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Almost Periodicity, Finite Automata Mappings, and Related Effectiveness Issues

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Abstract—We introduce a class of eventually almost periodic sequences where some suffix is almost periodic (i.e., uniformly recurrent). The class of generalized almost periodic sequences includes the class of eventually almost periodic sequences, and we prove this inclusion to be strict. We also prove that the class of eventually almost periodic sequences is closed under finite automata mappings and finite transductions. Moreover, we obtain an effective form of this result. In conclusion we consider some algorithmic questions related to the almost periodicity.

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1. INTRODUCTION

Almost periodic (in other terminology, uniformly recurrent) sequences were studied in the works of Morse and Hedlund [4, 5] and of many others (e.g., see [2, 7]). A sequence is *almost periodic* if every its factor occurs in it infinitely many times with bounded gaps. This notion first appeared in the field of symbolic dynamics, but then turned out to be interesting in connection with computer science, mathematical logic, combinatorics on words. Generalized almost periodic sequences were introduced by A. L. Semenov in [11] (under the name "almost periodic") while studying logical theories of unary functions over $\mathbb{N}$. A sequence is *generalized almost periodic* if each of its factors either occurs in it infinitely many times with bounded gaps or occurs only finitely many times. We introduce a new class of sequences called *eventually almost periodic*, where some suffix is almost periodic. Then we study some properties of this class.

This paper is organized as follows.

In Section 2 we give formal definitions of different generalizations of the periodicity notion. The class of generalized almost periodic sequences includes the class of eventually almost periodic sequences. We prove this inclusion to be strict (Theorem 2.1).

Section 3 is devoted to automata mappings. Generalized almost periodic sequences were studied in detail in [7, 12] (under the name "almost periodic"). In particular, the authors prove that the class of generalized almost periodic sequences is closed under finite automata mappings. The class of images of almost periodic sequences under finite automata mappings contains the class of eventually almost periodic sequences. The main result of the paper (Theorem 3.2) establishes the equality of the classes. In other words, Theorem 3.2 says that finite automata preserve the property of eventual almost periodicity. Moreover, an effective variant of this theorem is proved (Theorem 3.3). Then we consider a generalization of finite automata, i.e., finite transducers, and prove the same statement for them.

In Section 4 we deal with some algorithmic questions connected with almost periodicity[1)]. Namely, we prove that some properties of sequences related to almost periodicity do not have corresponding effective analogs (in contrast to Theorem 3.2 with effective version in Theorem 3.3). For instance, we

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[1)] Note that we usually use the term "almost periodicity" in two different ways: The first one comes from the formal definition of an almost periodic sequence, and the second one (that in particular is used here) is for the general property of being close to periodic that we formalize in three different ways. Some further remarks on terminology are given throughout the paper.

prove that given an eventually almost periodic sequence and its regulator, we can not find any prefix which is sufficient to cut from this sequence in order to obtain an almost periodic sequence. Actually, the results of these facts are reduced to concrete constructions in combinatorics on words.

Let us introduce some basic notions and notations.

Let us denote $\{0,1\}$ by $\mathbb{B}$, the set of natural numbers $\{0,1,2,\dots\}$ by $\mathbb{N}$. Let Σ be a finite alphabet with at least two symbols. We consider sequences over this alphabet, i.e., mappings $\omega : \mathbb{N} \to \Sigma$. The set of all such sequences forms the standard Cantor metric space. Denote this space by $\Sigma^{\mathbb{N}}$. Then $\lim\limits_{n\to\infty} x_n = \omega$, if $\forall i\, \exists n\, \forall m > n\;\; x_m(i) = \omega(i)$ (this definition works for finite x_n as well).

Let us denote by Σ^* the set of all finite words over Σ including the empty word Λ. If $i \le j$ are natural numbers, denote by $[i,j]$ the segment of $\mathbb{N}$ with ends in i and j, i.e., the set $\{i, i+1, i+2, \dots, j\}$. Also denote by $\omega[i,j]$ a subword $\omega(i)\omega(i+1)\dots\omega(j)$ of a sequence ω. A segment $[i,j]$ is an *occurrence* of a word $x \in \Sigma^*$ in a sequence ω if $\omega[i,j] = x$. We say that $x \ne \Lambda$ is a *factor* of ω if x occurs in ω. A word of the form $\omega[0,i]$ for some i is called a *prefix* of ω, and respectively a sequence of the form $\omega(i)\omega(i+1)\omega(i+2)\dots$ for some i is called a *suffix* of ω and is denoted by $\omega[i,\infty)$. Denote by $|x|$ the length of a word x. An occurrence $x = \omega[i,j]$ in ω is *k-aligned* if $k|i$. We imagine sequences going horizontally from left to right, so we use terms "to the left" and "to the right" when speaking of smaller and greater indices respectively.

2. ALMOST PERIODICITY

A sequence ω is *periodic* if for some T we have $\omega(i) = \omega(i+T)$ for each $i \in \mathbb{N}$. This T is called a *period* of ω. The class of all periodic sequences we denote by $\mathcal{P}$. Let us consider some extensions of this class.

A sequence ω is called *generalized almost periodic* if for every factor x of ω occurring in it infinitely many times, there exists a number l such that every l-length factor of ω contains at least one occurrence of x. We denote the class of all generalized almost periodic sequences by $\mathcal{GAP}$.

