ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2014, Vol. 58, No. 2, pp. 51–57. © Allerton Press, Inc., 2014.
Original Russian Text © D.S. Chistyakov, 2014, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2014, No. 2, pp. 61–68.

Homogeneous Maps of Abelian Groups

D. S. Chistyakov[1*]

[1] *Moscow Pedagogical State University, ul. Malaya Pirogovskaya 1, Moscow, 119991 Russia*

Received October 11, 2012

Abstract—In this paper we study Abelian groups whose homogeneous maps to other Abelian groups are homomorphisms. We consider these groups as modules over the ring of integers and over their endomorphism rings. We also study related issues.

DOI: 10.3103/S1066369X1402008X

Keywords and phrases: *Abelian group, homogeneous map, near-ring.*

1. STATEMENT OF THE PROBLEM

Let R be an associative ring with identity, and A and B be unitary left R-modules. A map $f : A \to B$ is said to be R-homogeneous provided that $f(ra) = rf(a)$ for all $r \in R$ and $a \in A$. We denote by $\mathcal{M}_R(A, B)$ the set of all R-homogeneous maps from A to B. In the case when $A = B$, we put $\mathcal{M}_R(A) = \mathcal{M}_R(A, A)$. The set $\mathcal{M}_R(A)$ is a near-ring under the operations of function addition and composition. Note that it contains the ring $E_R(A)$ of all R-endomorphisms of A: $E_R(A) \subseteq \mathcal{M}_R(A)$. If the inverse inclusion holds, i.e., every R-homogeneous self-map of A is an endomorphism, then an R-module A is said to be endomorphic. An Abelian group A is said to be endomorphic if it is an endomorphic module over its endomorphism ring (i.e., $\mathcal{M}_{E(A)}(A) = C(E(A))$ where $C(E(A))$ is the center of the ring $E(A)$). This paper is devoted to the study of Abelian groups whose all homogeneous maps to other Abelian groups are homomorphisms. These groups are considered as modules over the ring of integers and over their endomorphism rings. Papers [1–7] deal with this subject. In [1] the authors characterized a class of those rings R such that every R-module is endomorphic. Paper [2] is devoted to the study of endomorphic modules over integral, Dedekind and principal ideal domains. In [3] and [4] the endomorphic groups are described in the classes of torsion and torsion-free Abelian groups. It is an interesting problem to characterize those modules whose all homogeneous bijections into other modules are module isomorphisms. A solution to this problem for some classes of modules and Abelian groups is found in [5–7].

To construct nonadditive maps we need the following notions. A subset A of an R-module is said to be R-closed if $ra \in A$ for all $r \in R$ and $a \in A$. A nonempty subset A of a left R-module V is said to be strongly R-pure if the conditions $r \in R$, $v \in V$ and $0 \neq rv \in A$ imply $v \in A$.

In what follows, the word "group" is used to mean an Abelian group. We will make of use the notation and the terminology from [8–10].

2. $\mathbb{Z}$-HOMOGEMEOUS MAPS OF ABELIAN GROUPS

Let A and B be Abelian groups. It is straightforward to check the following properties of the group $M_{\mathbb{Z}}(A, B)$:

1. in the cases when either A is a torsion group and B is a torsion-free group, A is a p-group and B is a q-group (where p and q are distinct primes) or A is a divisible group and B is a reduced group, the equality $\mathcal{M}_{\mathbb{Z}}(A, B) = 0$ holds;

[*] E-mail: chistyakovds@yandex.ru.

2. if the equality $B[n] = 0$ holds for some $n \in \mathbb{N}$, then $\mathcal{M}_{\mathbb{Z}}(A, B)[n] = 0$ for any group A;

3. if B is a (divisible) torsion-free group, then $\mathcal{M}_{\mathbb{Z}}(A, B)$ is a (divisible) torsion-free group as well;

4. if A is a divisible group, then $\mathcal{M}_{\mathbb{Z}}(A, B)$ is a torsion-free group.

Theorem 1. *The equality* $\mathcal{M}_{\mathbb{Z}}(A, B) = \mathrm{Hom}(A, B)$ *holds for each group* B *if and only if* A *is a direct sum of locally cyclic torsion groups or a torsion-free group of rank* 1.

