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On Representation of Stinespring's Type for n-Tuples of Completely Positive Maps in Hilbert $C^\star$-Modules

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Abstract—Here we prove an analog of the Stinespring's theorem for n-tuples of completely positive maps in Hilbert $C^\star$-modules.

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INTRODUCTION

Completely positive maps acting in operator algebras comprise a useful instrument in quantum mechanics and quantum calculus theory study. Let A and B be some $C^\star$-algebras. A completely positive map $\varphi : A \to B$ is a linear map such that $[\varphi(a_{ij})]_{i,j=1}^n$ is a positive element of the $C^\star$-algebra $M_n(B)$ of $n \times n$-square matrices with elements from B, for all positive matrices $[(a_{ij})]_{i,j=1}^n$ in $M_n(A)$, $n \in \mathbb{N}$. V. F. Stinespring in [1] proved that any completely positive map $\varphi : A \to L(H)$ acting from A into the algebra $L(H)$ of linear bounded operators in the Hilbert space H can be represented as $\varphi(\cdot) = S^\star\pi(\cdot)S$, here π is a representation of the algebra A in the Hilbert space K and S is some linear bounded operator mapping H into K. The structure of n-completely positive maps which we assume to be square $n \times n$-matrices whose elements are linear positive maps from the $C^\star$-algebra A into $L(H)$ was studied in [2].

The Hilbert $C^\star$-modules are a generalization of both Hilbert spaces and $C^\star$-algebras. In [3] the authors proved an analog of the Stinespring's theorem for completely positive maps in Hilbert $C^\star$-modules. Later the work [4] eliminated some technical restrictions. In [5] the problem was considered from the viewpoint of the $C^\star$-correspondences theory. Papers [6, 7] contain covariant version of the Stinespring's theorem and noncommutative variant of the Radon—Nikodym theorem. In [8] the authors proved an analog of the Stinespring's theorem for Hilbert modules over local $C^\star$-algebras. Here we prove one analog of the Stinespring's theorem for n-tuples of completely positive maps acting in Hilbert $C^\star$-modules.

1. PRELIMINARY CONSIDERATIONS

Let us give certain preliminary information necessary for our constructions. Our aim here is to fix the terms and notation and to introduce the necessary notions. All the necessary properties of $C^\star$-algebras, Hilbert $C^\star$-modules and completely positive maps can be found in [9—12].

1.1. We denote Hilbert spaces by H_1, H_2, K_1, and K_2. The inner products and norms generated by these products relative to the spaces are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. We assume that the inner products are linear on the second variable and skew-linear with respect to the first one. The spaces of all linear bounded operators acting from H_1 into H_2 (H_1) are denoted by $L(H_1, H_2)$ and $L(H_1) := L(H_1, H_1)$. We denote $C^\star$-algebras by A, B, etc.

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Recall that for the $C^\star$-algebra A an element $x \in A$ is *positive*, if $x = x^\star$ and $\sigma(x) \subset \mathbb{R}_+$, here $\sigma(x)$ is the spectrum of x.

We denote by $M_n(A)$ the $\star$-algebra of all matrices over the algebra A. Here the matrices addition and multiplication and their multiplication by a basic field element are defined similarly to the case of the ordinary scalar matrices. It is known that $M_n(A)$ is also a $C^\star$-algebra ([10], theorem 3.4.2). A linear map $\varphi : A \to B$ is *completely positive* if the linear map $\varphi^n : M_n(A) \to M_n(B)$ given by the formula $\varphi^n([a_{ij}]_{i,j=1}^n) = [\varphi(a_{ij})]_{i,j=1}^n$ is positive for all $n \in \mathbb{N}$.

Any square $n \times n$-matrix of linear maps $(\varphi_{ij})_{i,j=1}^n$ from A to B can be considered as a linear map $[\varphi] : M_n(A) \to M_n(B)$ of the matrix algebras given by the relation $[\varphi]((a_{ij})_{i,j=1}^n) = (\varphi_{ij}(a_{ij}))_{i,j=1}^n$. We say that the matrix $(\varphi_{ij})_{i,j=1}^n$ is an n-*completely positive* mapping from A to B if $[\varphi]$ is a completely positive map from $M_n(A)$ to $M_n(B)$.