A sequence ω is called *almost periodic* if for every factor x of ω there exists a number l such that every l-length factor of ω contains at least one occurrence of x (and therefore x occurs in ω infinitely many times). Denote by $\mathcal{AP}$ the class of all almost periodic sequences. Clearly, in order to show the almost periodicity of a sequence, it is sufficient to check the mentioned condition only for all prefixes but not for all factors.

Let us introduce another definition for convenience. A sequence ω is *eventually almost periodic* if some its suffix is almost periodic. The class of all eventually almost periodic sequences we denote by $\mathcal{EAP}$.

Suppose $\omega \in \mathcal{EAP}$. Denote by $\mathrm{pr}(\omega)$ the minimal n such that $\omega[n,\infty) \in \mathcal{AP}$. Thus for each $m \geqslant \mathrm{pr}(\omega)$ we have $\omega[m,\infty) \in \mathcal{AP}$.

A function $\mathrm{R}_\omega : \mathbb{N} \to \mathbb{N}$ is an *almost periodicity regulator* of a sequence $\omega \in \mathcal{GAP}$, if

(1) every n-length subword occurring in ω infinitely many times occurs in every $\mathrm{R}_\omega(n)$-length factor of ω;

(2) every n-length word occurring finitely many times in ω does not occur in $\omega[\mathrm{R}_\omega(n),\infty)$.

The latter condition is important only for sequences in $\mathcal{GAP} \setminus \mathcal{AP}$. Note that regulator is not unique: every function greater than regulator is also a regulator. We will also use letters $f, g, \dots$ for regulators.

Clearly, $\mathcal{P} \subset \mathcal{AP} \subset \mathcal{EAP} \subset \mathcal{GAP}$. In fact, all these inclusions are strict. For instance, the famous Thue–Morse sequence $\omega_T = 0110100110010110\dots$ (see [1, 13] or Section 4) is an example of the sequence in $\mathcal{AP}$ but not in $\mathcal{P}$ (moreover, $\mathcal{AP}$ has cardinality continuum while $\mathcal{P}$ is countable, see [3] or [7] for the proofs). The inequality $\mathcal{AP} \subsetneq \mathcal{EAP}$ is obvious. The inequality $\mathcal{EAP} \subsetneq \mathcal{GAP}$ was first proved in [8]. We present here essentially the same proof, though with some differences in technical details. We will refer to some parts of this proof later.

Theorem 2.1. *There exists a binary sequence $\omega \in \mathcal{GAP} \setminus \mathcal{EAP}$.*

Proof. Let us construct a sequence of binary words

$$a_0 = 1,\ a_1 = 10011,\ a_2 = 1001101100011001001110011,$$

and so on. The word a_{n+1} is obtained from a_n in accordance with this rule:

$$a_{n+1} = a_n\overline{a}_n\overline{a}_n a_n a_n,$$

where $\overline{x}$ is a word obtained from x by changing every 0 to 1 and 1 to 0. Let

$$c_n = a_n a_n a_n a_n$$

and

$$\omega = c_0 c_1 c_2 c_3 \dots$$

Let us prove that $\omega \in \mathcal{GAP} \setminus \mathcal{EAP}$.

The length of a_n is 5^n, so the length of $c_0c_1\dots c_{n-1}$ is $4(1+5+\dots+5^{n-1}) = 5^n - 1$. For convenience, let

$$l_n = 5^n - 1 = |c_0c_1\dots c_{n-1}|.$$

Let us show that ω is generalized almost periodic. Suppose $x \neq \Lambda$ occurs in ω infinitely many times. Take n such that $|x| < 5^n$. Suppose $[i,j]$ is an occurrence of x in ω such that $i \geq l_n$. By construction, for every k we can represent $\omega[l_k,\infty)$ as a concatenation of words a_k and $\overline{a}_k$. Thus (by the assumption about i) the word x is a subword of either a_na_n, $a_n\overline{a}_n$, $\overline{a}_na_n$, or $\overline{a}_n\overline{a}_n$. Note that 10011 contains all words of length two (00, 01, 10 and 11), so a_{n+1} contains each of a_na_n, $a_n\overline{a}_n$, $\overline{a}_na_n$, $\overline{a}_n\overline{a}_n$. Hence x is a subword of a_{n+1}. Similarly, x is a subword of $\overline{a}_{n+1}$. In each $2|a_{n+1}|$-length factor of $\omega[l_{n+1},\infty)$, a_{n+1} or $\overline{a}_{n+1}$ occurs. Hence for $l = (5^{n+1}-1) + 2\cdot 5^{n+1}$ the word x occurs in every l-length factor of ω.

Now let us prove that for every $n \geq 1$ the word c_n does not occur in $\omega[l_{n+1},\infty)$. This would imply that c_n occurs, but only finitely many times in the suffix $\omega[l_n,\infty)$, i. e., this suffix is not almost periodic. Therefore ω is not eventually almost periodic.