Proof. Necessity. Let A be a torsion group and $A = \bigoplus\limits_{p \in P} A_p$ be the primary decomposition. If, for some $q \in P$, the primary component A_q is decomposable, then there exists a decomposition $A_q \cong C \bigoplus D$ where C is a cocyclic group being isomorphic to some subgroup D_1 of the group D.

This fact is evident provided that the q-component A_q is a divisible group (in this case $A_q \cong \bigoplus Z(q^\infty)$). If $A_q = R_q \bigoplus D_q$, where R_q is a reduced subgroup and D_q is a nontrivial divisible subgroup in A_q, then R_q contains a cyclic direct summand C (see, e.g, [10], corollary 27.3): $R_q = C \bigoplus C^*$. Let us take the subgroup $C^* \bigoplus D_q$ as the group D. The subgroup C is isomorphic to a subgroup of the group D_q. If A_q is a reduced group, then there exist finite or infinite decompositions

$$A_q = \langle a_1 \rangle \bigoplus \cdots \bigoplus \langle a_k \rangle \bigoplus A_k \ \ (k = 1, 2, \dots)$$

such that $A_k = \langle a_{k+1} \rangle \bigoplus A_{k+1}$ and $o(a_k) = q^{n_k}$ where $1 \leq n_1 < \cdots < n_k < \cdots$ (see, e.g., [10], the proof of theorem 108.1). Then C is a cyclic direct summand of the least order.

We claim that the set $H = \{(a, b) \mid a \in C, b \in D, o(a) = o(b)\}$ is strongly $\mathbb{Z}$-pure in A_q.

To show this, we assume that $n(a, b) \in H$ for some element $(a, b) \in A_q$, $n = q^l n_1$ where $(n_1, q) = 1$. If $o(a) = q^s$ and $o(b) = q^t$, then the equalities $o(na) = q^{s-\ell}$ and $o(nb) = q^{t-\ell}$ hold. Therefore, we have $q^{s-\ell} = q^{t-\ell}$ and $s = t$. Consequently, $(a, b) \in H$. Moreover, H is a $\mathbb{Z}$-closed subset of A_q.

Next we define a $\mathbb{Z}$-homogeneous map $f_q : A_q \to A_q$ by the following rule:

$$f_q(a, b) = \begin{cases} (a, b), & (a, b) \in H; \\ 0, & (a, b) \notin H. \end{cases}$$

For $(0, 0) \neq (a, b) \in H$, we have the correlations

$$(a, b) = f_q(a, b) \neq f_q(a, 0) + f_q(0, b) = 0.$$

Hence, $f_q \notin M_{\mathbb{Z}}(A_q)$. Further, let $f \in M_{\mathbb{Z}}(A, A) = M_{\mathbb{Z}}(\bigoplus\limits_{p \in P} A_p)$. Since $f(A_q) \subseteq A_q$, we consider f_q as the restriction of the map f to A_q and obtain that $f \notin M_{\mathbb{Z}}(A)$.

Let A be a torsion-free group with $r(A) > 1$. Assume that H is a pure subgroup of rank 1 generated by an element a from a maximal linearly independent system of elements in A. Then H is a strongly $\mathbb{Z}$-pure and $\mathbb{Z}$-closed subset of A. We construct a $\mathbb{Z}$-homogeneous map $f : A \to A$, which is not a group homomorphism, by the following formula:

$$f(a) = \begin{cases} a, & a \in H; \\ 0, & a \notin H. \end{cases}$$

If A is a mixed group, then its torsion part is a strongly $\mathbb{Z}$-pure and $\mathbb{Z}$-closed subset of A. By the method used above one can construct a $\mathbb{Z}$-homogeneous non-additive self-map of the group A.

Sufficiency. If $A = \bigoplus\limits_{p \in P} A_p$ is a torsion group and A_p is indecomposable for each $p \in P$, then A is a locally cyclic group. Therefore, as in the case when A is a torsion-free group of rank 1, the equality $M_{\mathbb{Z}}(A, B) = \mathrm{Hom}(A, B)$ is verified directly [2]. □

3. HOMOGENEOUS MAPS OF MIXED QUOTIENT DIVISIBLE GROUPS

The quotient divisible torsion-free groups were introduced by R. Beaumont and R. Pierce [12]. In [13] A. A. Fomin and W. Wickless defined mixed finite rank quotient divisible groups. They proved that the category of mixed quotient divisible groups and the category with torsion-free finite rank groups as objects and quasi-homomorphisms as morphisms are dual. Bearing in mind this duality, one can pose questions and solve problems in the class of mixed quotient divisible groups analogous to those in the theory of torsion-free Abelian groups. For example, one can try to extend the fully transitivity property to the case of mixed quotient divisible groups. Let us give relevant explanations.