Let $[\varphi_{ij}]_{i,j=1}^n$ be an n-completely positive map from A to B. Denote by D_{ij} the matrix whose all elements are zero with the exception of one element situated at (i, j) and being equal d. Then we have $[\varphi](D_{ij})^\star = [\varphi](D_{ij}^\star)$. Hence

$$\varphi_{ij}(d)^\star = \varphi_{ji}(d^\star).$$

A pre-Hilbert A-module is a complex vector space V which is also a right A-module with a sesquilinear form $\langle \cdot, \cdot \rangle : V \times V \to A$, meeting the following conditions:

(0.1) $\langle x, x \rangle \geq 0$ for all $x \in V$;

(0.2) $\langle x, y \rangle^\star = \langle y, x \rangle$ for all $x, y \in V$;

(0.3) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ for all $x \in V$;

(0.4) $\langle x, ya \rangle = \langle y, x \rangle a$ for all $x, y \in V$; $a \in A$.

We say that V is a *Hilbert A-C**-module or simply a $C^\star$-module if V is the Banach space with respect to the norm

$$\|x\| := \sqrt{\langle x, x \rangle}, \ \ x \in V.$$

A Hilbert $C^\star$-module V is *complete* if the two-sided closed ideal $\langle V, V \rangle_A$ generated by $\{\langle x, y \rangle_A : x, y \in V\}$ coincides with A. Further we assume that all the considered Hilbert $C^\star$-modules are complete. Note that the space $L(H_1, H_2)$ is the Hilbert $L(H_1)$-module for all Hilbert spaces H_1, H_2 with respect to the following operations:

(0.6) Exterior product $(T, S) \mapsto TS : L(H_1, H_2) \times L(H_1) \to L(H_1, H_2)$;

(0.7) Inner product $(T, S) \mapsto T^\star S : L(H_1, H_2) \times L(H_1, H_2) \to L(H_1)$.

A representation of the Hilbert $C^\star$-module V on the pair of Hilbert spaces H_1 and H_2 is a mapping $\Psi : V \to L(H_1, H_2)$ such that there exists a $\star$-representation π of the algebra A in the Hilbert space H_1 and for all $x, y \in V$ the relation $\langle \Psi(x), \Psi(y) \rangle = \pi(\langle x, y \rangle)$ holds true. If the $C^\star$-modulus V is complete, the representation π relative to Ψ is unique. A representation $\Psi : V \to L(H_1, H_2)$ is *nondegenerate* if $[\Psi(V)(H_1)] = H_2$ and $[\Psi(V)^\star(H_2)] = H_1$ (here $[Y]$ is a closed subspace of the Hilbert space Z generated by $Y \subset Z$). A linear map $\Phi : V \to L(H_1, H_2)$ is a *completely positive* map of Hilbert $C^\star$-modules if there exists a linear completely positive mapping of $C^\star$-algebras $\varphi : A \to L(H_1)$ such that $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in V$.

2. MAIN RESULT

Consider the Hilbert $C^\star$-module V over some $C^\star$-algebra A and let H_1 and H_2 be Hilbert spaces. Let Φ_i, $i \in \{1, \ldots, n\}$, be certain mappings $\Phi_i : V \to L(H_1, H_2)$.

A set of n-maps $\Phi = (\Phi_1, \ldots, \Phi_n)$ is *completely positive* if there exists a n-completely positive map $[\varphi]$ from A to $L(H_1)$ such that $[\langle \Phi_i(x), \Phi_j(y) \rangle]_{i,j=1}^n = [\varphi_{ij} \langle x, y \rangle]_{i,j=1}^n$ for all $x, y \in V$. Let us formulate the main result.

Theorem 2.1. *Let A be a unital $C^\star$-algebra, V be a Hilbert $C^\star$-module, $[\varphi_{ij}]_{i,j=1}^n : A \to L(H_1)$ be some n-completely positive map and $\Phi = (\Phi_1, \dots, \Phi_n)$, $\Phi_i : V \to L(H_1, H_2)$, $i \in \{1, \dots, n\}$ stand for $[\varphi]$-completely positive n-tuple. Then there exists the following data set*: $(\pi, S_1, \dots, S_n, K_1)$, $(\Psi, W_1, \dots, W_n, K_2)$, *here*

(1) K_1 *and* K_2 *are Hilbert spaces*;

