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On Canonical Almost Geodesic Mappings of the First Type of Affinely Connected Spaces

V. E. Berezovskii[1*] and J. Mikeš[2]**

[1]*Uman National University of Horticulture, ul. Institutskaya 1, Uman, 20305 Ukraine*

[2]*Palacky University, 17. listopadu 12, Olomouc, 77146 Czech Republic*

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Abstract—In this paper, we study special cases of canonical almost geodesic mappings of the first type of affinely connected spaces. The basic equations of mappings in question are reduced to a closed system of Cauchy type in covariant derivatives, and the number of essential parameters in the general solution of this system is estimated. We give an example of such mappings from a flat space onto another flat space. The mappings constructed send straight lines of the first space into parabolas in the second space. These almost geodesic mappings of the first type do not belong to the classes of mappings of the second and third types.

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1. INTRODUCTION

In 1960s, N. S. Sinyukov [1] studied almost geodesic mappings of Riemannian and affinely connected spaces. The basic results of these studies were published in [2] and [3].

The theory of almost geodesic mappings was developed in many papers, e.g., [4–14]. Almost geodesic mappings of the first type in the sense of N. S. Sinyukov studied by V. E. Berezovskii and J. Mikeš [4–7], N.V. Yablonskaya [15]. This line of investigation follows in particular A. Z. Petrov's plan [16] of modelling physical processes with the use of mappings and transformations.

In this paper, we study special cases of canonical almost geodesic mappings of the first type of affinely connected spaces. The basic equations of mappings in question are reduced to a closed system of Cauchy type in covariant derivatives. We estimate the number of essential parameters in the general solution of this system and give examples of mappings under study.

2. CANONICAL ALMOST GEODESIC MAPPINGS

Recall the basic notions of the theory of almost geodesic mappings of affinely connected spaces outlined in [2] and [3].

Consider an n-dimensional torsion-free affinely connected space A_n endowed with a coordinate system $x = (x^1, x^2, \dots, x^n)$. We assume that $n > 2$. All functions which appear in the paper are assumed to be of sufficiently high differentiability class.

A curve in an affinely connected space A_n is called an *almost geodesic* line if there exists a parallel field of two-dimensional planes along it containing its tangent vectors.

A diffeomorphism $f : A_n \to \overline{A}_n$ is called an *almost geodesic mapping* if it maps all geodesics of A_n into almost geodesic lines of $\overline{A}_n$.

*E-mail: Berez.Volod@rambler.ru.
**E-mail: Josef.Mikes@upol.cz.

In order for a mapping $f : A_n \to \overline{A}_n$ to be almost geodesic, it is necessary and sufficient that the connection deformation tensor $P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ satisfy the conditions

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\,\lambda^h + b\,P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta$$

in terms of a common for f coordinate system, where $A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij}P^h_{\alpha k}$, $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are the connection objects of A_n and $\overline{A}_n$, respectively, λ^h is a vector, a and b are functions of x^h and λ^h. Here and in what follows the sign " , " placed as a subscript denotes the covariant derivative with respect to the affine connection of A_n.

N. S. Sinyukov [1–3] singled out the three types of almost geodesic mappings: π_1, π_2, and π_3. We have proved [4] that, for $n > 5$, there are no other types of such mappings.

Almost geodesic mappings belonging to the type π_1 are characterized by the following conditions imposed on the deformation tensor: $A^h_{(ijk)} = a_{(ij}\delta^h_{k)} + b_{(i}P^h_{jk)}$, where a_{ij} is a symmetric tensor, b_i a covector, δ^h_i are Kronecker's deltas, $(i\,j\,k)$ denotes the symmetrization with respect to the indices $i\,j\,k$ without division.

If $b_i \equiv 0$, the mapping is called *canonical*.

It is known [2, 3] that any almost geodesic mapping of the type π_1 can be represented as the composition of a canonical almost geodesic mapping of type π_1 and a geodesic mapping.