For an element $g \in G$ and a prime number p we define m_p as the least nonnegative integer such that the element $p^{m_p} g$ is divisible by any power of p in the group G. If such an integer does not exist, we let $m_p = \infty$.

The sequence $(m_p) = (m_{p_1}, m_{p_2}, \dots, m_{p_n}, \dots)$ is called the co-characteristic of the element g in the group G and is denoted by $\operatorname{cochar}(g)$. The type containing this co-characteristic is called the cotype and denoted by $\operatorname{cotype}(g)$.

Let (m_p) and (k_p) be two co-characteristics. We set

- $(m_p) = (k_p) \Leftrightarrow m_{p_i} = k_{p_i}$,
- $(m_p) \geq (k_p) \Leftrightarrow m_{p_i} \geq k_{p_i}$

for all primes p_i.

The set of all co-characteristics is a lattice with respect to the componentwise operations:

- $(m_p) \wedge (k_p) = (\min\{m_{p_i}, k_{p_i}\})$,
- $(m_p) \vee (k_p) = (\max\{m_{p_i}, k_{p_i}\})$.

It is straightforward to check the following properties of the co-characteristics and the cotypes:

1. $\operatorname{cochar}(g) = \operatorname{cochar}(-g)$ for all $g \in G$,
2. $\operatorname{cochar}(p_i g) = \begin{cases} \operatorname{cochar}(g), & m_{p_i} = 0; \\ (m_{p_1}, m_{p_2}, \dots, m_{p_i} - 1, \dots), & m_{p_i} \neq 0, \ \text{where } \infty - 1 = \infty, \end{cases}$
3. $\operatorname{cochar}(g + h) \leq \operatorname{cochar}(g) \vee \operatorname{cochar}(h)$ for all $g, h \in G$,
4. if $\varphi : G \to H$ is a homomorphism, then $\operatorname{cochar}_G(g) \geq \operatorname{cochar}_H(\varphi(g))$ for all $g \in G$,
5. elements of the group G, which are linearly dependent over $\mathbb{Z}$, have the same cotypes.

A group G is said to be quotient divisible if it does not contain nonzero torsion divisible subgroups, but it contains a free finite-rank subgroup F such that the quotient G/F is torsion divisible.

A linearly independent set of elements that generates the subgroup F is called a basis of the quotient divisible group G. The rank of the group F is called the rank of the group G. Note that the quotient divisible groups are studied in [12–14].

A quotient divisible group G is said to be fully transitive if for any pair of elements $0 \neq x, y \in G$ such that $\operatorname{cochar}(x) \geq \operatorname{cochar}(y)$ there exists $\varphi \in E(G)$ satisfying the condition $\varphi(x) = y$. One can easily verify that rank-1 quotient divisible groups are fully transitive [14] and endomorphic. The latter follows from the isomorphism between a rank-1 quotient divisible group and its endomorphism group.

A direct sum of rank-1 quotient divisible groups with the same cotype is fully transitive. Indeed, let G be a direct sum of rank-1 quotient divisible groups with the same cotype and let $x, y \in G$ be elements satisfying the inequality $\operatorname{cochar}(x) \geq \operatorname{cochar}(y)$. Then we have $G = \langle x \rangle_* \oplus G_x = \langle y \rangle_* \oplus G_y$. Taking into account the above mentioned remark on the rank-1 quotient divisible groups, we obtain the equality $\varphi(x) = y$ for some $\varphi \in E(G)$.

In [7] the following theorem is proved.

Theorem 2. *Let V be a distributive R-module. Then for any R-module W the equality $M_R(V,W) = \mathrm{Hom}_R(V,W)$ holds.*

In particular, this theorem and the results of [15] imply the following. A homogeneous map from a separable torsion-free group or from a torsion group to another group is linear because the named groups are distributive modules over their endomorphism rings (i.e., they are endo-distributive groups).

We will prove below that a direct sum of the rank-1 quotient divisible groups is endo-distributive (see also [16]).

