ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2014, Vol. 58, No. 2, pp. 39–44. © Allerton Press, Inc., 2014.
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# The Classical Galois Closure for Universal Algebras

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Received October 9, 2012

**Abstract**—We study the Galois correspondence between subgroups of groups of universal algebras automorphisms and subalgebras of fixed points of these automorphisms.

**DOI:** 10.3103/S1066369X14020066

Keywords and phrases: *automorphisms, subalgebras, Galois-correspondence.*

The algebraic structures automorphisms relationship with the sets of their fixed points is related to a traditional algebraic subjects whose origin can be traced to the classical Galois theory which establishes the correspondence between subgroups of the Galois fields automorphisms groups and fixed points subfields of these automorphisms. In somewhat more general setting of the universal algebras the similar study was presented in the B. I. Plotkin monograph [1]. Nevertheless, later the universal algebras specialists concentrated more on the Galois-correspondence between clones of the relations on the fixed set and functions clones which do not change these relations (see, e.g., [2]). Here we consider the Galois-correspondence between the universal algebra $\mathfrak{A}$ subgroups of the automorphism group $\operatorname{Aut}\mathfrak{A}$ and the fixed points automorphism subalgebras from these groups in the classical Galois theory setting.

Let us denote by $\operatorname{Sub}\mathfrak{A}$ a lattice of subalgebras of some universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$. For any $f \in \operatorname{Aut}\mathfrak{A}$ we say that $\operatorname{Fix} f$ is the set $\{a \in A \mid f(a) = a\}$ of all fixed points for the automorphism $f$, and $\operatorname{Fix} G = \bigcap_{f\in G} \operatorname{Fix} f$ is the set of all fixed points for all automorphisms from the subgroup $G$ of the group $\operatorname{Aut}\mathfrak{A}$. Clearly, for any $f \in \operatorname{Aut}\mathfrak{A}$ and any subgroup $G$ of the group $\operatorname{Aut}\mathfrak{A}$ the sets $\operatorname{Fix} f$ and $\operatorname{Fix} G$ are subalgebras of the algebra $\mathfrak{A}$. We denote by $\operatorname{Stab}\mathfrak{B}$ the set of functions $f \in \operatorname{Aut}\mathfrak{A}$ such that $\mathfrak{B} \subseteq \operatorname{Fix} f$ for any subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$. Again it seems clear that $\operatorname{Stab}\mathfrak{B}$ is the subgroup of the group $\operatorname{Aut}\mathfrak{A}$. Thus we have the Galois mappings (analogs of the relative mapping from the classical Galois theory for the fields):

$$\operatorname{Stab}: \operatorname{Sub}\mathfrak{A} \to \operatorname{Sub}\operatorname{Aut}\mathfrak{A},$$
$$\operatorname{Fix}: \operatorname{Sub}\operatorname{Aut}\mathfrak{A} \to \operatorname{Sub}\mathfrak{A}.$$

Let us consider the operation of *Galois-closure* corresponding to these mappings on the lattice $\operatorname{Sub}\mathfrak{A}$ of the algebra $\mathfrak{A}$ subalgebras: We put in correspondence to any subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$ the closure of this subalgebra $\overline{\mathfrak{B}} = \operatorname{Fix}\operatorname{Stab}\mathfrak{B}$. The definition of subalgebra $\overline{\mathfrak{B}}$ implies that for any automorphisms $f$ and $g$ of the algebra $\mathfrak{A}$ the equality $f\restriction\mathfrak{B}=g\restriction\mathfrak{B}$ yields the equality $f\restriction\overline{\mathfrak{B}}=g\restriction\overline{\mathfrak{B}}$. Here $f \restriction \mathfrak{B}$ is the restriction of the function $f$ to subalgebra $\mathfrak{B}$. Particularly, the identity of automorphism $f$ for $\mathfrak{A}$ on its subalgebra $\mathfrak{B}$ implies this automorphism identity also on $\overline{\mathfrak{B}}$.

Note the principal properties of the Galois-closure operation (these properties are implicit consequences of definition of the operation):

1) $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$,

2) $\overline{\overline{\mathfrak{B}}} = \overline{\mathfrak{B}}$,

3) $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \to \overline{\mathfrak{B}}_1 \subseteq \overline{\mathfrak{B}}_2$.