(2) $\Psi : V \to L(K_1, K_2)$ *is a representation of* V *in the Hilbert spaces* K_1 *and* K_2, $\pi : A \to L(K_1)$ *is a $\star$-homomorphism, associated with* Ψ, $S_i : H_1 \to K_1$ *are isometric linear operators*, $W_i : H_2 \to K_2$ *are coisometric linear operators for any* $i \in \{1, \dots, n\}$ *such that*

$$\varphi_{ij}(a) = S_i^\star \pi_A(a) S_j \ \ \textit{for any} \ \ a \in A; \ \ i, j \in \{1, \dots, n\},$$

and

$$\Phi_i(x) = W_i^\star \Psi(x) S_i \ \ \textit{for any} \ \ x \in V \ \ \textit{and every} \ \ i \in \{1, \dots, n\}.$$

Proof. Let us show the existence of π, K_1 and $S_1, \dots, S_n$. Denote by $(A \otimes H_1)^n$ the direct sum of n-copies of the algebraic tensor product $A \otimes H_1$. Note that any element of the vector space $(A \otimes H_1)^n$ can be written as follows:

$$\left(\sum_{s=1}^{m_1} a_{1s} \otimes \xi_{1s}, \dots, \sum_{s=1}^{m_n} a_{ns} \otimes \xi_{ns} \right) \ \text{and} \ \ m = \max\{m_1, \dots, m_n\}.$$

Completion of these sums with zeros makes it possible to state that any element of $(A \otimes H_1)^n$ can be represented as $\sum\limits_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n$. Consider now the map $\langle \cdot, \cdot \rangle_0 : (A \otimes H_1)^n \times (A \otimes H_1)^n \to \mathbb{C}$ given by the formula

$$\Big\langle \sum_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n, \sum_{t=1}^{l} (b_{jt} \otimes \eta_{jt})_{j=1}^n \Big\rangle_0 = \sum_{s,t=1}^{m,l} \sum_{i,j=1}^{n} \langle \xi_{is}, \varphi_{ij}(a_{is}^\star b_{jt}) \eta_{jt} \rangle$$

on the vector space $(A \otimes H_1)^n$.

This mapping is $\mathbb{C}$-linear with respect to the second variable and skew-linear by the first one. Moreover we have the equality

$$\begin{aligned} \bigg(\Big\langle \sum_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n, \ \sum_{t=1}^{l} (b_{jt} \otimes \eta_{jt})_{j=1}^n \Big\rangle_0 \bigg)^\star &= \sum_{s,t=1}^{m,l} \sum_{i,j=1}^{n} \big(\langle \xi_{is}, \varphi_{ij}(a_{is}^\star b_{jt}) \eta_{jt} \rangle \big)^\star \\ &= \sum_{s,t=1}^{m,l} \sum_{i,j=1}^{n} \langle \eta_{jt}, (\varphi_{ij}(a_{is}^\star b_{jt}))^\star \xi_{is} \rangle = \Big\langle \sum_{t=1}^{l} (b_{jt} \otimes \eta_{jt})_{j=1}^n, \sum_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n \Big\rangle_0 \end{aligned}$$

for all $(a_{is} \otimes \xi_{is})_{i=1}^n$, $(b_{jt} \otimes \eta_{jt})_{j=1}^n \in (A \otimes H_1)^n$. Finally we have one more important property of the map $\langle \cdot, \cdot \rangle_0$ which allows us to define an inner product on the appropriate factor-space $(A \otimes H_1)^n$ by the relation

$$\Big\langle \sum_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n, \sum_{s=1}^{m} (a_{is} \otimes \xi_{is})_{i=1}^n \Big\rangle_0 \geq 0.$$

Nonnegativity of this relation is provided due to complete positivity of $[\varphi]$ acting from $M_n(A)$ to $M_n(L(H_1))$. Put

$$M := \{\zeta \in (A \otimes H_1)^n : \langle \zeta, \zeta \rangle_0 = 0\}.$$

Now the Cauchy–Schwartz inequality implies that M is a subspace of $(A \otimes H_1)^n$. Then it is possible to define the scalar product $\langle \zeta_1 + M, \zeta_2 + M \rangle := \langle \zeta_1, \zeta_2 \rangle_0$ on the factor-space $(A \otimes H_1)^n / M$. We denote the completion of $(A \otimes H_1)^n / M$ with respect to the norm given by this inner product by K_1. Also denote by ξ_i the element of $(A \otimes H_1)^n$ whose ith component equals $1 \otimes \xi$ and all the other components vanish. We now define the maps $S_i : H_1 \to K_1$ by the formula

