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Sharp Estimates of Hardy Constants for Domains with Special Boundary Properties

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Abstract—We investigate the behavior of Hardy constants in domains whose boundaries have at least one regular point.

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1. Introduction. Let Ω be a domain, i.e., an open connected set in the Euclidean space $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, and let $C_0^1(\Omega)$ stand for the family of continuously differentiable functions $f : \Omega \to \mathbb{R}$ with compact supports in the domain Ω. There are many inequalities which connect integrals over the domain for functions $f \in C_0^1(\Omega)$ and their gradients in various Lebesgue spaces with weights. Using these inequalities and the standard closure by norm one obtains inequalities in corresponding Sobolev spaces. In the books by S. L. Sobolev [1] and V. Maz'ya [2] one can find basic results with historical remarks.

The following Hardy inequality presents a branch of the above extensive topic:

$$\int_\Omega \frac{|\nabla f|^p}{\delta^{s-p}}\,dx \geqslant c_p(s,\Omega)\int_\Omega \frac{|f|^p}{\delta^s}dx \quad \forall f \in C_0^1(\Omega), \tag{1}$$

where ∇f is the gradient of the function f, $\delta = \operatorname{dist}(x, \partial\Omega)$ is the distance from the point $x \in \Omega$ to the boundary of the domain, p and s are fixed numerical parameters, $p \in [1,\infty)$ and $s \in (1,\infty)$, $c_p(s,\Omega)$ is the greatest constant which is appropriate at this inequality. Clearly, $c_p(s,\Omega) \in [0,\infty)$.

It is known that for any convex subdomain $\Omega \subset \mathbb{R}^n$ one has the equality $c_p(p,\Omega) = ((p-1)/p)^p$. For $n = 1$ this fact is a consequence of the classical Hardy results, and for the case $n \geq 2$ it is proved in the papers [3, 4]. If $n \geq 2$ and $s > n$, then $c_p(s,\Omega) \geq ((s-n)/p)^p$ for arbitrary domains Ω (see [5, 6]). In the case $1 < s \leq n$ there are several interesting results and many open problems, and the theory of inequality (1) for $s \in (1,n]$ is far from being complete (see [7–13]).

Following E. B. Davies [7], the constant $c = 1/c_p(s,\Omega)$ will be called a strong Hardy constant. In Item 2 we will define S-regular boundary points and prove the sharp lower estimate for the strong Hardy constant in the situation when there is at least one S-regular boundary point of the domain Ω. In Item 3, using the Davies definitions [7] about boundary properties of domains, we obtain an extension of Davies' results on the strong and weak Hardy constants to the case of arbitrary $s > 1$.

2. Main result.

Definition 1. A point $y \in \partial\Omega$ will be called *bilaterally accessible by balls* if there exist two non-empty balls (discs in the case $n = 2$) $B_1 = B_1(a_1, r_1)$ and $B_2 = B_2(a_2, r_2)$ such that $B_1 \subset \Omega$, $B_2 \subset \mathbb{R}^n \setminus \Omega$ and $y \in \overline{B_1} \bigcap \overline{B_2}$ (i.e., $\overline{B_1} \bigcap \overline{B_2} = \{y\}$).

Let $\varepsilon \in (0, 1/4)$ and let y and x_0 be fixed points in $\mathbb{R}^n$, $r_0 := |x_0 - y| > 0$. We define the frustum of a spherical cone $S_y(x_0, \varepsilon)$ as a set of points $x \in \mathbb{R}^n$, satisfying two following conditions:

$$r_0(1 - 2\sqrt{\varepsilon}) < |x - x_0| < r_0, \quad \cos(\sqrt{\varepsilon}) < (x - x_0, y - x_0)/(r_0|x - x_0|) \leq 1.$$

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Definition 2. A point $y \in \partial\Omega$ will be called *S-regular* if there exist a point $x_0 \in \mathbb{R}^n$, constants $\varepsilon_0 \in (0, 1/4)$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the following conditions are fulfilled: 1) $S_y(x_0, \varepsilon) \subset \Omega$; 2) $r_0 - |x - x_0| \le \operatorname{dist}(x, \partial\Omega) \le r_0(1 + C\varepsilon) - |x - x_0|$ for $r_0 = |y - x_0|$ and any point $x \in S_y(x_0, \varepsilon)$.