Theorem 3. *A direct sum of homogeneous fully transitive quotient divisible groups is endo-distributive.*

Proof. Let G be a homogeneous fully transitive quotient divisible group. We claim that for all $x, y, z \in G$ the equality

$$E(G)x \cap (E(G)y + E(G)z) = E(G)x \cap E(G)y + E(G)x \cap E(G)z$$

holds. Note that

$$E(G)x = \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x)\}\,,$$
$$E(G)y = \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(y)\}\,.$$

Let $h \in E(G)x \cap E(G)y$. Then we have the correlations: $\mathrm{cochar}(h) \le \mathrm{cochar}(x)$ and $\mathrm{cochar}(h) \le \mathrm{cochar}(y)$. Therefore, we obtain immediately the condition $\mathrm{cochar}(h) \le \mathrm{cochar}(x) \wedge \mathrm{cochar}(y)$, whence

$$h \in \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x) \wedge \mathrm{cochar}(y)\}\,.$$

On the other hand, if $h \in \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x) \wedge \mathrm{cochar}(y)\}$, then we have

$$\mathrm{cochar}(h) \le \mathrm{cochar}(x) \quad \text{and} \quad \mathrm{cochar}(h) \le \mathrm{cochar}(y).$$

There exist $\varphi, \psi \in E(G)$ satisfying the condition $h = \varphi(x) = \psi(y)$. This means that $h \in E(G)x \cap E(G)y$. Thus, we obtain the following equality:

$$E(G)x \cap E(G)y = \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x) \wedge \mathrm{cochar}(y)\}\,.$$

Let $h \in E(G)x + E(G)y$ so that $h = \varphi(x) + \psi(y)$ for some $\varphi, \psi \in E(G)$. This implies

$$\mathrm{cochar}(h) = \mathrm{cochar}(\varphi(x) + \psi(y)) \le \mathrm{cochar}(\varphi(x)) \vee \mathrm{cochar}(\psi(y)).$$

Since the cocharacteristic does not increase under an endomorphism action, we have

$$\mathrm{cochar}(h) \le \mathrm{cochar}(x) \vee \mathrm{cochar}(y).$$

Let $h \in \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x) \vee \mathrm{cochar}(y)\}$. For definiteness, we put $\mathrm{cochar}(x) \vee \mathrm{cochar}(y) = \mathrm{cochar}(x)$. Then there exists $\varphi \in E(G)$ such that

$$h = \varphi(x) \in E(G)x + E(G)y.$$

This yields the equality

$$E(G)x + E(G)y = \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(x) \vee \mathrm{cochar}(y)\}\,.$$

For all $x, y, z \in G$, we have

$$(E(G)x + E(G)y) \cap E(G)z = \{g \in G \mid \mathrm{cochar}(g) \le \mathrm{cochar}(z) \wedge (\mathrm{cochar}(x) \vee \mathrm{cochar}(y))\}\,,$$

$$E(G)x \cap E(G)z + E(G)y \cap E(G)z \\ = \{g \in G \mid \mathrm{cochar}(g) \le (\mathrm{cochar}(x) \wedge \mathrm{cochar}(z)) \vee (\mathrm{cochar}(y) \wedge \mathrm{cochar}(z))\}\,.$$

One can easily see that the right-hand sides of the last two equations are equal. Hence, we obtain the required equality

$$(E(G)x + E(G)y) \cap E(G)z = E(G)x \cap E(G)z + E(G)y \cap E(G)z.$$

Finally, since a direct sum of endo-distributive groups is endo-distributive, the conclusion of the theorem follows. □

In particular, this theorem implies that if G is a direct sum of homogeneous fully transitive quotient divisible groups, then an $E(G)$-homogeneous map $f : G \to H$ is additive for any $E(G)$-module H.

4. HOMOGENEOUS MAPS OF FINITE RANK TORSION-FREE GROUPS COINCIDING WITH THEIR PSEUDO-SOCLES

This Section is concerned with the torsion-free groups of finite rank. For the reader's convenience and for the sake of completeness we recall some notions and results.

Let A and B be torsion-free groups.

We say that A is quasi-contained in B if $nA \subseteq B$ for some positive integer n. If A is quasi-contained in B and B is quasi-contained in A (i.e., $nA \subseteq B$, $mB \subseteq A$ for some $n, m \in \mathbb{N}$), then A is quasi-equal to B ($A \doteq B$). A quasi-equality $A \doteq \bigoplus_{i \in I} A_i$ is called a quasi-decomposition or a quasi-direct decomposition of the group A. Furthermore, the subgroups A_i are called quasi-summands of the group A. A group A is said to be strongly indecomposable if it does not have a nontrivial quasi-decomposition.