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The latter property, in particular, implies the following inclusions: $\overline{\mathfrak{B}}_1 \vee \overline{\mathfrak{B}}_2 \subseteq \overline{\mathfrak{B}_1 \vee \mathfrak{B}_2}$ and $\overline{\mathfrak{B}_1 \wedge \mathfrak{B}_2} \subseteq \overline{\mathfrak{B}}_1 \wedge \overline{\mathfrak{B}}_2$, here the symbols $\wedge$ and $\vee$ denote inf and sup in the lattice Sub $\mathfrak{A}$, respectively. Note that at the same time the relations $\subseteq$ here cannot be replaced by the equality relations.

Indeed, let us consider the algebra $\mathfrak{A} = \langle A; \sigma \rangle$ of the signature $\sigma$ which consists of one one-place function $f$. Put $A = \{0_1, 1_1, 0_2, 1_2, 0_3, 1_3\}$ and $f(0_i) = 1_i$, $f(1_i) = 1_i$ for any $i \in \{1, 2, 3\}$. Then $\mathfrak{B}_1 = \{0_1, 1_1\}$, $\mathfrak{B}_2 = \{0_2, 1_2\}$ are subalgebras of the algebra $\mathfrak{A}$ and it seems clear that $\overline{\mathfrak{B}}_1 = \mathfrak{B}_1$, $\overline{\mathfrak{B}}_2 = \mathfrak{B}_2$. At the same time $\overline{\mathfrak{B}_1 \vee \mathfrak{B}_2} = \mathfrak{A}$. Assume also that $\mathfrak{C}_1 = \{0_1, 1_1, 0_3, 1_3\}$, $\mathfrak{C}_2 = \{0_2, 1_2, 0_3, 1_3\}$. Again it seems clear that $\overline{\mathfrak{C}}_1 = \mathfrak{A}$, $\overline{\mathfrak{C}}_2 = \mathfrak{A}$, but $\overline{\mathfrak{C}_1 \wedge \mathfrak{C}_2} = \{0_3, 1_3\}$, so in this case $\overline{\mathfrak{B}_1 \vee \mathfrak{B}_2} \neq \overline{\mathfrak{B}}_1 \vee \overline{\mathfrak{B}}_2$ and $\overline{\mathfrak{C}_1 \wedge \mathfrak{C}_2} \neq \overline{\mathfrak{C}}_1 \wedge \overline{\mathfrak{C}}_2$.

We say that a subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$ is Galois-closed if $\overline{\mathfrak{B}} = \mathfrak{B}$.

Note that the set-theoretical intersection of any number of Galois-closed subalgebras of the algebra $\mathfrak{A}$ is again a Galois-closed subalgebra from $\mathfrak{A}$. Thus the set of Galois-closed subalgebras of the algebra $\mathfrak{A}$ is a complete lattice with respect to the set-theoretical inclusion relation.

The example similar to the given above allows us to state that Galois-closure is not a local notion, i.e., Galois-closure of a subalgebra may not coincide with Galois-closures union of its finitely generated subalgebras. Assume again that the signature $\sigma$ is still a one-place function $f$. Put $A = \{0_i, 1_i \mid i \in \omega\} \cup \{0_\omega, 1_\omega\}$ and at the same time put $f(0_i) = 1_i$, $f(1_i) = 1_i$ for $i \in \omega$ and $f(0_\omega) = 1_\omega$, $f(1_\omega) = 1_\omega$. Let $\mathfrak{B} = \{0_i, 1_i \mid i \in \omega\}$ and $\mathfrak{B}^j$ $(j \in J)$ be the set of all finitely generated subalgebras of the algebra $\mathfrak{B}$. Since there exists $n \in \omega$ such that $\mathfrak{B}^j \subseteq \{0_i, 1_i \mid i \leqslant n\}$ for any $j \in J$, it can be immediately noted that also $\overline{\mathfrak{B}^j} \subseteq \{0_i, 1_i \mid i \leqslant n\}$. Thus $\bigcup\limits_{j \in J} \overline{\mathfrak{B}^j} = \mathfrak{B}$ but $\overline{\mathfrak{B}} = \mathfrak{A}$, so the Galois-closure of the subalgebra $\mathfrak{B}$ does not coincide with Galois-closures union of finitely generated subalgebras of algebra $\mathfrak{B}$.

