a nonzero kernel*).

Proof. Let us first show that the conditions (i) and (ii) are equivalent. The implication (i)⇒(ii) is obvious. Conversely, let an action S on $Z(N)$ be exact. The kernel of the natural action L on N is an ideal in L, in fact it is its centralizer $Z_L(N)$. Due to (ii) this centralizer has a nonzero intersection with S. Then it has a trivial semisimple part (due to the Maltsev theorem on conjugacy of all semisimple parts of Lie algebra L), i.e., it is a solvable ideal. Hence, it is contained in the nilradical R of the Lie algebra and coincides with $Z_R(N)$. Due to the Proposition above we get $Z_L(N) = Z(N)$. Thus, conditions (i) and (ii) in Theorem 2 are equivalent.

Let us go directly to the proof of Theorem 2. Due to the said above it suffices to consider the statement of Theorem 2 in case (i).

Let the kernel of action L on N be equal $Z(N)$. Then the intersection of this kernel with S is trivial, i.e., S on N acts exactly. But then S also acts exactly on R, i.e., the Lie algebra L will be exact.

Conversely, let L be an exact Lie algebra. Let us demonstrate that $\ker(\mathrm{ad}_L|_N) \cap R \subset N$. We have $\ker(\mathrm{ad}_L|_N) = Z_L(N)$, and $Z_L(N) \cap R = Z_R(N)$. As we showed in the proof of Theorem 1, the equality $Z_R(N) = Z(N)$ holds. In particular, $Z_R(N) \subset N$, which in combination with the equalities above gives the required inclusion $\ker(\mathrm{ad}_L|_N) \cap R \subset N$.

From $\ker(\mathrm{ad}_L|_N) \cap R \subset N$ it follows that under factorization by R we get an embedding $\ker(\mathrm{ad}_L|N) \hookrightarrow L/R = S$. The image under this embedding is an ideal in the semisimple Lie algebra S and thus this image is also a semisimple Lie algebra. Then we may assume that the subalgebra $\ker(\mathrm{ad}_L)$ is contained in S and, due to exactness, the Lie algebra must be trivial. This completes the proof of Theorem 2.

Now consider a construction analogous to that given above for the case of solvable Lie algebras.

Consider a subalgebra $\mathrm{ad}_L \subset \mathrm{Der}(L)$ and its algebraic closure $\langle \mathrm{ad}_L \rangle \subset \mathrm{Der}(L)$. This closure is contained in the algebraic Lie algebra $\mathrm{Der}(L)$. The Lie algebra $\langle \mathrm{ad}_L \rangle$ is an algebra and so it allows the Chevalley decomposition $\langle \mathrm{ad}_L \rangle = F + U$, where F is the maximal reductive (but not necessarily Abelian, as it was in the case of solvable Lie algebras) subalgebra, and U is a nilpotent ideal. In turn, a reductive Lie algebra F has the decomposition (a unique one) into a direct sum $F = S \oplus T$ of a semisimple ideal S and a torus T (maximal Abelian central Lie subalgebra consisting of semisimple elements).

Consider a Lie algebra $A(L) = T + L$ (semidirect sum), an algebraic splitting for L (the notion of splitting includes the natural embedding of the Lie algebra L into $A(L)$). Note that in this construction only the radical of the Lie algebra L is actually splitted, and the semisimple part of the Lie algebra L remains intact. Thus we actually get that the algebraic splitting $A(L)$ of the Lie algebra L can be written as $A(L) = S + A(R)$. Some additional information on this splitting can be found in [3]. We have $A(L) = F + W$, where W is a nilpotent ideal.

Further, for an arbitrary nilpotent Lie algebra N consider the Lie algebra $\mathrm{Der}(N)$ of its differentiations, and let Φ be a maximal reductive subalgebra in the Lie algebra $\mathrm{Der}(L)$.

Let, finally, $H(L) = \Phi + N$ (a semidiract sum). It is clear that $H(L) \in \mathcal{L}(N)$. The given constructions for L are direct analogs of the above given ones for the case of solvable Lie algebras. It is of no surprise that the results in the general case are similar. Making the arguments similar to those in the proof of Theorem 1, we get

Theorem 3. *Let N be an arbitrary nilpotent (complex, finite-dimensional) Lie algebra, and $L \in \mathcal{L}(N)$ be some exact Lie algebra. Then* $\dim(L) \leq \dim H(N)$.

In particular, the Lie algebra $H(N)$ is d-maximal in $\mathcal{L}(N)$ and is strongly maximal in $\mathcal{L}(N)$.

Proof. Here are the basic stages of the proof, in essence repeating (with minor modifications) the proof of Theorem 1.

Consider an algebraic splitting $A(L) = T + L$ of our Lie algebra L. Let N be its nilradical. Then consider a homomorphism of the restraint $\lambda : T \to \mathrm{Der}(N)$ of differentiations from T onto characteristic ideal N. We will further need (similar to Lemma 1)

Lemma 2. *For exact Lie algebra L the homomorphism $\lambda : T \to \mathrm{Der}(N)$ is monomorphic.*

Proof. Consider an arbitrary element $f \in \ker \lambda$. By its definition $f(N) = \{0\}$. Similarly to the proof of Lemma 1, we demonstrate that $f(R) = \{0\}$, where R is a radical of the Lie algebra L. The action T on S is trivial. Hence we get $f = 0$. □

The further arguments in the proof of Theorem 3 are conducted in the same way as in the proof of Theorem 1. Here we must take into account that the action S on N has a nonzero kernel (due to exactness of the Lie algebra L and Theorem 2). We also use the fact that all maximal reductive Lie subalgebras in an algebraic Lie algebra $\mathrm{Der}(N)$ are conjugate. □

In particular, we get an exact estimate for the dimension of exact maximal Lie algebras from $\mathcal{L}(N)$: $\dim(L) \leq \dim(N) + r$, where r is $r_s(\mathrm{Rad}(\mathrm{Der}(N)))$, i.e., semisimple rank (defined above) for the radical of the Lie algebra $\mathrm{Der}(N)$. Hence, the inequality $r \leq \dim N/[N,N]$ holds, which generalizes the estimate obtained in [1] for the case of solvable Lie algebras (but only in the case when Lie algebra L is exact).

There is no uniqueness of maximal exact Lie algebra in $\mathcal{L}(N)$, because for solvable (and, hence, automatically exact) Lie algebras we give above examples of non-uniqueness of maximal Lie algebras in $\mathcal{R}(N)$.

Let us briefly note that for arbitrary Lie algebras one can prove a statement analogous to Proposition proved above for the case of solvable Lie algebras.

In conclusion, let us say a few words about the generalization of results of this paper onto the case of an arbitrary perfect field of characteristic 0 (e.g., for the case of real-valued finite-dimensional Lie algebras). Of course, Theorem 2 holds in this case, too. As for Theorems 1 and 3, here we must make a minor modification of the statements.

The Chevalley decomposition, serving as the base of the proof of Theorems 1 and 3, holds for a perfect field of characteristic 0, too. However, say, for a solvable Lie algebra R the Lie subalgebra $F \subset \mathrm{Der}(R)$ is not necessarily diagonalizable. Further, in the construction of the Lie algebra $H(L)$ we must use the same terms, as in the case of an algebraically closed field, but the action F is not necessarily diagonalizable, that requires some alterations in the proofs of Theorems 1 and 3.

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