3. SPECIAL CANONICAL ALMOST GEODESIC MAPPINGS

If a mapping between affinely connected spaces A_n and $\overline{A}_n$ satisfies the conditions

$$3P^h_{ij,k} = -P^\alpha_{(ij}P^h_{k)\alpha} + a_{(ij}\delta^h_{k)}, \tag{1}$$

then $A^h_{ijk} = -\frac{1}{3}\,P^\alpha_{kj}P^h_{i\alpha} - \frac{1}{3}\,P^\alpha_{ki}P^h_{j\alpha} + \frac{2}{3}\,P^\alpha_{ij}P^h_{\alpha k} + \frac{1}{3}\,a_{(ij}\delta^h_{k)}$ and, consequently, $A^h_{(ijk)} = a_{(ij}\delta^h_{k)}$. Therefore, such mappings are a partial case of canonical almost geodesic mappings of the first type π_1.

We will consider relations (1) as a system of differential equations in covariant derivatives with respect to the unknown deformation tensor P^h_{ij} and tensor a_{ij}. To find the integrability conditions of this system, we differentiate (1) with respect to x^m and then take the skew-symmetric part of the obtained relation with respect to the indices k and m. Taking into account the Ricci identities, we obtain

$$\begin{aligned} a_{i[k,m]}\delta^h_j + a_{j[k,m]}\delta^h_i + a_{ij,[m}\delta^h_{k]} &= 3\,(-P^\alpha_{ij}R^h_{\alpha km} + P^h_{\alpha(i}R^\alpha_{j)km}) \\ &- \frac{1}{3}\,\bigl[P^\beta_{\alpha m}P^\alpha_{(ij}P^h_{k)\beta} - P^\beta_{\alpha k}P^\alpha_{(ij}P^h_{m)\beta} + 2(a_{j[m}P^h_{k]i} + a_{i[m}P^h_{k]j}) \\ &+ a_{\alpha(i}P^\alpha_{j)k}\delta^h_m - a_{\alpha(i}P^\alpha_{j)m}\delta^h_k + a_{\alpha m}P^\alpha_{k(i}\delta^h_{j)} - a_{\alpha k}P^\alpha_{m(i}\delta^h_{j)}\bigr], \end{aligned} \tag{2}$$

where $[i\,j]$ denotes alternation with respect to the indicated indices and R^h_{ijk} is the Riemann tensor of A_n.

Contracting integrability conditions (2) with respect to h and m, we obtain

$$\begin{aligned} a_{ik,j} + a_{jk,i} - (n+1)\,a_{ij,k} &= 3\,(-P^\alpha_{ij}R_{\alpha k} + P^\beta_{\alpha(i}R^\alpha_{j)k\beta}) \\ &- \frac{1}{3}\,\bigl[P^\beta_{\alpha i}P^\alpha_{(kj}P^\gamma_{\gamma)\beta} - P^\beta_{\alpha k}P^\alpha_{(ij}P^\gamma_{\gamma)\beta} + (n+2)\,a_{\alpha(i}P^\alpha_{j)k} - 2\,(a_{k\alpha}P^\alpha_{ij} + a_{k(i}P^\alpha_{j)\alpha})\bigr]. \end{aligned} \tag{3}$$

Alternation of Eqs. (3) with respect to i and k gives

$$\begin{aligned} a_{jk,i} = a_{ij,k} + \frac{1}{n+2}\bigl[&P^\alpha_{j[k}R_{i]\alpha} + P^\alpha_{\beta j}R^\beta_{[ik]\alpha} + P^\alpha_{\beta[i}R^\beta_{|j|k]\alpha} \\ &- \frac{1}{3}(P^\alpha_{\beta i}P^\beta_{(kj}P^\gamma_{\gamma)\alpha} - P^\alpha_{\beta k}P^\beta_{(ij}P^\gamma_{\gamma)\alpha} + (n+4)\,a_{\alpha[i}P^\alpha_{k]j} + 2\,a_{j[i}P^\alpha_{k]\alpha})\bigr]. \end{aligned} \tag{4}$$