$$S_i(\xi) = \xi_i + M.$$

Denote by $\xi_{a,i}$ an element of the space $(A \otimes H_1)^n/M$ such that its ith component is $a \otimes \xi$ and all the other components vanish. Let $a \in A$. Consider the linear map $\pi(a) : (A \otimes H_1)^n \to (A \otimes H_1)^n$ given by the relation $\pi(a)(a_i \otimes \xi_i)_{i=1}^n = (aa_i \otimes \xi_i)_{i=1}^n$. The operator $\pi(a)$ can be extended by continuity to linear map from K_1 to K_1. We keep the notation $\pi(a)$ for this extension. We prove that $\pi(a)$ is a representation of the algebra A in the Hilbert space K_1 by the scheme similar to that of theorem 3.3.2 in [13]. Direct calculation shows us that $\pi(a_i)S_i\xi_i = \xi_{i,a} + M$. Thus the subspace K_1 generated by elements $\pi(a_i)S_i\xi_i$, $i \in \{1, \dots, n\}$, $\xi_i \in H_1$, $a_i \in A$ is exactly $(A \otimes H_1)^n/M$.

Consider the space $K_2 := [\{\Psi_i(V)(H_1)\}]$, $i = 1, \dots, n$. The operator $\Psi : V \to L(K_1, K_2)$ can be given by the formula

$$\Psi(x)\Big(\sum_{s=1}^m \pi(a_{1s})S_1\xi_{1s}, \dots, \sum_{s=1}^m \pi(a_{ns})S_n\xi_{ns}\Big) := \sum_{s=1}^m \Phi_1(xa_{1s})\xi_{1s} + \dots + \sum_{s=1}^m \Phi_n(xa_{ns})\xi_{ns} = \sum_{i=1}^n \sum_{s=1}^m \Phi_i(xa_{is})\xi_{is},$$

here $x \in V$, $a_{is} \in A$, $\xi_{is} \in H_1$, $1 \le s \le m$, $m \in \mathbb{N}$. Let us prove that the linear map $\Psi(x)$ is bounded. Indeed we have the relations

$$\begin{aligned}
\Big\|\Psi(x)\Big(\sum_{s=1}^m \pi(a_{1s})S_1\xi_{1s}, \dots, \sum_{s=1}^m \pi(a_{ns})S_n\xi_{ns}\Big)\Big\|^2 &= \Big\|\sum_{s=1}^m \Phi_1(xa_{1s})\xi_{1s} + \dots + \sum_{s=1}^m \Phi_n(xa_{ns})\xi_{ns}\Big\|^2 \\
= \Big\langle \sum_{s=1}^m \sum_{i=1}^n \Phi_i(xa_{is})\xi_{is}, \sum_{r=1}^m \sum_{j=1}^n \Phi_j(xa_{jr})\xi_{jr}\Big\rangle &= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, \Phi_i(xa_{is})^\star \Phi_j(xa_{jr})\xi_{jr}\rangle \\
= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, \varphi_{ij}(\langle xa_{is}, xa_{jr}\rangle)\xi_{jr}\rangle &= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \xi_{is}, S_i^\star \pi(a_{is}^\star \langle x, x\rangle a_{jr})S_j\xi_{jr}\rangle \\
&= \sum_{s,r=1}^m \sum_{i,j=1}^n \langle \pi(a_{is})S_i(\xi_{is}), \pi(\langle x, x\rangle)\pi(a_{jr})S_j\xi_{jr}\rangle \\
&= \Big\langle \sum_{s=1}^m \sum_{i=1}^n \pi(a_{is})S_i(\xi_{is}), \pi(\langle x, x\rangle)\Big(\sum_{r=1}^m \sum_{j=1}^n \pi(a_{jr})S_j\xi_{jr}\Big)\Big\rangle \\
&\le \|\pi(\langle x, x\rangle)\| \Big\|\Big(\sum_{r=1}^m \sum_{i=1}^n \pi(a_{i,r})S_i\xi_{i,r}\Big)\Big\|^2 \le \|x\|^2 \Big\|\Big(\sum_{r=1}^m \sum_{i=1}^n \pi(a_{i,r})S_i\xi_{i,r}\Big)\Big\|^2.
\end{aligned}$$