We say that A is quasi-isomorphic to B ($A \sim B$) if there exist subgroups A', B' and numbers $m, n \in \mathbb{N}$ such that

$$mA \subseteq A' \subseteq A, \ \ nB \subseteq B' \subseteq B \ \text{ and } \ A' \cong B'.$$

Note that if A and B are two torsion-free groups of finite rank, then these groups are quasi-isomorphic if and only if A is isomorphic to some subgroup of finite index in the group B.

The ring $\mathbb{Q} \otimes E(A)$ is called the quasi-endomorphism ring of the group A. We denote it by $\mathcal{E}(A)$. It is well-known that

$$\mathcal{E}(A) = \{\alpha \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q} \otimes A) \mid (\exists n \in \mathbb{N})(n\alpha \in E(A))\}.$$

Moreover, the divisible hull $\mathbb{Q} \otimes A$ of the group A is a left module over the ring $\mathcal{E}(A)$. Namely, we let $(r \otimes \varphi)(q \otimes a) = rq \otimes \varphi(a)$ for all $r, q \in \mathbb{Q}$, $\varphi \in E(A)$, $a \in A$.

A torsion-free group G is said to be quasi-E-locally cyclic if for any elements $a, b \in G$ there exists an element $c \in G$ such that $a, b \in \mathcal{E}(G)c$.

For a torsion-free group A the pure subgroup generated by all minimal pure fully invariant subgroups (pfi-subgroups) of A is called the pseudo-socle of A. It is denoted by $\mathrm{Soc}\, A$. A torsion-free group A is said to be irreducible if it does not contain proper pfi- subgroups.

The next theorem is true.

Theorem 4 ([9], theorem 7.3). *Let A be a torsion-free group of finite rank. The following conditions are equivalent*:

(1) $A = \mathrm{Soc}\, A$,

(2) $\mathcal{E}(A)$ *is a classically semisimple ring,*

(3) $E(A)$ *is a semi-prime ring,*

(4) $A \doteq G = \bigoplus_{i=1}^{n} A_i$, *where every group A_i is fully invariant in the group G, $A_i = \bigoplus_{j=1}^{n_i} A_{ij}$, $A_{ij} \sim A_{ik}$ and $\mathcal{E}(A_{ij})$ is a division ring ($i = \overline{1, n}$, $j, k = \overline{1, n_i}$).*

Before stating the main result of this Section, we will prove the following assertion being of interest by itself.

Lemma. *Let A be a torsion-free group. If the $\mathcal{E}(A)$-module $\mathbb{Q} \otimes A$ is endomorphic, then the group A is also endomorphic.*

Proof. We have a near-ring monomorphism

$$\Phi : \mathbb{Q} \otimes \mathcal{M}_{E(A)}(A) \to \mathcal{M}_{\mathcal{E}(A)}(\mathbb{Q} \otimes A),$$

given by the formula $\Phi(q \otimes f)(s \otimes a) = qs \otimes f(a)$ for $q, s \in \mathbb{Q}$, $f \in \mathcal{M}_{E(A)}(A)$, $a \in A$.

Let us take $q_1, q_2, s \in \mathbb{Q}$, $f_1, f_2 \in \mathcal{M}_{E(A)}(A)$. Then we have the correlations:

$$\begin{aligned}\Phi(q_1 \otimes f_1 + q_2 \otimes f_2)(s \otimes a) &= \Phi\left(\tfrac{1}{t} \otimes m_1 f_1 + \tfrac{1}{t} \otimes m_2 f_2\right)(s \otimes a)\\ &= \Phi\left(\tfrac{1}{t} \otimes (m_1 f_1 + m_2 f_2)\right)(s \otimes a) = \tfrac{s}{t} \otimes (m_1 f_1 + m_2 f_2)(a)\\ &= \tfrac{s}{t} \otimes (m_1 f_1(a) + m_2 f_2(a)) = \tfrac{s}{t} \otimes m_1 f_1(a) + \tfrac{s}{t} \otimes m_2 f_2(a)\\ &= q_1 s \otimes f_1(a) + q_2 s \otimes f_2(a) = \Phi(q_1 \otimes f_1)(s \otimes a) + \Phi(q_2 \otimes f_2)(s \otimes a)\end{aligned}$$

for some $t, m_1, m_2 \in \mathbb{Z}$.