Replacing indices i and j in (4), we obtain

$$a_{ik,j} = a_{ij,k} + \frac{1}{n+2}\big[P^{\alpha}_{i[k}R_{j]\alpha} + P^{\alpha}_{\beta i}R^{\beta}_{[jk]\alpha} + P^{\alpha}_{\beta[j}R^{\beta}_{|i|k]\alpha} - \frac{1}{3}(P^{\alpha}_{\beta j}P^{\beta}_{(ki}P^{\gamma}_{\gamma)\alpha} - P^{\alpha}_{\beta k}P^{\beta}_{(ji}P^{\gamma}_{\gamma)\alpha} + (n+4)\,a_{\alpha[j}P^{\alpha}_{k]i} + 2\,a_{i[j}P^{\alpha}_{k]\alpha})\big]. \quad (5)$$

Substituting (4) and (5) into (3), we obtain

$$(n-1)\,a_{ij,k} = P^{\beta}_{\alpha\gamma}P^{\alpha}_{(ij}P^{\gamma}_{\gamma)\beta} + \frac{1}{n+2}\big[3(nP^{\alpha}_{ij}R_{\alpha k} - nP^{\alpha}_{\beta(i}R^{\beta}_{j)k\alpha} + P^{\alpha}_{k(i}R_{|\alpha|j)} - P^{\alpha}_{\beta(i}R^{\beta}_{|k|j)\alpha} - P^{\alpha}_{\beta k}R^{\beta}_{(ij)\alpha}) + \frac{1}{3}(-nP^{\alpha}_{\beta k}P^{\beta}_{(ij}P^{\gamma}_{\gamma)\alpha} + (n^2+3n)\,a_{\alpha(i}P^{\alpha}_{j)k} - 2(n+1)\,(a_{k(i}P^{\alpha}_{j)\alpha} + 4(a_{k\alpha}P^{\alpha}_{ij} - a_{ij}P^{\alpha}_{k\alpha}) - P^{\alpha}_{\beta i}P^{\beta}_{(kj}P^{\gamma}_{\gamma)\alpha} - P^{\alpha}_{\beta j}P^{\beta}_{(ki}P^{\gamma}_{\gamma)\alpha}))\big]. \quad (6)$$

One can easily see that Eqs. (1) and (6) are closed system of differential equations in covariant derivatives in A_n of the Cauchy type with respect to the unknown functions $P^h_{ijk}(x)$ and $a_{ij}(x)$ which must satisfy in addition the algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x). \quad (7)$$

Thus we have proved the following

Theorem. *In order for an affinely connected space A_n to admit an almost geodesic mapping defined by Eqs. (1) onto an affinely connected space $\overline{A}_n$, it is necessary and sufficient that there exist a solution of mixed system of differential equations in covariant derivatives of the Cauchy type* (1), (6) *and* (7) *with respect to the unknown functions $P^h_{ij}(x)$ and $a_{ij}(x)$.*

From the properties of this system it follows that the number of essential parameters in its general solution does not exceed $\frac{1}{2}\,n(n+1)^2$.

Consider further Eqs. (1) in the case when the tensor a_{ij} vanishes identically. In this case, Eqs. (1) take the form

$$P^h_{ij,k} = -\frac{1}{3}\,P^{\alpha}_{(ij}P^h_{k)\alpha}. \quad (8)$$

The integrability conditions of system (8) are

$$P^{\alpha}_{ij}R^h_{\alpha km} + P^h_{\alpha(i}R^{\alpha}_{j)km} = \frac{1}{9}\,(P^{\alpha}_{\beta m}P^{\beta}_{(ij}P^h_{k)\alpha} - P^{\alpha}_{\beta k}P^{\beta}_{(ij}P^h_{m)\alpha}).$$