Let us describe in detain the elements of the Galois-closure $\overline{\mathfrak{B}}$ for subalgebra $\mathfrak{B}$ of the arbitrary universal algebra $\mathfrak{A}$. It is possible to obtain a complete description in the case of no more than countable algebra $\mathfrak{A}$ of no more than countable signature. In the general case the problem remain open.

We denote by $\sigma_B$ the signature $\sigma$ extension by addition to this signature new constants $c_b$ for $b \in B$, and by $\mathfrak{A}_B$ the corresponding algebra $\mathfrak{A}$ extension to signature $\sigma_B$ by interpretation of the constant $c_b$ $(b \in B)$ by element $b$ for any algebra $\mathfrak{A} = \langle A; \sigma \rangle$ and any subset $B \subseteq A$.

The definition of computation formulas $L_{\omega_1\omega}$ can be found, for example, in [3, 4]. Traditionally we say that the basic formulas are logical formulas of the first order $(L_{\omega\omega})$.

Again we treat the quantification $\exists! x$ in common sense ("there exists a unique $x$") by assuming that

$$\exists! x\, \varphi(x) \Leftrightarrow \exists x\, \varphi(x) \,\&\, \forall x, y\, (\varphi(x) \,\&\, \varphi(y) \rightarrow x = y).$$

Here $\varphi(x)$ is either a basic or $L_{\omega_1,\omega}$-formula according to the context. The $\exists!$-basic ( $\exists!$-$L_{\omega_1\omega}$-) formula $\varphi(x)$ for the algebra $\mathfrak{A} = \langle A; \sigma \rangle$ is any $(L_{\omega_1\omega}$-) basic formula of signature $\sigma$ such that $\mathfrak{A} \models \exists! x\, \varphi(x)$.

Now the description of elements of the arbitrary finite or countable universal algebra subalgebras of no more than countable signature Galois-closures allows us to state the following result.

**Theorem.** a) *For any finite universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ and any its subalgebra $\mathfrak{B} = \langle B; \sigma \rangle$ the element $d$ of $A$ belongs to $\overline{\mathfrak{B}}$ if and only if $\mathfrak{A}_B \models \varphi(d)$ for some $\exists!$-basic formula $\varphi(x)$ for the algebra $\mathfrak{A}_B$ of signature $\sigma_B$.*

b) *For any no more than countable universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ of no more than countable signature and any of its subalgebras $\mathfrak{B} = \langle B; \sigma \rangle$ the element $d$ of $A$ belongs to $\overline{\mathfrak{B}}$ if and only if $\mathfrak{A}_B \models \varphi(d)$ for some $\exists!$-$L_{\omega_1\omega}$-formula $\varphi(x)$ for the algebra $\mathfrak{A}_B$ of signature $\sigma_B$.*

**Proof.** First note that if the element $d \in A$ can be uniquely defined in $\mathfrak{A}_B$ by some $L_{\omega_1,\omega}$ -formula then any automorphism of the algebra $\mathfrak{A}$ fixing an element of $B$ also does not shift $d$. Thus we obtain the sufficiency part of the theorem for items a) and b).

Let us now turn to the necessity part of the theorem. Consider two cases: $d \in B$ and $d \in A \setminus B$. In the first case the formula $\varphi(x)$ part is given to the relation $x = c_d$. We confine ourselves only to the case $d \in A \setminus B$.

a) It is well-known that there exists a basic formula $\varphi_{\mathfrak{A}}$ of signature $\sigma$ for any finite algebra $\mathfrak{A} = \langle A; \sigma\rangle$ of the finite signature $\sigma$ such that for any algebra $\mathfrak{C} = \langle C; \sigma\rangle$ the formula $\varphi_{\mathfrak{A}}$ turns to be true on $\mathfrak{C}$ if and only if $\mathfrak{C}$ is isomorphic to $\mathfrak{A}$. Let now $\mathfrak{A} = \langle A; \sigma\rangle$ be any finite algebra of the arbitrary (not necessarily finite) signature $\sigma$. Fix now a finite signature $\sigma' \subseteq \sigma$. Let us denote by $\mathfrak{A}^{\sigma'}$ the simplification of the algebra $\mathfrak{A}$ to the algebra of signature $\sigma'$. So we are able to state that $g$ is an isomorphism of $\mathfrak{A}^{\sigma'}_C$ onto $\mathfrak{D}_{g(C)}$ if and only if $\mathfrak{D}_{g(C)} \models \varphi_{\mathfrak{A}^{\sigma'}_C}$ for any $C \subseteq A$, any finite signature $\sigma' \subseteq \sigma$ and any bijection $g$ of the set $A$ to the basic set $D$ of the algebra $\mathfrak{D} = \langle D; \sigma'\rangle$.