Two simple geometrical assertions are valid: a) *If the point $y \in \partial\Omega$ is bilaterally accessible by balls, then it is an S-regular point.* b) *There exist a domain Ω and an S-regular point $y \in \partial\Omega$ such that y is not bilaterally accessible by balls.*

Our main result is the following

Theorem 1. *Suppose that $n \ge 2$, $p \in [1, \infty)$, $s \in (1, \infty)$ and the domain $\Omega \subset \mathbb{R}^n$ has at least one S-regular boundary point $y \in \partial\Omega$, then $c_p(s, \Omega) \le ((s-1)/p)^p$.*

Scheme of the proof of Theorem 1. First we consider the case $n = 2$. Since the Hardy constant is an invariant of the linear conformal transformations, without loss of generality we can suppose that $y = (1, 0)$ and $x_0 = (0, 0)$. In this case the set $A(\varepsilon) := S_{(1,0)}((0, 0), \varepsilon)$ will be presented as the set $A(\varepsilon) = \{(r, \theta) \in \mathbb{R}^2 : 1 - 2\sqrt{\varepsilon} =: \rho < r < 1, |\theta| < \sqrt{\varepsilon} =: \alpha\}$ in the polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$. According to the definition of S-regular point, the distance function $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ satisfies the inequalities $1 - r \le \delta(x) \le 1 - r + C\varepsilon$ for any point $x \in A(\varepsilon)$. It is evident that inequality (1) is valid for any function $f = U_\varepsilon \in C_0^1(A(\varepsilon)) \subset C_0^1(\Omega)$. We will consider functions $U_\varepsilon(r, \theta) = \varphi(r)\psi(\theta)$, where $\varphi \in C_0^1(\rho, 1)$ and $\psi \in C_0^1(-\alpha, \alpha)$. We will suppose that φ is an arbitrary function belonging to the above family and that the function ψ is chosen as one special representative of the family. Namely, consider first the function $\psi_0(\theta) = \cos(\frac{\pi}{2\alpha}\theta)$, satisfying the equality

$$\int_{-\alpha}^{\alpha} |\psi_0'(\theta)|^p \, d\theta = \Big(\frac{\pi}{2\alpha}\Big)^p \int_{-\alpha}^{\alpha} |\psi_0(\theta)|^p \, d\theta.$$

Taking into account the facts that $\pi/2 < 2$, $\psi_0(-\alpha) = \psi_0(\alpha) = 0$, $\psi_0 \in C^1[-\alpha, \alpha]$, it is not difficult to show that there exists a function $\psi \in C_0^1(-\alpha, \alpha)$, not equal to zero identically and satisfying the inequality

$$\int_{-\alpha}^{\alpha} |\psi'(\theta)|^p \, d\theta \le \frac{2^p}{\alpha^p} \int_{-\alpha}^{\alpha} |\psi(\theta)|^p \, d\theta. \tag{2}$$

We will suppose that this function $\psi(\theta)$ (for which inequality (2) is valid) is taken as the factor in the formula for U_ε.