Obviously, the deformation tensor $P^h_{ij}(x)$ satisfying Eqs. (1) has an additional property

$$P^h_{ij,k} = P^h_{ik,j}. \quad (9)$$

4. CANONICAL ALMOST GEODESIC MAPPINGS OF A FLAT SPACE

In conclusion, we demonstrate the existence of a solution of Eq. (9) in a flat space. We assume that in a flat space A_n an affine coordinate system $x^1, x^2, \ldots, x^n$ is considered. Then the Christoffel symbols Γ^h_{ij} are zero, and covariant derivatives are partial derivatives. Thus,

$$P^h_{ij,k} = \partial P^h_{ij}/\partial x^k.$$

From conditions (9) it follows that the deformation tensor P^h_{ij} has the following more specific structure

$$P^h_{ij} = \frac{\partial^2 \varphi^h(x)}{\partial x^i \partial x^j}, \quad (10)$$

where $\varphi^h(x)$ are some functions. On the other hand, expressions (10) imply conditions (9).

We restrict ourselves to demonstration of a particular solution of Eq. (6) without investigation of the difficult problem on the general solution.

Obviously, if

$$\varphi^h(x) = (x^h - c^h) \cdot \ln|x^h - c^h|, \tag{11}$$

where c^h are constants, then the deformation tensor P^h_{ij} defined by conditions (10) is a solution of Eqs. (8).

In terms of the coordinates $x^1, x^2, \dots, x^n$, for the deformation tensor P^h_{ij}, we have

$$P^h_{hh} = \frac{1}{x^h - c^h}, \quad h = 1, 2, \dots, n.$$

All the other components of P^h_{ij} are zero.

One can easily check that a mapping with such deformation tensor will not belong to the types π_2 and π_3.

Note that the types of almost geodesic mappings π_1, π_2, and π_3 may intersect. In particular, mappings from the intersection of π_1 and π_2 preserve linear complexes of geodesics, and mappings from the intersection of π_1 and π_3 preserve quadratic complexes of geodesics [4].

5. AN EXAMPLE OF ALMOST GEODESIC MAPPINGS OF THE FIRST TYPE OF A FLAT SPACE

We give an example of almost geodesic mapping of the first type defined by Eqs. (8) of a flat space A_n onto a flat space $\overline{A}_n$.

Let $x^1, x^2, \dots, x^n$ and $\overline{x}^1, \overline{x}^2, \dots, \overline{x}^n$ be affine coordinates in A_n and $\overline{A}_n$, respectively. The mapping

$$\overline{x}^h = \frac{1}{2}\, c^h_\alpha\, (x^\alpha - c^\alpha)^2 + x^h_\circ, \tag{12}$$

where c^h_i, c^h, and $x^h_\circ$ are constants, $x^h \neq c^h$, and $\det|c^h_i| \neq 0$, define an almost geodesic mapping of the first type of A_n onto $\overline{A}_n$.

It is a matter of direct verification to show that the deformation tensor P^h_{ij} in terms of the coordinate system $x^1, x^2, \dots, x^n$ is of the form (11).

Such a mapping sends straight lines of the space A_n, which are defined by the equations $x^h = a^h + b^h\, t$, where t is a parameter, into parabolas of the space $\overline{A}_n$ defined by equations $\overline{x}^h = D^h + E^h\, t + F^h\, t^2$, where

$$D^h = \frac{1}{2}\, c^h_\alpha\, (a^\alpha - c^\alpha)^2, \quad E^h = c^h_\alpha\, (a^\alpha - c^\alpha)\, b^\alpha, \quad F^h = \frac{1}{2}\, c^h_\alpha\, (b^\alpha)^2.$$

The exception is the case of straight lines for which the vectors E^h and F^h are collinear. Such straight lines are mapped into straight lines.

Formulas (12) give a family of almost geodesic transformations of the type π_1 of a flat space whose parameters are the coefficients c^h_i, c^h and $x^h_\circ$.

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