Let $\mathfrak{B} = \langle B; \sigma\rangle$ be an arbitrary subalgebra of the algebra $\mathfrak{A}$ and $d \in \overline{\mathfrak{B}}$. We denote by $\varphi^{\sigma'}(x)$ the formula which is the formula $\varphi_{\mathfrak{A}^{\sigma'}_{B\cup\{d\}}}$ with the constant $c_d$ replaced by the variable $x$. Since $d \in \overline{B}$, any automorphism of the algebra $\mathfrak{A}$ fixing elements of $B$ also does not shift $d$. Then, since the formula $\varphi_{\mathfrak{A}^{\sigma'}_{B\cup\{d\}}}$ describes the algebra $\mathfrak{A}^{\sigma'}_{B\cup\{d\}}$ up to isomorphism and the bijection set of $A$ (a set of automorphisms of potential algebra) is finite, we can find a finite signature $\sigma' \subseteq \sigma$ such that the formula $\varphi^{\sigma'}(x)$ is $\exists!$-basic for algebra $\mathfrak{A}_B$ and we have $\mathfrak{A}_B \models \varphi^{\sigma'}(d)$. This completes the proof of item a).

b) By D. Scott theorem [5] there exists $L_{\omega_1\omega}$-formula $\varphi_{\mathfrak{A}}$ such that for any no more than countable algebra $\mathfrak{C}$ of signature $\sigma$ the formula $\varphi_{\mathfrak{A}}$ validity is equivalent to isomorphism of $\mathfrak{A}$ and $\mathfrak{C}$ for any no more than countable algebra $\mathfrak{A} = \langle A; \sigma\rangle$ of no more than countable signature. Let $\mathfrak{B} = \langle B; \sigma\rangle$ be a subalgebra of no more than countable algebra $\mathfrak{A} = \langle A; \sigma\rangle$ of no more than countable signature $\sigma$ and $d \in \overline{\mathfrak{B}}$. As $\varphi(x)$ we fix the formula deduced from the formula $\varphi_{\mathfrak{A}_{B\cup\{d\}}}$ by replacing the constant $c_d$ with the variable $x$. Again by D. Scott theorem, the inclusion $d \in \overline{\mathfrak{B}}$ and notions similar to given in the proof of item a) the formula $\varphi(x)$ is a $\exists!$-$L_{\omega_1\omega}$-formula for algebra $\mathfrak{A}_B$, and at the same time $\mathfrak{A}_B \models \varphi(d)$. Indeed, if we have $\mathfrak{A}_B \models \varphi(f)$ for some $f \in A \setminus B$, then there exists an automorphism $\psi$ of the algebra $\mathfrak{A}$ which fixes elements of $B$ and such that $\psi(d) = f$. Since $d \in \overline{\mathfrak{B}}$, we have $d = f$. □

It seems interesting to compare item a) of Theorem 1 with Poschel–Kaluznin theorem [6]. This theorem describes relations which belong to the relation clone generated by an arbitrary relation system on some finite set as relations defined by $\exists$-basic formulas of signature consisting of the relations of the initial system.

We call a subalgebra $\mathfrak{B} = \langle B; \sigma\rangle$ of the algebra $\mathfrak{A}$ the $\exists!$-*elementary* ($\exists!$-$L_{\omega_1\omega}$-) *subalgebra* of the algebra $\mathfrak{A}$ if for any $\exists!$-basic ($\exists!$-$L_{\omega_1\omega}$-) for the algebra $\mathfrak{A}_B$ formula $\varphi(x)$ of signature $\sigma_B$ and any element $d$ from $\mathfrak{A}$ such that $\mathfrak{A}_B \models \varphi(d)$ the inclusion $d \in B$ holds.