Furthermore, the follwing equalities hold:

$$\begin{aligned}\Phi((q_1 \otimes f_1)(q_2 \otimes f_2))(s \otimes a) &= \Phi(q_1 q_2 \otimes f_1 f_2)(s \otimes a) = q_1 q_2 s \otimes f_1 f_2(a)\\ &= \Phi(q_1 \otimes f_1)\Phi(q_2 \otimes f_2)(s \otimes a).\end{aligned}$$

Moreover, the monomorphism Φ preserves identity elements.

Note that the near-ring $\mathcal{M}_{E(A)}(A)$ is embedded into the near-ring $\mathbb{Q} \otimes \mathcal{M}_{E(A)}(A)$ by means of the trivial identification of the elements $f \in \mathcal{M}_{E(A)}(A)$ and $1 \otimes f \in \mathbb{Q} \otimes \mathcal{M}_{E(A)}(A)$.

If the $\mathcal{E}(A)$-module $\mathbb{Q} \otimes A$ is endomorphic, then for all $a_1, a_2 \in A$ and $f \in \mathcal{M}_{E(A)}(A)$ we have

$$\begin{aligned}\Phi(1 \otimes f)(1 \otimes (a_1 + a_2)) &= \Phi(1 \otimes f)(1 \otimes a_1 + 1 \otimes a_2)\\ &= \Phi(1 \otimes f)(1 \otimes a_1) + \Phi(1 \otimes f)(1 \otimes a_2) = 1 \otimes f(a_1) + 1 \otimes f(a_2).\end{aligned}$$

On the other hand, the equality

$$\Phi(1 \otimes f)(1 \otimes (a_1 + a_2)) = 1 \otimes f(a_1 + a_2)$$

holds. Thus, we get the required equality $f(a_1 + a_2) = f(a_1) + f(a_2)$. □

Note, in the case when A is an endo-finite torsion-free group of finite rank (i.e., A is a finitely generated $E(A)$-module) and the ring $E(A)$ is semi-prime, then the equality $\mathbb{Q} \otimes \mathcal{M}_{E(A)}(A) = \mathcal{M}_{\mathcal{E}(A)}(\mathbb{Q} \otimes A)$ holds. This fact is valid because the group A is a quasi-direct sum of fully invariant strongly irreducible groups ([9], theorem 11.4). Therefore, it is endomorphic [3]. In this case the $\mathcal{E}(A)$-module $\mathbb{Q} \otimes A$ is irreducible and, whence, distributive. Applying lemma 8.4 from [9] we obtain the equalities:

$$\mathbb{Q} \otimes \mathcal{M}_{E(A)}(A) = \mathbb{Q} \otimes C(E(A)) = E_{\mathcal{E}(A)}(\mathbb{Q} \otimes A) = \mathcal{M}_{\mathcal{E}(A)}(\mathbb{Q} \otimes A).$$

In notation of Theorem 4 we have

Theorem 5. *Let A be a torsion-free group of finite rank coinciding with its pseudo-socle. Then the group A is endomorphic if and only if either all $n_i > 1$ or if $n_k = 1$, then the group A_k is an irreducible strongly indecomposable group.*

Proof. Necessity. Assume that the equality $n_k = 1$ holds for some $k \in \{1, \ldots, n\}$ and the group A_k is not irreducible. Then it has a proper minimal pfi-subgroup H. Let us consider the equation $\varphi(x) = h$ for some $\varphi \in E(A_k)$, $x \in A_k$, $h \in H$. Since $\mathcal{E}(A_k)$ is a division ring, we get $\varphi^{-1} \in \mathcal{E}(A_k)$. In addition, we have $m\varphi^{-1} \in E(A_k)$ for some $m \in \mathbb{N}$. As a consequence, we get $mx = m\varphi^{-1}h \in H$, $x \in H$, and the subgroup H is strongly $E(A_k)$-pure and $E(A_k)$-closed in the group A_k. Let us construct an $E(A_k)$-homogeneous map $f : A_k \to A_k$ by the following rule:

$$f(a) = \begin{cases} a, & x \in A_k; \\ 0, & x \in A_k \setminus H. \end{cases}$$

Take two elements $0 \neq x \in H, y \in A_k \setminus H$. Then we have the correlation $f(x+y) = 0 \neq x = f(x) + f(y)$. The summands A_i are fully invariant in the group G and $f(A_k) = f(e_k A_k) = e_k f(A_k) \subseteq A_k$ (where $e_k : G \to A_k$ is the projection). Therefore $f \in \mathcal{M}_{E(G)}(G)$. It is proved in [3] that the property of being endomorphic is invariant for groups which are quasi-equal. Hence, the group A is not endomorphic.