Next, we have

$$|\nabla U_\varepsilon|^p = \bigg(\Big(\frac{\partial U_\varepsilon}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial U_\varepsilon}{\partial \theta}\Big)^2\bigg)^{p/2} \le \bigg(\Big|\frac{\partial U_\varepsilon}{\partial r}\Big| + \frac{1}{r}\Big|\frac{\partial U_\varepsilon}{\partial \theta}\Big|\bigg)^p. \tag{3}$$

Now, consider the basic inequality (1) for functions $f(x) = U_\varepsilon(r, \theta) = \varphi(r)\psi(\theta)$ with the above choice of $\psi(\theta)$. Using (3) and the inequalities $1 - r < \delta < 1 - r + C\varepsilon$, we obtain

$$c_p(s, \Omega) \iint_{A(\varepsilon)} \frac{|\varphi(r)\psi(\theta)|^p}{(1 - r + C\varepsilon)^s} r \, dr \, d\theta \le \iint_{A(\varepsilon)} \frac{(|\varphi'(r)||\psi(\theta)| + |\varphi(r)||\psi'(\theta)|/r)^p}{\min\{(1-r)^{s-p}, (1 - r + C\varepsilon)^{s-p}\}} r \, dr \, d\theta.$$

We apply the Minkowski inequality for integrals to estimate the right-hand side. Next, we divide both sides of the resulting inequality by the integral $\int_{-\alpha}^{\alpha} |\psi(\theta)|^p \, d\theta$. Taking into account (2) and the inequalities $\rho < r < 1$, we obtain

$$\bigg(\int_\rho^1 \frac{\rho c_p(s, \Omega)|\varphi(r)|^p dr}{(1 - r + C\varepsilon)^s}\bigg)^{1/p} \le \bigg(\int_\rho^1 \frac{|\varphi'(r)|^p dr}{\min\{(1-r)^{s-p}, (1 - r + C\varepsilon)^{s-p}\}}\bigg)^{1/p} + \frac{2}{\rho\alpha}\bigg(\int_\rho^1 \frac{|\varphi(r)|^p \, dr}{\min\{(1-r)^{s-p}, (1 - r + C\varepsilon)^{s-p}\}}\bigg)^{1/p}.$$

From this we obtain a certain inequality for arbitrary function $g \in C_0^1(-1,1)$ using the relation $1-\rho = 2\sqrt{\varepsilon}$ and the changes of variables and functions according to the formulas $t = 1 - \frac{1-r}{\sqrt{\varepsilon}}$, $g(t) := \varphi(r)$. Clearly, this new inequality will contain an arbitrary parameter $\varepsilon \in (0, \varepsilon_0)$. As $\varepsilon \to 0$ one has the inequality

$$\sqrt[p]{c_p(s,\Omega)}\left(\int_{-1}^{1} \frac{|g(t)|^p}{(1-t)^s}\,dt\right)^{1/p} \le \left(\int_{-1}^{1} \frac{|g'(t)|^p}{(1-t)^{s-p}}\,dt\right)^{1/p} + 2\left(\int_{-1}^{1} \frac{|g(t)|^p}{(1-t)^{s-p}}\,dt\right)^{1/p},$$

which is valid for any function $g \in C_0^1(-1,1)$, consequently, it will also valid for all functions g_0, absolutely continuous on the segment $[-1,1]$ and satisfying the boundary conditions $g_0(\pm 1) = 0$. Consider an arbitrary number $\delta \in (0,1)$ and the function defined by equalities $g_0(t) = 1 + t$ for $-1 \le t \le 0$, $g_0(t) = (1-t)^{\frac{s-1+\delta}{p}}$ for $0 < t \le 1$. Applying our integral inequality to the function g_0, we obtain $\sqrt[p]{c_{p,s}}/\delta \le (s-1+\delta)(1/\delta+1)/p + C_1$, where C_1 is a constant. As $\delta \to 0$ one has the desired inequality $\sqrt[p]{c_{p,s}} \le (s-1)/p$.

The scheme of the proof is the same in the case $n \ge 3$. We suppose that the point x_0 is at the origin and that $y = (1, 0, \ldots, 0)$, and we use the generalized polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\theta_1$, $\ldots$, $x_n = r\sin\theta\sin\theta_1\ldots\sin\theta_{n-2}$. As test functions we consider functions $U_\varepsilon(r,\theta) = \varphi(r)\psi(\theta)$, where $\varphi \in C_0^1(\rho, 1)$ and $\psi \in C_0^1(-\alpha, \alpha)$. For such functions one has Eqs. (3). Other details of the proof are the same as in the case $n = 2$ except some nuances.