Let $\nu = \omega[l_{n+1},\infty)$. As it was already noted above, for each k such that $1 \leq k \leq n+1$, ν is a concatenation of words a_k and $\overline{a}_k$. Assume c_n occurs in ν and let $[i,j]$ be one of this occurrences. For $n \geq 1$ the word c_n begins with a_1, hence $[i,i+4]$ is an occurrence of a_1 in ν. We see that $a_1 = 10011$ occurs in $a_1a_1 = 1001110011$, $a_1\overline{a}_1 = 1001101100$, $\overline{a}_1a_1 = 0110010011$, or $\overline{a}_1\overline{a}_1 = 0110001100$ only in 0-th or 5-th position. Thus $[i,j]$ is 5-aligned, and hence ν and c_n can be considered as constructed of "letters" a_1 and $\overline{a}_1$, and we assume that c_n occurs in ν. Now it is easy to prove by induction on m that $[i,j]$ is 5^m-aligned for $1 \leqslant m \leqslant n$, i. e., we can consider ν and c_n as constructed from "letters" a_m and $\overline{a}_m$, and assume that c_n occurs in ν. The base for $m=1$ is already proved. If we know this statement for $m=k$, we can represent ν and c_n as constructed from a_k and $\overline{a}_k$, and assume c_n occurs in ν. Then in order to prove the statement for $m = k+1$, we can repeat the same argument as for $m=1$ changing 1 and 0 to a_m and $\overline{a}_m$ and taking into account that c_n begins with a_m for each $1 \leq m \leq n$.

Therefore we have shown that $[i,j]$ is 5^n-aligned, hence if we consider ν and c_n as being constructed from "letters" a_n and $\overline{a}_n$, then $c_n = a_na_na_na_n$ occurs in ν. But note that in every sequence constructed by concatenation of words $a_1 = 10011$ and $\overline{a}_1 = 01100$ there is no any occurrence of 0000 or 1111. That is why c_n also can not occur in ν. This is the contradiction. □

Moreover, it is quite easy to modify the proof in order to show that $\mathcal{GAP} \setminus \mathcal{EAP}$ has cardinality continuum. For instance, for each sequence $\tau : \mathbb{N} \to \{4,5\}$ we can construct ω_τ in the same way as in the proof of Theorem 2.1, but instead of c_n we take

$$c_n^{(\tau)} = \underbrace{a_na_n\dots a_n}_{\tau(n)}.$$

Obviously, all ω_τ are different for different τ and hence there exists continuum of various τ.

3. FINITE AUTOMATA MAPPINGS

It seems interesting to understand whether some transformations of sequences preserve the property of almost periodicity. The simplest type of algorithmic transformation is a finite automaton mapping. Another motivation, less philosophical, is that finite automata mappings were one of the main tools in [12] while studying almost periodicity and proving the decidability criterion for first-order and monadic theories of unary functions over $\mathbb{N}$.

A *finite automaton* is a tuple $F = \langle \Sigma, \Delta, Q, \widetilde{q}, f\rangle$, where Σ and Δ are finite sets called input and output alphabets respectively, Q is a finite set of states, $\widetilde{q} \in Q$ is the initial state, and

$$f : Q \times \Sigma \to Q \times \Delta$$

is the transition function. For $\alpha \in \Sigma^{\mathbb{N}}$ consider the sequence $\langle p_n, \beta(n)\rangle_{n=0}^{\infty}$, where $p_n \in Q$, $\beta(n) \in \Delta$, and assume $p_0 = \widetilde{q}$ and $\langle p_{n+1}, \beta(n)\rangle = f(p_n, \alpha(n))$ for each n. Then we call $\beta = F(\alpha)$ a finite automaton mapping of α. If $[i, j]$ is an occurrence of a word x in α, and $p_i = q$, then we say that automaton F comes to this occurrence of x being in the state q.

In [7, 12] the following statement was proved.

Theorem 3.1. *If F is a finite automaton and $\omega \in \mathcal{GAP}$, then $F(\omega) \in \mathcal{GAP}$.*

A counterpart of this statement for eventually almost periodic sequences was proved in [8]. We present here this proof (a little modified though), for the analysis of this proof motivates the main result of this Section.

Theorem 3.2. *If F is a finite automaton and $\omega \in \mathcal{EAP}$, then $F(\omega) \in \mathcal{EAP}$.*

Proof. Obviously, it is enough to prove the theorem for $\omega \in \mathcal{AP}$, since prefix does not matter.

Let $\omega \in \mathcal{AP}$. By Theorem 3.1, $F(\omega) \in \mathcal{GAP}$. Suppose $F(\omega)$ is not eventually almost periodic. This means that for every natural N there exists a word that has an occurrence in $F(\omega)[N, \infty)$ which is the rightmost occurrence of this word into the sequence. Indeed, if we remove the prefix $[0, N]$ from $F(\omega)$, we do not get an almost periodic sequence, hence there exists a word occurring in this sequence only finitely many times. Then take its rightmost occurrence.

Let $[i_0, j_0]$ be the rightmost occurrence of a word y_0 in $F(\omega)$. For some l_0 the word $x_0 = \omega[i_0, j_0]$ occurs in every l_0-length factor in ω, due to the property of almost periodicity. If F comes to i_0 being in the state q_0, then F never comes to righter occurrences of x being in the state q_0, since otherwise the automaton would output y_0 completely.

Now let $[r, s]$ be the rightmost occurrence of some word a in $F(\omega)$, where $r > i_0 + l_0$. The factor $\omega[r - l_0, r]$ contains an occurrence $[r', s']$ of the word x_0. By definition of r, we have $r' > i_0$. Thus let

$$i_1 = r', \;\; j_1 = s, \;\; x_1 = \omega[i_1, j_1], \;\; y_1 = F(\omega)[i_1, j_1].$$

Since a does not occur in $F(\omega)[r, \infty)$, then y_1 does not occur in $F(\omega)[i_1, \infty)$, for it contains a as a subword. Therefore if the automaton comes to the position i_1 being in the state q_1, then it never comes to righter occurrences of x_1 being in the state q_1. Since x_1 begins with $\omega[r', s'] = x_0$, and $r' > i_0$, then $q_1 \neq q_0$. We have found the word x_1 such that the automaton F never comes to occurrences of x_1 to the right of i_1 being in the state q_0 or q_1.