Sufficiency. Note that the irreducible groups are endomorphic [3]. By Theorem 4, if $n_k > 1$, then $\mathcal{E}(A_k)$ is isomorphic to the matrix ring over a division ring. Every module over the matrix ring is endomorphic, therefore $\mathbb{Q} \otimes A_k$ is an endomorphic $\mathcal{E}(A_k)$-module. By Lemma, the group A_k is endomorphic. Then the group G is endomorphic as a direct sum of endomorphic groups [3]. Therefore, in view of the quasi-equality $A \doteq G$, the group A is also endomorphic. □

As an application of Lemma, we point to the following result. If A is a quasi-E-locally cyclic group, then $\mathbb{Q} \otimes A$ is a locally cyclic $\mathcal{E}(A)$-module. Since the locally cyclic modules are endomorphic the group A is endomorphic as well.

ACKNOWLEDGMENTS

The author is grateful to O. V. Lyubimtsev for posing the problem and to A. V. Tsarev for his attention to the author's work and for useful discussions.

The work is supported by the Federal Program "Scientific and Scientific-Pedagogical Personnel of Innovative Russia for 2009–2013 years", the agreement 14V87.21.0363.

REFERENCES

1. P. Fuchs, C. J. Maxson, and G. Pilz, "On Rings for Which Homogeneous Maps Are Linear," Proc. Amer. Math. Soc. **112** (1), 1–7 (1991).
2. J. Hausen and J. A. Johnson, "Centralizer Near-Rings that are Rings," J. Austr. Math. Soc. **59**, 173–183 (1995).
3. D. S. Chistyakov and O. V. Lyubimtsev, "Abelian Groups as Endomorphic Modules over their Endomorphism Ring," Fundam. i Prikl. Mat. **13** (1), 229–233 (2007).
4. D. S. Chistyakov, "Endomorphic Indecomposable Torsion-Free Abelian Groups of Rank 3," Vestnik Tomsk. Gos. Univ. Ser. Mat. Mekh. **13** (1), 61–66 (2011).
5. A. B. van der Merwe, "Unique Addition Modules," Commun. Algebra **27** (9), 4103–4115 (1999).
6. O. V. Lyubimtsev and D. S. Chistyakov, "Abelian Groups as UA-Modules over the Ring $\mathbb{Z}$," Mat. Zametki **87** (3), 412–416 (2010).
7. D. S. Chistyakov, "Abelian Groups as UA-Modules over their Endomorphism Rings," Mat. Zametki **91** (6), 934–941 (2012).
8. A. A. Tuganbaev, *Theory of Rings. Arithmetic Modules and Rings* (MTsNMO, Moscow, 2009) [in Russian].
9. P. A. Krylov, A. V. Mikhalev, and A. A. Tuganbaev, *Abelian Groups and their Endomorphism Rings* (Faktorial Press, Moscow, 2006) [in Russian].
10. L. Fuchs, *Infinite Abelian Groups* (Academic Press, New York–London, 1970; Mir, Moscow, 1974), Vol. 1.
11. L. Fuchs, *Infinite Abelian Groups*" (Academic Press, New York–London, 1973; Mir, Moscow, 1977), Vol. 2.
12. R. Beaumont and R. Pierce, "Torsion-Free Rings," Illinois J. Math. **5**, 61–98 (1961).
13. A. A. Fomin and W. Wickless, "Quotient Divisible Abelian Groups," Proc. Amer. Math. Soc. **126**, 45–52 (1998).
14. O. I. Davydova, "Rank-1 Qutient Divisible Groups," Fundam. i Prikl. Mat. **13** (3), 25–33 (2007).
15. A. A. Tuganbaev, "Distributive Rings and Endo-Distributive Modules,"Ukrainsk. Mat. Zhurn. **38**(1), 63–67 (1986).
16. E. V. Tarakanov, "Endo-Distributive Modules over Dedekind Rings," in *Abelian Groups and Modules* (Tomsk. Gos. Univ., Tomsk, 1990), pp. 83–107.

Translated by R. N. Gumerov