**Corollary 1.** a) For any finite universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$ its subalgebra $\mathfrak{B} = \langle B; \sigma\rangle$ is Galois-closed if and only if $\mathfrak{B}$ is an $\exists!$-basic subalgebra of the algebra $\mathfrak{A}_B$.

b) For any no more than countable universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$ of no more than countable signature $\sigma$ its subalgebra $\mathfrak{B} = \langle B; \sigma\rangle$ is Galois-closed if and only if $\mathfrak{B}$ is an $\exists!$-$L_{\omega_1\omega}$-subalgebra of the algebra $\mathfrak{A}_B$.

In dual way the mappings $\mathrm{Stab} : \mathrm{Sub}\,\mathfrak{A} \to \mathrm{Sub}\,\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Fix} : \mathrm{Sub}\,\mathrm{Aut}\,\mathfrak{A} \to \mathrm{Sub}\,A$ define the operation of $\mathfrak{A}$-closure on the subgroups lattice $\mathrm{Sub}\,\mathrm{Aut}\,\mathfrak{A}$ of the group $\mathrm{Aut}\,\mathfrak{A}$. We denote by $\overline{G}^{\mathfrak{A}}$ the group $\mathrm{Stab}\,\mathrm{Fix}\,G$ consisting of automorphisms $f$ of the algebra $\mathfrak{A}$ such that $\mathrm{Fix}\,f \supseteq \bigcap_{g\in G} \mathrm{Fix}\,g$ for any subgroup $G$ of the group $\mathrm{Aut}\,\mathfrak{A}$. Thus the mappings $\mathrm{Stab} : \mathrm{Sub}\,\mathfrak{A} \to \mathrm{Sub}\,\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Fix} : \mathrm{Sub}\,\mathrm{Aut}\,\mathfrak{A} \to \mathrm{Sub}\,\mathfrak{A}$ define mutually-inverse (dual) mappings between partially ordered (complete lattices) sets $\mathrm{CSubAut}\mathfrak{A}$ of $\mathfrak{A}$-closed subgroups of the group $\mathrm{Aut}\,\mathfrak{A}$ and $G\,\mathrm{Sub}\,\mathfrak{A}$ Galois-closed subalgebras of the algebras $\mathfrak{A}$ (or, due to Corollary 1, similarly $\exists!$-basic subalgebras of the finite algebras $\mathfrak{A}$ and $\exists!$-$L_{\omega_1\omega}$-subalgebras of the countable algebras $\mathfrak{A}$ of no more than countable signature, respectively). This duality in itself is exactly the generalization of the classical Galois duality between subfields of separable normal extensions $K$ of the arbitrary fields $k$ and subgroups of groups of $k$-automorphisms of fields $K$ for the case of arbitrary universal algebras. Note also that normal $\mathfrak{A}$-closed subgroups of the group $\mathrm{Aut}\,\mathfrak{A}$ naturally correspond to the fixed Galois-closed subalgebras $\mathfrak{B}$ of the algebra $\mathfrak{A}$, i.e., algebras such that for any $g \in \mathrm{Aut}\,\mathfrak{A}$ we have $g(\mathfrak{B}) = \mathfrak{B}$.

The element $d$ of the algebra $\mathfrak{A} = \langle A; \sigma \rangle$ is *separable from the subalgebra* $\mathfrak{B} = \langle B; \sigma \rangle$ if $d$ does not belong to the Galois-closure of the algebra ($d \notin \overline{\mathfrak{B}}$). In other words, if there exist some automorphisms $g_1$ and $g_2$ of the algebra $\mathfrak{A}$ coinciding on $\mathfrak{B}$ and such that $g_1(d) \neq g_2(d)$ or in one more equivalent formulation, if there exists $f \in \mathrm{Aut}\,\mathfrak{A}$ such that $f$ is an identity on $\mathfrak{B}$ but $f(d) \neq d$. The latter automorphism $f$ is the separating automorphism of $d$ from $\mathfrak{B}$. We also call the algebra $\mathfrak{A}$ *separable* if all its subalgebras are Galois-closed, i.e., if any its element is separable from any of its subalgebra which does not contain this element. Clearly this separable universal algebras notion is related and similar to the same notion in the field theory (see, e.g., [7]).