Let $m = |Q|$. Arguing in the same manner, for $k < m$ we construct the words $x_k = \omega[i_k, j_k]$ and corresponding different states q_k, such that F never comes to occurrences of x_k in $\omega[i_k, \infty)$ in the states $q_0, q_1, \ldots, q_k$. For $k = m$ we get the contradiction. $\square$

Note that this proof is non-effective in the following sense. Suppose we know $\omega \in \mathcal{AP}$ and its almost periodicity regulator R_ω. Then by Theorem 3.2 an upper bound on $\mathrm{pr}(F(\omega))$ exists for $F(\omega) \in \mathcal{EAP}$, but the presented proof does not allow us to obtain any such bound.

The following effective version of Theorem 3.2 was announced in [9].

For a function g denote $\underbrace{g \circ g \circ \cdots \circ g}_{n}$ by g^n.

Theorem 3.3. *Let F be a finite automaton with n states and $\omega \in \mathcal{AP}$. Then*

$$\mathrm{pr}(F(\omega)) \leqslant \mathrm{R}_\omega^n(1) + \mathrm{R}_\omega^{n-1}(1) + \cdots + \mathrm{R}_\omega(1).$$

In order to prove this theorem, first we consider a particular type of automata called reversible for which the statement of the theorem is simple. Then we introduce a construction in combinatorics on words which allows us to reduce the general situation to the case of reversible automata.

A finite automaton $F = \langle \Sigma, \Delta, Q, \widetilde{q}, f\rangle$ is *reversible*, if for every $q \in Q$ and $a \in \Sigma$ there exist unique $q' \in Q$ and $b \in \Delta$, such that $f(q', a) = \langle q, b\rangle$. In other words, in such an automaton each letter of the input alphabet Σ performs a permutation on Q (output alphabet does not matter). Given a state, we can reconstruct the sequence of previous states from the sequence of previous input letters (that is what reversibility means).

Theorem 3.4. *If F is a reversible finite automaton and $\omega \in \mathcal{AP}$, then $F(\omega) \in \mathcal{AP}$.*

Proof. Suppose x occurs in ω, and F comes to this occurrence being in the state q. Our goal is to prove that the next time when F comes to x in ω being in the state q, is at some distance from the previous such situation, and we can upper-bound this distance in terms of $|x|$ and R_ω. This means the same bound for this distance works for each situation when F comes to x being in the state q. So this is sufficient for our purpose.

Let $x = x_0 = \omega[r, s]$ be a factor of ω; the automaton comes to this occurrence being in some state q. Let $[i_0, j_0]$ be the next occurrence of x_0 in ω, so that $j_0 \leqslant r + \mathrm{R}_\omega(|x|)$ (the segment $\omega[r+1, r+\mathrm{R}_\omega(|x|)]$ contains an occurrence of x). If F comes to this occurrence being in the state q, then we are done. Otherwise F comes to i_0 being in the state $q_0 \neq q$. Let $x_1 = \omega[r, j_0]$, and let $[i_1, j_1]$ be the next occurrence of x_1 in ω, so that $j_1 \leqslant r + \mathrm{R}_\omega(\mathrm{R}_\omega(|x|) + 1)$, for $|x_1| = \mathrm{R}_\omega(|x|) + 1$. Suppose the automaton comes to the position $i_1 + i_0 - r$ being in the state q_1. If $q_1 = q$, then we are done, for $\omega[i_1 + i_0 - r, j_1] = x_0$. If $q_1 = q_0$, then F comes to the position i_1 being in the state q, due to reversibility of F, and we are done. Otherwise $q_1 \neq q$ and $q_1 \neq q_0$. Similarly, for $x_2 = \omega[r, j_1]$ and its occurrence $[i_2, j_2]$ in ω such that $i_2 > r$ and $j_2 \leqslant r + \mathrm{R}_\omega(\mathrm{R}_\omega(\mathrm{R}_\omega(|x|) + 1) + 1)$, in the worst case F comes to the position $i_2 + i_1 + i_0$ being in the state q_2 where $q_2 \neq q$, $q_2 \neq q_0$, and $q_2 \neq q_1$. Arguing in the same manner, for $k < m = |Q|$ we construct the words $x_0, x_1, \ldots, x_k$ with occurrences $[i_0, j_0], [i_1, j_1], \ldots, [i_k, j_k]$, and different states $q_0, q_1, \ldots, q_{k-1}$ such that in the worst case F cannot come to the position $i_k + i_{k-1} + \cdots + i_0 - kr$ being in the states $q, q_0, \ldots, q_{k-1}$. Thus for $k = m$ we are done for sure, and the bound for the distance is $f(f(\ldots(|x|)\ldots))$, where $f = \mathrm{R}_\omega + 1$ and the number of iterations is m. ☐

For $\omega \in \Sigma^{\mathbb{N}}$, $\nu \in \Delta^{\mathbb{N}}$ let us define $\omega \times \nu \in (\Sigma \times \Delta)^{\mathbb{N}}$ so that $(\omega \times \nu)(i) = \langle \omega(i), \nu(i)\rangle$.