Hence the operator $\Psi(x)$ is bounded on the dense subspace and consequently can be extended to the whole space K_1. Again we keep the notation for the extended operator. Let us now show that the mapping Ψ is a representation of Hilbert $C^\star$-modules. Consider $x, y \in V$; $a_{is}, b_{jr} \in A$; $\xi_{is}, \eta_{jr} \in H_1$; $1 \le i, j \le n$; $1 \le s \le l$, $1 \le r \le m$; $n, m \in \mathbb{N}$. Then the following holds:

$$\Big\langle \Psi(x)^\star \Psi(y)\Big(\sum_{r=1}^m \sum_{j=1}^n \pi(b_{j,r})S_j\eta_{j,r}\Big), \sum_{s=1}^l \sum_{i=1}^n \pi(a_{i,s})S_i\xi_{i,s}\Big\rangle = \Big\langle \sum_{r=1}^m \sum_{j=1}^n \Phi_j(yb_{jr})\eta_{jr}, \sum_{s=1}^l \sum_{i=1}^n \Phi_i(xa_{is})\xi_{is}\Big\rangle$$

$$= \sum_{s=1}^{l}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\langle \Phi_i(xa_{is})^\star\Phi_j(yb_{jr})\eta_{jr},\xi_{is}\rangle = \sum_{s=1}^{l}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\langle \varphi_{ij}(\langle xa_{is}, yb_{jr}\rangle)\eta_{jr},\xi_{is}\rangle$$

$$= \sum_{s=1}^{l}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\langle S_i^\star\pi(a_{is}^\star\langle x,y\rangle a_{jr})S_j\eta_{jr},\xi_{is}\rangle$$

$$= \Big\langle \pi(\langle x,y\rangle)\Big(\sum_{r=1}^{m}\sum_{j=1}^{n}\pi(b_{j,r})S_j\eta_{j,r}\Big), \sum_{s=1}^{l}\sum_{i=1}^{n}\pi(a_{i,s})S_i\xi_{i,s}\Big\rangle.$$

Thus the equality $\Psi(x)^\star\Psi(y) = \langle\Psi(x),\Psi(y)\rangle = \pi(\langle x,y\rangle)$ holds true on the dense set. Hence by continuity the operators $\langle\Psi(x),\Psi(y)\rangle$ and $\pi(\langle x,y\rangle)$ coincide on the whole space K_1. Note that $K_2\subset H_2$. Let us denote the closed subspace $[\Phi_i(V)(H_1)]$ of H_2 by K_{2i}. Let $W_i := P_{K_{2i}}$, $i\in\{1,\dots,n\}$ be the orthogonal projection onto the space K_{2i}. Then the operator $W_i^\star : K_{2i}\to H_2$ is an inclusion operator. Hence $W_iW_i^\star = I_{K_{2i}}$ for any $i\in\{1,\dots,n\}$. Now for all $x\in V$ and $\xi\in H_1$ we have $\Phi_i(x)(\xi) = W_i^\star\Psi(x)S_i(\xi)$, $i\in\{1,\dots,n\}$. □

Let $[\varphi]$ and Φ meet the assumptions of Theorem 2.1. We say that the data set $(\pi, S_1,\dots,S_n,K_1)$, $(\Psi, W_1,\dots,W_n,K_2)$ is a *Stinespring representation* for (φ,Φ) if conditions (1)–(2) of Theorem 2.1 hold. We denote by K_{2i} (K'_{2i}), $i\in\{1,\dots,n\}$ the closed subspaces $[\Psi(V)S_i(H_1)]$ $([\Psi'(V)S'_i(H_1)])$. This representation is *minimal* if

1) $K_1 = [\{\pi(A)S_i(H_1) : i=1,\dots,n\}]$,

2) $K_2 = [\{\Psi(V)S_i(H_1) : i=1,\dots,n\}]$.