Let us show for instance that any finite Boolean algebra $\mathfrak{B} = \langle B; \wedge, \vee, \neg, 0, 1 \rangle$ is separable. Let $\mathfrak{C} = \langle C; \wedge, \vee, \neg, 0, 1 \rangle$ be a proper subalgebra of the algebra $\mathfrak{B}$ and $d \in B \setminus C$. Let $\{b_1, \ldots, b_n\}$ be all atoms of the algebra $\mathfrak{B}$, and $\{c_1, \ldots, c_m\}$ be all atoms of the algebra $\mathfrak{C}$. Let us prove the existence of an automorphism $f$ of the algebra $\mathfrak{B}$ such that $f$ is identical on $\mathfrak{C}$ and $f(d) \neq d$. Since $d \in B \setminus C$, there exists $i \leqslant m$ such that $c_i \wedge d \neq 0$ and $c_i \wedge \neg d \neq 0$. Conversely, $d$ is a union of $c_i$, i.e., it belongs to $\mathfrak{C}$. Since $b_1, \ldots, b_n$ is the set of all atoms of $\mathfrak{B}$, there exist $i_1 \neq i_2$ such that $b_{i_1} \leqslant c_i \wedge d$ and $b_{i_2} \leqslant c_i \wedge \neg d$. Let $f \in \mathrm{Aut}\,\mathfrak{B}$ be induced by equalities $f(b_j) = b_j$ for $j \notin \{i_1, i_2\}$, $f(b_{i_1}) = b_{i_2}$, $f(b_{i_2}) = b_{i_1}$. It seems clear then that $f$ is identical on $\mathfrak{C}$ and $f(d) \neq d$.

Assume now that $\mathfrak{B} = \langle B; \wedge, \vee, \neg, 0, 1 \rangle$ is an infinite Boolean algebra which is not non-atomic. Let us show that in this situation $\mathfrak{B}$ is not separable. Let us consider the following two cases: a) $\mathfrak{B}$ contains only finite number of atoms: $a_0, \ldots, a_n$, b) $\mathfrak{B}$ contains an infinite set of atoms: $a_0, \ldots, a_n, \ldots$.

a) Put $a = a_1 \vee \cdots \vee a_n$. Then $a$ is fixed under any Boolean algebra automorphism and, consequently, $a$ belongs to the Galois-closure of any subalgebra. Thus $a \notin \mathfrak{C}$ and $a \in \overline{\mathfrak{C}}$ for $\mathfrak{C} = \langle \{0, 1\}; \wedge, \vee, \neg, 0, 1 \rangle$, so $\mathfrak{B}$ is not separable.

b) Let $C$ be a collection of all finite atom disjunctions of $\mathfrak{B}$ different from $a_0$ and the complements of the similar elements. Then $a_0$ does not belong to the subalgebra $\mathfrak{C} = \langle C; \wedge, \vee . \neg, 0, 1 \rangle$ of the algebra $\mathfrak{B}$. Let $f$ be an automorphism of the algebra $\mathfrak{B}$ identical on the subalgebra $\mathfrak{C}$ and consequently on all atoms of the algebra $\mathfrak{B}$ different from $a_0$. Then this automorphism is identical also on $a_0$. Hence $a_0 \in \overline{\mathfrak{C}}$ and the algebra $\mathfrak{B}$ is not separable.

Finally, let us show that any non-atomic Boolean algebra is also non-separable. Recall that a subalgebra $\mathfrak{C}$ of the Boolean algebra $\mathfrak{B}$ is *dense* in $\mathfrak{B}$ if there exists $c \in \mathfrak{C} \setminus \{0\}$ such that $c \leqslant b$ for any $b \in \mathfrak{B} \setminus \{0\}$. The subset $D$ of the Boolean algebra $\mathfrak{B}$ is said to be *nonredundant* if for any $d \in D$ the element $d$ does not belong to the subalgebra generated by the set $D \setminus \{d\}$. According to R. McCansy ([8], statement 4.23) any maximal with respect to inclusion nonredundant subset of the Boolean algebra $\mathfrak{B}$ generates some dense in $\mathfrak{B}$ subalgebra. So let $\mathfrak{B}$ be some non-atomic Boolean algebra and $D$ be some its maximal nonredundant subset with respect to inclusion. Hence there exists a dense in $\mathfrak{B}$ subalgebra $\langle D \rangle_{\mathfrak{B}}$ generated by the set $D$. Fix an arbitrary element $d \in D$, then $d \notin \langle D \setminus \{d\} \rangle_{\mathfrak{B}}$ and nonatomicity of $\mathfrak{B}$ implies that $\langle D \setminus \{d\} \rangle_{\mathfrak{B}}$ is still dense in $\mathfrak{B}$. Thus, $\mathfrak{C} = \langle D \setminus \{d\} \rangle_{\mathfrak{B}}$ is a proper dense subalgebra of the algebra $\mathfrak{B}$. It can be shown by direct computation that if $f \in \mathrm{Aut}\,\mathfrak{B}$ is identical on the dense in $\mathfrak{B}$ subalgebra, then $f$ is also identical on $\mathfrak{B}$. Thus we arrive to the following statement.