Corollary 3.1. *If $\omega \in \mathcal{AP}$ and $\nu \in \mathcal{P}$, then $\omega \times \nu \in \mathcal{AP}$.*

Proof. The operation "$\times$" with a periodic sequence can be simulated by a cyclic finite automaton that is obviously reversible. ☐

Now consider the following construction. Let $\omega \in \Sigma^{\mathbb{N}}$, and suppose $a \in \Sigma$ occurs in ω infinitely many times. Cut ω into blocks of the form xa, where $x \in (\Sigma \setminus \{a\})^*$, i.e., into blocks containing a symbol a on the end and not containing any other occurrences of a. To do this, we need to cut after each occurrence of a. If a occurs in ω with bounded gaps, then the number of all such blocks is finite (for example, if $\omega \in \mathcal{GAP}$, then the length of these blocks is not greater than $\mathrm{R}_\omega(1)$). Encode these blocks by symbols of some finite alphabet, denote this alphabet by $\mathrm{b}_{a,\omega}(\Sigma)$. Thus we obtained a new sequence in this alphabet from ω. Delete the first symbol of this sequence. The result is called an *a-split* of ω and is denoted by $\mathrm{s}_a(\omega)$. For example, 0-split of the sequence 3200122403100110 ... is (0)(12240)(310)(0)(110) ...

Lemma 3.1. *Let $\omega \in \mathcal{AP}$, and suppose $a \in \Sigma$ occurs in ω. Then $\mathrm{s}_a(\omega) \in \mathcal{AP}$.*

Proof. Let k be the maximal length of the a-split blocks. Consider a prefix x of $\mathrm{s}_a(\omega)$. The corresponding word y in ω is not longer than $k|x|$. Let $z = ay$, $|z| \leqslant k|x| + 1$. The word z occurs in ω, and the first symbol of the first occurrence of z intersects with the last symbol of the first block which is then deleted in the construction. Therefore z occurs in every factor of length $l = \mathrm{R}_\omega(k|x| + 1)$ in ω. The first and the last symbols of z are a, so every such occurrence is well-aligned with respect to the a-split of ω. Hence for every occurrence of z in ω there is an occurrence of x in $\mathrm{s}_a(\omega)$. Therefore x occurs in each factor of length $\mathrm{R}_\omega(k|x| + 1)$ in $\mathrm{s}_a(\omega)$. □

Now we can prove the promised theorem.

Proof of Theorem 3.3. Let $F = \langle \Sigma, \Delta, Q, \widetilde{q}, f\rangle$ and $|Q| = n$. We construct an algorithm to compute some

$$l \geqslant \mathrm{pr}(F(\omega)),$$

and simultaneously prove

$$l \leqslant \mathrm{R}^n_\omega(1) + \mathrm{R}^{n-1}_\omega(1) + \cdots + \mathrm{R}_\omega(1).$$

Let us assume that every automaton in the proof has the maximum possible output alphabet "input alphabet"×"the set of states" (the general case can be obtained from this one by projection). For example, for F this is $\Sigma \times Q$. Correspondingly, the transition function f prints the pair of a current state and an input symbol to the output. In what follows, we omit the second component of the transition function value, i.e., for instance, instead of $f(p, a) = \langle q, b\rangle$ we write simply $f(p, a) = q$ with $f(p, a) = \langle q, \langle p, a\rangle\rangle$ in mind.

Let $\omega_0 = \omega$. We assume every symbol of Σ occurs in ω_0, otherwise we restrict F only to the symbols occurring in ω_0; to determine these symbols effectively, we can read first $\mathrm{R}_{\omega_0}(1)$ symbols of ω_0.

If F is reversible, by Theorem 3.4 we get $\mathrm{pr}(F(\omega_0)) = 0$. Otherwise there exists a symbol $a_0 \in \Sigma$ that accomplishes a non-one-to-one mapping of Q, so that the set

$$Q_1 = \{q : \exists q'\ f(q', a_0) = q\}$$

is a proper subset of Q. Consider

$$\omega_1 = \mathrm{s}_{a_0}(\omega_0),$$

which is almost periodic by Lemma 3.1. Note that starting with any state on ω_0, the automaton F comes to any block of the a_0-split of ω_0 being in the state of the set Q_1, for every such block has a_0 in the end.

Let us construct a new automaton F_1 (the construction is effective over F). Let the input alphabet of F_1 be $\mathrm{b}_{a_0,\omega_0}(\Sigma)$, the set of states be Q_1, and the value of the transition function on $x \in \mathrm{b}_{a_0,\omega_0}(\Sigma)$, $q \in Q_1$ be the output of F when starting in the state q and given the word x represented in symbols of Σ as input. Let the initial state of F_1 be the state of F that is reached after the work on the prefix of ω until the first occurrence of a_0 (the prefix which we delete in order to obtain $\mathrm{s}_{a_0}(\omega_0)$ from ω_0). Now the work of F_1 on ω_1 simulates the work of F on ω_0. Note that ω_1 is obtained from ω by deleting not more than $\mathrm{R}_{\omega_0}(1)$ first symbols, counting in the alphabet Σ.