It is evident that any convex domain $\Omega \ne \mathbb{R}^n$ has boundary points bilaterally accessible by balls. Consequently, applying Theorem 1 to the convex domains one obtains the following assertion which extends the known equality ([3, 4]) $c_p(p,\Omega) = ((p-1)/p)^p$ to the general case taking into account the known estimate [5] $c_p(s,\Omega) \ge ((s-1)/p)^p$.

Theorem 2. *For all values of parameters $p \in [1,\infty)$, $s \in (1,\infty)$ and any convex domain $\Omega \subset \mathbb{R}^n$, $\Omega \ne \mathbb{R}^n$, the equality $c_p(s,\Omega) = ((s-1)/p)^p$ is valid.*

3. Generalization of results by E. B. Davies. We will consider the following Hardy type inequality

$$\int_\Omega \frac{|f(x)|^2}{\delta(x)^s}\,dx \le c\int_\Omega \frac{|\nabla f(x)|^2}{\delta(x)^{s-2}}\,dx + a\int_\Omega \frac{|f(x)|^2}{\delta(x)^{s-2}}\,dx \quad \forall f \in C_0^1(\Omega). \tag{4}$$

In [7] the case $s = 2$ of this inequality is considered. Following [7], we define the weak Hardy constant as the infimum of all $c > 0$ such that inequality (4) is valid with some $a < \infty$. We call the strong Hardy constant the smallest number c, for which the inequality is valid with $a = 0$. Following [7], it is possible to show that the weak Hardy constant has a local nature in some sense for any fixed $s \in (1,\infty)$.

We need the following definitions and notation from [7]. Suppose that $C_z := \{\widetilde{x} \in \mathbb{R}^{n-1} : -z < x_r < z,\ r = \overline{2,n}\}$, and $F : C_z \to \mathbb{R}$ satisfies the conditions $F(0) = 0$ and $|F(\widetilde{x}) - F(\widetilde{y})| \le \gamma|x-y|$ for some $\gamma \ge 0$ and all $\widetilde{x}, \widetilde{y} \in C_z$. We will use the following notation: $V_z := \{(x_1,\widetilde{x}) \in \mathbb{R}^n : \widetilde{x} \in C_z, F(\widetilde{x}) \le x_1 < (1+\gamma)z\}$, $V_z^0 := \{(x_1,\widetilde{x}) \in \mathbb{R}^n : \widetilde{x} \in C_z, F(\widetilde{x}) < x_1 < (1+\gamma)z\}$, $\partial V_z := \{(x_1,\widetilde{x}) \in \mathbb{R}^n : \widetilde{x} \in C_z, F(\widetilde{x}) = x_1\}$.

Definition 3 ([7]). A point $P \in \partial\Omega$ is called *Lipschitz point with exponent* γ, if there exist $\gamma \ge 0$, $z > 0$ and a homeomorphism τ, which transforms some neighborhood $D \subset \overline{\Omega}$ of the point P on V_z and has the following properties: $\tau(P) = 0$, $D \bigcap \partial\Omega$ is transformed into ∂V_z and τ is a diffeomorphism of $D \bigcap \Omega$ onto V_z^0. Besides, for the metric $g_{ij}(x)$ on V_s^0 induced from Ω, one has $\lim\limits_{x\to 0} g_{ij}(x) = \delta_{ij}$ for all i, j.

A point is said to be *regular* if it is a Lipschitz point with exponent $\gamma = 0$.

To prove the following theorem we use the constructions from [7], where the case $s = 2$ is considered.

Theorem 3. *Let $1 < s < \infty$. If the boundary of the domain Ω has at least one regular point P, then for the weak Hardy constant in the inequality* (4) *the estimate*

$$c \ge 4/(s-1)^2$$

is valid.

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