**Statement.** *A Boolean algebra is separable if and only if it is finite.*

Let us also note similar questions on Fix-definable automorphisms and automorphic definable subalgebras of the universal algebras (the questions posed for Boolean algebras by S. S. Goncharov and solved by D. E. Pal'chunov and A. V. Trofimov [9]). The automorphism $g$ of the universal algebra $\mathfrak{A}$ is said to be Fix-*definable* if for any $f \in \mathrm{Aut}\,\mathfrak{A}$ the equality $\mathrm{Fix}\, f = \mathrm{Fix}\, g$ implies the equality $f = g$. Due to relation $\mathrm{Fix}\, f^{-1} = \mathrm{Fix}\, f$ any Fix-definable automorphism of algebra $\mathfrak{A}$ is the involution of this algebra (second order automorphism). In [9] the authors prove that for Boolean algebras the converse also holds true: Any Boolean algebra involution is Fix-definable. Similar result was found also for the case of the distributive nets (for the finite latices by the author and in the general case by D. E. Pal'chunov and A. V. Trofimov).

We say that the subalgebra $\mathfrak{B}$ of the universal algebra $\mathfrak{A}$ is *automorphic definable* if $\mathfrak{B} = \mathrm{Fix}\, f$ for some automorphism $f$ of the algebra $\mathfrak{A}$. The paper [9] gives the description of the automorphic definable subalgebras of the Boolean algebras. It seems clear that any automorphic definable subalgebra of the algebra $\mathfrak{A}$ is Galois-closed in $\mathfrak{A}$. We say that the algebra $\mathfrak{A}$ is the algebra with *automorphic definable subalgebras* if this is the case for all the subalgebras of $\mathfrak{A}$.

Thus any algebra with automorphic definable subalgebras is separable. It is also interesting whether the converse holds true: Is any separable algebra automorphic definable?

Let us show that this statement is false. Let a signature $\sigma$ consist of only one one-place function $f$. Fix $\mathfrak{A} = \langle \{0, 1\}; f \rangle$, here $f(0) = 1$ and $f(1) = 0$. Let $\mathfrak{B}$ be the Boolean algebra of all finite and cofinite subsets of the set $\omega$ and $\mathfrak{A}^{\mathfrak{B}}$ be a Boolean $\mathfrak{B}$-degree of the algebra $\mathfrak{A}$, i.e., the subalgebra of direct degree $\mathfrak{A}^{\omega}$ of the algebra $\mathfrak{A}$ consisting of all $g \in \mathfrak{A}^{\omega}$ for which the set $\{i \in \omega \mid g(i) = 0\}$ is either finite or cofinite. It is now clear that $\mathfrak{A}^{\mathfrak{B}}$ is separable.

Let us define for any universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ a quasi-order $\leqslant$ on the set $A$ as follows: Put $a \leqslant b$ for any $a, b \in A$ if and only if $\overline{\langle \{a\} \rangle_{\mathfrak{A}}} \subseteq \overline{\langle \{b\} \rangle_{\mathfrak{A}}}$. Here, as in the previously given considerations, $\langle B \rangle_{\mathfrak{A}}$ for $B \subseteq A$ denotes a subalgebra of the algebra $\mathfrak{A}$ generated by the set $B$. Clearly, $a \leqslant b$ if and only if for any $g_1, g_2 \in \operatorname{Aut}\mathfrak{A}$ the equality $g_1(b) = g_2(b)$ implies the equality $g_1(a) = g_2(a)$ or otherwise if for any $g \in \operatorname{Aut}\mathfrak{A}$ the equality $g(b) = b$ implies the equality $g(a) = a$.