We have the sequence ω_1 (in the alphabet larger than initial) and the automaton F_1 with the set of states less than initial. If F_1 is not reversible, then we can repeat the procedure from the last paragraph. Thus we obtain the sequence ω_2 in some alphabet $\mathrm{b}_{a_1,\omega_1}(\mathrm{b}_{a_0,\omega_0}(\Sigma))$, and the automaton F_2 with the set of states less than previous, working on ω_2. The sequence ω_2 is obtained from ω_1 by deleting not more than $\mathrm{R}_{\omega_1}(1)$ first symbols, counting in the alphabet $\mathrm{b}_{a_0,\omega_0}(\Sigma)$. Therefore ω_2, written in the initial alphabet Σ, is obtained from ω by deleting not more than $\mathrm{R}_{\omega_0}(\mathrm{R}_{\omega_0}(1)) + \mathrm{R}_{\omega_0}(1)$ first symbols, counting in the alphabet Σ.

An automaton with a single state (and with arbitrary input alphabet) is always reversible. Hence after k repetitions of the described procedure for some $k \leqslant n$, we come to the situation where the reversible automaton F_k works on the almost periodic sequence ω_k in some alphabet (after each repetition of the procedure the number of states decreases). The symbols of this alphabet encode the blocks of the initial sequence. Thus $F_k(\omega_k) \in \mathcal{AP}$.

Writing ω_k back in the alphabet Σ, we get some suffix ω' obtained from ω by deleting a prefix not longer than

$$\mathrm{R}_\omega^k(1) + \mathrm{R}_\omega^{k-1}(1) + \cdots + \mathrm{R}_\omega(1) \leqslant \mathrm{R}_\omega^n(1) + \mathrm{R}_\omega^{n-1}(1) + \cdots + \mathrm{R}_\omega(1).$$

It only remains to check why $F_k(\omega_k) \in \mathcal{AP}$ implies $F(\omega') \in \mathcal{AP}$. Let us explain this in a simple case when the automaton F_1 obtained after the first iteration of the procedure is reversible (the general situation can be reduced to this case by induction). Then ω' is obtained from ω by deleting first symbols until the first occurrence of a_0. Let the initial state of F_1 (in which F comes to ω') be q. To show $F(\omega') \in \mathcal{AP}$, it is necessary and sufficient to check whether for every prefix of ω', the occurrences of its copies in ω' to which F comes being in the state q, are sufficiently regular, i.e., these copies occur in each l-length factor for some l (one direction is obvious, the other follows from our requirement for automata always to output the pair $\langle$input symbol, current state$\rangle$).

Let x be a prefix of ω' that ends with a_0 (an arbitrary prefix is contained in some such prefix). We can correctly split it into blocks ending with a_0. Let us denote this split by y. The automaton F_1 is reversible, so $F_1(\omega_1) \in \mathcal{AP}$. By the necessary and sufficient condition of the previous paragraph, F_1 comes to y being in the state q in each t-length factor for some t. Every such situation corresponds in ω' to coming F to some occurrence of x being in the state q, and this happens in each tk-length factor, where $k \leqslant \mathrm{R}_\omega(1)$ is the maximal length of the blocks. $\square$

It is not possible to improve significantly the upper bound on $\mathrm{pr}(F(\omega))$ in Theorem 3.3 and to get rid of the number of iterations proportional to the number of states. It follows from the construction in [10] after small modifications.

It is interesting that now we have two different proofs of Theorem 3.2, and the connection between them is not clear.

The results on finite automata mappings can be extended to more general class of mappings preformed by finite transducers.

Let Σ and Δ be finite alphabets. The mapping $h : \Sigma^* \to \Delta^*$ is called a *homomorphism*, if for any $u, v \in \Sigma^*$ we have $h(uv) = h(u)h(v)$. Clearly, a homomorphism is completely determined by its values on single-letter words. Let $\omega \in \Sigma^{\mathbb{N}}$. By definition, put

$$h(\omega) = h(\omega(0))h(\omega(1))h(\omega(2))\ldots$$

Suppose $h : \Sigma^* \to \Delta^*$ is a homomorphism, $\omega \in \Sigma^{\mathbb{N}}$ is generalized almost periodic. In [7] it was shown that if $h(\omega)$ is infinite, then it is generalized almost periodic. Therefore if ω is almost periodic, and $h(\omega)$ is infinite, then $h(\omega)$ is also almost periodic. Indeed, it suffices to show that every v occurring in $h(\omega)$ occurs infinitely many times. But there exists some factor u of ω such that $h(u)$ contains v, and by the definition of almost periodicity u occurs in ω infinitely many times. Clearly, for $\omega \in \mathcal{EAP}$ we have $h(\omega) \in \mathcal{EAP}$, if $h(\omega)$ is infinite.

A natural generalization of a finite automaton is a *finite transducer* (see [7, 14] for more detail). The difference is in that now we allow it to output an arbitrary word (including the empty one) over an output alphabet after reading only one character from the input. Formally, we only change the definition of the transition function. Now it has the form

$$f : Q \times \Sigma \to Q \times \Delta^*.$$

If the sequence $\langle p_n, v_n\rangle_{n=0}^{\infty}$, where $p_n \in Q$, $v_n \in \Delta^*$, is the mapping of α, then the output is the sequence $v_0 v_1 v_2 \ldots$

Actually, we can decompose the mapping performed by a finite transducer, into two: the first one is a finite automaton mapping and another one is a homomorphism. Each of these mappings preserves the class $\mathcal{GAP}$, so we get the consequence: Finite transducers map generalized almost periodic sequences to generalized almost periodic sequences. Similarly, by Theorem 3.2 and the arguments above we also get the following

Corollary 3.2. *Let F be a finite transducer, $\omega \in \mathcal{EAP}$. Suppose $F(\omega)$ is infinite. Then $F(\omega) \in \mathcal{EAP}$.*

4. EFFECTIVENESS

Lots of interesting algorithmic questions naturally appear in connection with almost periodicity, i.e., if one can check some property or find some characteristic algorithmically being given a sequence. Sometimes these questions are just effectiveness issues for corresponding noneffective results, for example, Theorem 3.3 is an effective variant of Theorem 3.2. Further, we mainly deal with the case where the answers to these questions are negative. We prove that some properties do not have effective analogs.