Theorem 2.2. *Let $[\varphi]$ and Φ be similar to those in Theorem* 2.1. *Assume that $(\pi,S_1,\dots,S_n,K_1)$, $(\Psi,W_1,\dots,W_n,K_2)$ and $(\pi',S'_1,\dots,S'_n K'_1)$, $(\Psi',W'_1,\dots,W'_n,K_2)$ are minimal Stinespring representations. Then there exist unitary operators $U_1 : K_1\to K'_1$, $U_2 : K_2\to K'_2$ such that*

(1) $U_1S_i = S'_i\ \ \forall i\in\{1,\dots,n\}$; $U_1\pi(a) = \pi'(a)U_1\ \ \forall a\in A$.

(2) $U_2W_i = W'_i\ \ \forall i\in\{1,\dots,n\}$; $U_2\Psi(x) = \Psi'(x)U_1\ \ \forall x\in V$.

The diagram is commutative

$$\begin{array}{ccccccccc}
H_1 & \xrightarrow{S_i} & K_1 & \xrightarrow{\pi(a)} & K_1 & \xrightarrow{\Psi(x)} & K_2 & \xleftarrow{W_i} & H_2\\
\big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow{\scriptstyle U_1} & & \big\downarrow{\scriptstyle U_1} & & \big\downarrow{\scriptstyle U_2} & & \big\downarrow{\scriptstyle \mathrm{Id}}\\
H_1 & \xrightarrow{S'_i} & K'_1 & \xrightarrow{\pi'(a)} & K'_1 & \xrightarrow{\Psi'(x)} & K'_2 & \xleftarrow{W'_i} & H_2
\end{array}$$

for all $a\in A$, $x\in V$, $i\in\{1,\dots,n\}$.

Proof. Let us prove existence of the unitary operator $U_1 : K_1\to K'_1$. First define U_1 on the dense linear subset generated by the set $\{\pi(A)S_i(H_1); i=1,\dots,n\}$:

$$U_1\Big(\sum_{s=1}^{m}\sum_{i=1}^{n}\pi(a_{is})S_i(\xi_{is})\Big) := \Big(\sum_{s=1}^{m}\sum_{i=1}^{n}\pi'(a_{is})S'_i(\xi_{is})\Big),$$

here $a_{is}\in A$, $\xi_{is}\in H_1$, $m\in\mathbb{N}$. Direct computation shows that the operator U_1 is an isometry. We denote the extension of the operator U_1 by continuity onto the space K_1 by the same symbol. Then U_1 is a unitary operator meeting condition (1). Let us now define the operator U_2 on the dense subset (linear space) generated by the set $\{\Psi(V)S_i(H_1) : i=1,\dots,n\}$:

$$U_2\Big(\sum_{s=1}^{m}\Psi(x_{1s})S_1\xi_{1s} + \dots + \sum_{s=1}^{m}\Psi(x_{ns})S_n\xi_{ns}\Big) := \sum_{s=1}^{m}\Psi'(x_{1s})S'_1\xi_{1s} + \dots + \sum_{s=1}^{m}\Psi'(x_{ns})S'_n\xi_{ns},$$

here $x_{is} \in V$, $\xi_{is} \in H_1$, $m \in \mathbb{N}$. Since the operators S_i, S'_i are isometries for any $i \in \{1, \ldots, n\}$, we have

$$U_2\biggl(\sum_{s=1}^{m} \Psi(x_{is}) S_i \xi_{ns}\biggr) = \sum_{s=1}^{m} \Psi'(x_{is}) S'_i \xi_{ns},$$

and, consequently, $U_2(K_{2i}) = K'_{2i}$. It is possible now to write down the following:

$$\begin{aligned}
&\biggl\|\biggl(\sum_{s=1}^{m} \Psi'(x_{1s}) S'_1 \xi_{1s} + \cdots + \sum_{s=1}^{m} \Psi'(x_{ns}) S'_n \xi_{ns}\biggr)\biggr\|^2 \\
&\quad = \biggl\langle \sum_{i=1}^{n}\sum_{s=1}^{m} \Psi'(x_{is}) S'_i \xi_{is}, \sum_{j=1}^{n}\sum_{r=1}^{m} \Psi'(x_{jr}) S'_j \xi_{jr}\biggr\rangle = \sum_{s,r=1}^{m}\sum_{i,j=1}^{n} \langle \Psi'(x_{is}) S'_i \xi_{is}, \Psi'(x_{jr}) S'_j \xi_{jr}\rangle \\
&\qquad = \sum_{s,r=1}^{m}\sum_{i,j=1}^{n} \langle \xi_{is}, S'^{\star}_i \pi'(\langle x_{is}, x_{jr}\rangle) S'_j(\xi_{jr})\rangle \sum_{s,r=1}^{m}\sum_{i,j=1}^{n} \langle \xi_{is}, \varphi_{ij}(\langle xa_{is}, xa_{jr}\rangle)(\xi_{jr})\rangle \\
&\qquad = \sum_{s,r=1}^{m}\sum_{i,j=1}^{n} \langle \xi_{is}, S^{\star}_i \pi(\langle x_{is}, x_{jr}\rangle) S_j(\xi_{jr})\rangle \sum_{s,r=1}^{m}\sum_{i,j=1}^{n} \langle \Psi(x_{is}) S_i \xi_{is}, \Psi(x_{jr}) S_j \xi_{jr}\rangle \\
&\qquad\qquad = \biggl\langle \sum_{i=1}^{n}\sum_{s=1}^{m} \Psi(x_{is}) S_i \xi_{is}, \sum_{j=1}^{n}\sum_{r=1}^{m} \Psi(x_{jr}) S_j \xi_{jr}\biggr\rangle \\
&\qquad\qquad\qquad = \biggl\|\sum_{s=1}^{m} \Psi(x_{1s}) S_1 \xi_{1s} + \cdots + \sum_{s=1}^{m} \Psi(x_{ns}) S_n \xi_{ns}\biggr\|^2.
\end{aligned}$$

Hence U_2 is an isometry and the operator U_2 can be extended to the whole space K_2. We again keep the notation for the extended operator. The operator U_2 is unitary. Note that $(\pi, S_1, \ldots, S_n, K_1)$, $(\Psi, W_1, \ldots, W_n, K_2)$ $(\pi', S'_1, \ldots, S'_n K'_1)$, $(\Psi', W'_1, \ldots, W'_n, K_2)$ are Stinespring representations for $([\varphi], \Phi)$. Hence for any $i \in \{1, \ldots, n\}$ we have

$$\Phi_i(x) = W^{\star}_i \Psi(x) S_i = W'^{\star}_i \Psi'(x) S'_i = W'^{\star}_i U_2 \Psi(x) S_i.$$

So

$$(W^{\star}_i - W'^{\star}_i U_2)\Psi(x) S_i = 0 \Rightarrow (W^{\star}_i - W'^{\star}_i U_2)\Psi(x) S_i(\xi) = 0 \ \forall x \in V,\ \ \xi \in H_1,\ \ i \in \{1, \ldots, n\}.$$

Thus $U_2 W_i = W'_i$ for any $i \in \{1, \ldots, n\}$. Finally we prove that $U_2\Psi(x) = \Psi'(x)U_1$ on the dense subspace

$$\biggl\{\sum_{s=1}^{m}\sum_{i=1}^{n} \pi(a_{is}) S_i(\xi_{is});\ a_{is} \in A,\ \xi_{is} \in H_1,\ m \in \mathbb{N}\biggr\}.$$

Recall that any representation $\Psi : V \to L(K_1, K_2)$ of Hilbert $C^{\star}$-modules is A-linear in the following sense: $\Psi(xa) = \Psi(x)\pi(a)$ for all $x \in V$ and $a \in A$. Since Ψ and Ψ' are representations of the Hilbert $C^{\star}$-modules associated with π and π', respectively, we obtain

$$\begin{aligned}
&U_2\Psi(x)\biggl(\sum_{s=1}^{m}\sum_{i=1}^{n} \pi(a_{is}) S_i(\xi_{is})\biggr) = U_2\biggl(\sum_{s=1}^{m}\sum_{i=1}^{n} \Psi(xa_{is}) S_i \xi_{is}\biggr) \\
&\quad = \sum_{s=1}^{m}\sum_{i=1}^{n} \Psi'(xa_{is}) S'_i \xi_{is} = \Psi'(x)\biggl(\sum_{s=1}^{m}\sum_{i=1}^{n} \pi'(a_{is}) S'_i(\xi_{is})\biggr) = \Psi'(x)U_1\biggl(\sum_{s=1}^{m}\sum_{i=1}^{n} \pi(a_{is}) S_i(\xi_{is})\biggr).
\end{aligned}$$

Thus by continuity the equality $U_2\Psi(x) = \Psi'(x)U_1$ holds on all the space K_1. □

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