The element $a$ of the algebra $\mathfrak{A} = \langle A; \sigma \rangle$ is $\exists!$-*basic* ($\exists!$-$L_{\omega_1\omega}$-) element in $\mathfrak{A}$ if there exists a $\exists!$-basic ($\exists!$-$L_{\omega_1\omega}$-) formula $\varphi(x)$ in the signature of the algebra $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi(a)$. It seems clear that any $\exists!$-basic or $\exists!$-$L_{\omega_1\omega}$-basic element of the algebra $\mathfrak{A}$ is the minimal element with respect to the quasi-order $\langle A, \leqslant \rangle$. Let us show that the converse does not hold true. Put $\mathfrak{A} = \langle \{0_0, 1_0, 0_1, 1_1\}; f(x) \rangle$, then $f(0_i) = 1_i$, $f(1_i) = 1_i$ for $i = 0, 1$. Clearly, since there exists an automorphism $g$ of the algebra $\mathfrak{A}$ such that $g(0_0) = 0_1$, $g(0_1) = 0_0$, $g(1_0) = 1_1$, $g(1_1) = 1_0$, the algebra $\mathfrak{A}$ contains neither $\exists!$-basic nor $\exists!$-$L_{\omega_1\omega}$-elements. On the other hand, for any $a, b \in \mathfrak{A}$ we have $a \leqslant b$, i.e., all the elements of the algebra are minimal with respect to the quasi-order $\langle A; \leqslant \rangle$.

Finally, Theorem 1 implies that the following holds true.

**Corollary 2.** a) For any finite universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ its element $a$ is minimal with respect to the quasi-order $\mathfrak{A} = \langle A; \leqslant \rangle$ if and only if there exists a $\exists!$-basic formula $\varphi_b(x, b)$ of signature $\sigma$ with the parameter $b$ such that $\mathfrak{A} \models \varphi_b(a, b)$ for any $b \in \mathfrak{A}$.

b) For any no more than countable universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ of no more than countable signature $\sigma$ its element $a$ is minimal with respect to the quasi-order $\langle A; \leqslant \rangle$ if and only if there exists a $\exists!$-$L_{\omega_1\omega}$-formula $\varphi_b(x, b)$ of signature $\sigma$ with the parameter $b$ such that $\mathfrak{A} \models \varphi_b(a, b)$ for any $b \in \mathfrak{A}$.

The element $a \in \mathfrak{A}$ is maximal in $\langle A; \leqslant \rangle$ if and only if any automorphism of the algebra $\mathfrak{A}$ identical on $a$ is also identical on the whole algebra $\mathfrak{A}$. Thus, for instance, for any one-generated algebra $\mathfrak{A} = \langle A; \sigma \rangle$ its generator is maximal in the quasi-order $\langle A; \leqslant \rangle$. The other example of algebras $\mathfrak{A}$ which allow the existence of the maximal element with respect to the quasi-order $\langle A; \leqslant \rangle$ is given by stiff algebras (i.e., algebras which possess only trivial automorphism). All elements of these algebras are maximal. Note here that there exist stiff Boolean algebras of any noncountable cardinality (for example, [8]).

The description of the maximal elements with respect to the quasi-order $\langle A; \leqslant \rangle$ is similar to one given for the minimal elements.

**Corollary 3.** The element $a$ of the finite (countable) algebra $\mathfrak{A} = \langle A; \sigma \rangle$ (of no more than countable signature) is the maximal element with respect to the quasi-order $\langle A; \leqslant \rangle$ if and only if any element of $\mathfrak{A}$ is $\exists!$-basic ($\exists!$-$L_{\omega_1\omega}$-) element in the algebra $\mathfrak{A}_{\{a\}}$.

It also seems interesting to describe properties of $\mathfrak{A}$ which can be written only in terms of the quasi-order $\langle A; \leqslant \rangle$.

## ACKNOWLEDGMENTS

The author wishes to express his gratitude to the referee whose numerous remarks significantly improved the article presentation.

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*Translated by P. N. Ivan'shin*