Formally, we consider an algorithm with an oracle for a sequence as input. This algorithm halts on every oracle and outputs a finite binary word or any other constructive object. The main property of such an algorithm is continuity: it outputs the answer on having read only finite number of symbols from the sequence. Thus, in order to prove non-effectiveness, we only need to show discontinuity.

If we have only a sequence, then we cannot recognize almost any property of this sequence. For example it is even impossible to understand whether the symbol 1 occurs in a given binary sequence: If an algorithm checks some finite number of symbols and all these symbols are 0, then it can not guarantee that 1 does not occur further. The question about algorithmic decidability becomes more interesting if we allow to provide some additional information on input. In case of generalized almost periodic sequences it is natural to add an almost periodicity regulator.

It is easy to decode unambiguously functions $\mathbb{N} \to \mathbb{N}$ and also pairs $\langle$sequence, function$\rangle$ by binary sequences. That is why we can correctly consider algorithms with a generalized almost periodic sequence ω and its regulator f on input.

From this point of view the above problem can be solved effectively: After reading the first $f(1)$ symbols of the sequence we can say whether or not 1 occurs in it, and moreover after reading the next $f(1)$ symbols we can say whether 1 occurs in it finitely or infinitely many times.

The following several theorems are examples of problems concerning almost periodicity which do not have effective analogs. It is especially interesting to compare the results of Theorem 4.1 and Theorem 3.3. Theorem 4.4 is also connected with Theorem 3.3. All the following theorems were announced in [9].

We say $f_n \to f$ for $f_n, f : \mathbb{N} \to \mathbb{N}$ if $\forall i\, \exists n\, \forall m > n\;\; f_m(i) = f(i)$.

Theorem 4.1. *Given $\omega \in \mathcal{EAP}$ and its regulator f, it is impossible to compute algorithmically any $l \geqslant \mathrm{pr}(\omega)$.*

Let us recall that ω_T is the Thue–Morse sequence. This sequence can be obtained as follows. Let $a_0 = 0$, $a_{n+1} = a_n\overline{a}_n$, and $\omega_T = \lim\limits_{n\to\infty} a_n$. Notice that $|a_n| = 2^n$. The Thue–Morse sequence has lots of interesting properties (see [1]), but we are interested in the following one: ω_T is cube-free, i.e., for any $a \in \mathbb{B}^*$, $a \neq \Lambda$ the word aaa does not occur in ω_T. This was first proved in [13].

Proof of Theorem 4.1. It suffices to construct $\omega_n \in \mathcal{EAP}$, $\omega \in \mathcal{AP}$ with regulators f_n and f such that $\omega_n \to \omega$, $f_n \to f$, but $\mathrm{pr}(\omega_n) \to \infty$. Indeed, suppose the mentioned algorithm exists and it outputs some $l \geqslant 0$ (arbitrary for $\omega \in \mathcal{AP}$) given $\langle \omega, f\rangle$ on the input. During the computation of l the algorithm reads only finite number of symbols in ω and of values of f. Hence there exists $N > l$ such that the algorithm does not know any $\omega(k)$ or $f(k)$ for $k > N$. Since $\mathrm{pr}(\omega_n) \to \infty$, there exists n such that $\mathrm{pr}(\omega_n) > N$. The algorithm works on the input $\langle \omega_n, f_n\rangle$ in the same way as it works on the input $\langle \omega, f\rangle$, and then outputs l, but $\mathrm{pr}(\omega_n) > N > l$.

Let $\omega = \omega_T$, $\omega_n = a_na_na_n\omega$. Notice that $\mathrm{pr}(\omega_n) \geqslant 2^n$. Indeed, if $\mathrm{pr}(\omega_n) < 2^n$, then $a_na_n\omega = a_na_na_n\overline{a}_n\overline{a}_na_n\ldots \in \mathcal{AP}$, and hence $a_na_na_n$ occurs in ω_T, which is the contradiction with the cube-freeness of the Thue–Morse sequence.

It only remains to show that we can find regulators f_n and f for ω_n and ω such that $f_n \to f$. It suffices to find the same regulator g for all ω_n (then we can increase it and obtain the same regulator for all ω_n and for ω too). Fix some R_ω and let $g = 4\mathrm{R}_\omega$. Let a k-length word v occur in $\omega_n = a_na_na_n\omega$ infinitely many times. Let us take the factor $\omega[i,j]$ of length $4\mathrm{R}_\omega(k)$ and show that v occurs in it. If $j \geqslant 3\cdot 2^n + \mathrm{R}_\omega(k)$, then v occurs in the factor $\omega[3\cdot 2^n, 3\cdot 2^n + \mathrm{R}_\omega(k)]$ (by the definition of R_ω). If $j < 3\cdot 2^n + \mathrm{R}_\omega(k)$, then $i \leqslant 3\cdot 2^n - 3\mathrm{R}_\omega(k)$. But $i \geqslant 0$, therefore $\mathrm{R}_\omega(k) \leqslant 2^n = |a_n|$. Then $\omega_n[i, i+\mathrm{R}_\omega(k)]$ is contained in a_na_n. But a_na_n occurs in ω, so $\omega_n[i, i+\mathrm{R}_\omega(k)]$ occurs in ω as well. Therefore v occurs in ω.

However g is not yet what we need. We should watch the words occurring in ω_n finitely many times. Obviously, if some v occurs in ω_n finitely many times, then $|v| = k > 2^n$ (otherwise v occurs in the concatenation of two consecutive words a_n or $\overline{a}_n$, and thus in ω). Therefore this can happen only for finite number of different n. Considering all the situations when words of length k occur in some ω_n finitely many times, we probably increase the value $g(k)$, but only finitely many times. Thus the required bound for the regulators exists. $\square$

We have already seen that $\mathcal{EAP} \subsetneq \mathcal{GAP}$ (Theorem 2.1). Using the same construction, we can show that it is impossible to separate these classes effectively.

Theorem 4.2. *Given $\omega \in \mathcal{GAP}$ and its regulator f, it is impossible to determine algorithmically whether $\omega \in \mathcal{EAP}$.*

In [7] the following universal method for constructing almost periodic sequences was presented. This method is based on block algebra on words introduced in [6] and then studied in [3].

A sequence $\langle A_n, l_n\rangle$, where $A_n \subset \Sigma^*$ for a finite alphabet Σ, $l_n \in \mathbb{N}$, is called a *strong Σ-scheme*, if the following conditions hold:

(1) all the words in A_n have length l_n;

(2) every word $u \in A_{n+1}$ has the form $u = v_1 v_2 \dots v_k$, where $v_i \in A_n$, and for every $w \in A_n$ there exists i such that $v_i = w$.

A sequence $\alpha \in \Sigma^{\mathbb{N}}$ is said to be generated by a strong Σ-scheme $\langle A_n, l_n\rangle$ if for every i and n

$$\alpha[il_n, (i+1)l_n - 1] \in A_n.$$

It is easy to see (due to compactness) that every strong scheme generates some sequence. In [7], it is proved that every sequence generated by a strong scheme is almost periodic. Moreover, every almost periodic sequence is generated by some strong scheme.

Proof of Theorem 4.2. It suffices to construct $\omega_n \in \mathcal{EAP}$, $\omega \in \mathcal{GAP} \setminus \mathcal{EAP}$ with the same regulator f for all ω_n such that $\omega_n \to \omega$.

Let $a_0 = 1$, and then according to the rule: $a_{n+1} = a_n \overline{a}_n \overline{a}_n a_n a_n$. Denote $a_n a_n a_n a_n$ by c_n. As in the proof of Theorem 2.1, let us denote $l_n = 5^n - 1 = |c_0 c_1 \dots c_{n-1}|$. Consider $\omega = c_0 c_1 c_2 c_3 \dots$ and $\nu = \lim\limits_{n\to\infty} a_n$. As it was proved in [8] (see also Theorem 2.1), $\omega \in \mathcal{GAP} \setminus \mathcal{EAP}$. Let $\omega_n = c_0 c_1 \dots c_n \nu$. The sequence ν is generated by the strong scheme $\langle \{a_n, \overline{a}_n\}, 5^n\rangle$, hence $\nu \in \mathcal{AP}$. Therefore $\omega_n \in \mathcal{EAP}$. Obviously, $\omega_n \to \omega$, and it only remains to find a common regulator f. We will get a finite number of conditions of the form $f(k) \geqslant \alpha$, then we can take the maximum among all these α.

Let a k-length word $v = \omega_n[i, j]$ occur in $\omega_n = c_0 c_1 \dots c_n \nu$ infinitely many times. Then v occurs in ν, hence in some a_m as well. Therefore v occurs in ω infinitely many times and it suffices to take $f(k) \geqslant \mathrm{R}_\omega(k) + \mathrm{R}_\nu(k)$.

Let a k-length word $v = \omega_n[i, j]$ occur in ω_n finitely many times. Then $i < l_n$. If $j > l_n$, then $k > 5^n$, since otherwise v would occur in some a_m and hence would occur in ν infinitely many times. But the inequality $k > 5^n$ holds only for finite number of different n, and this yields only finitely many conditions on $f(k)$. Now suppose $j \leqslant l_n$. But then v occurs in $c_0 c_1 \dots c_n$ and occurs in ω finitely many times (otherwise v would occur in some a_m). Therefore in this case it is sufficient to take $f(k) \geqslant \mathrm{R}_\omega(k)$. $\square$

The following theorem shows that it is even impossible to separate effectively $\mathcal{AP}$ and $\mathcal{P}$.

Theorem 4.3. *Given $\omega \in \mathcal{AP}$ and its regulator f, it is impossible to determine algorithmically whether $\omega \in \mathcal{P}$.*

Proof. It suffices to construct $\omega_n \in \mathcal{P}$, $\omega \in \mathcal{AP} \setminus \mathcal{P}$ with a common regulator f for all ω_n such that $\omega_n \to \omega$.

Every almost periodic sequence can be obtained from a strong Σ-scheme $\langle A_n, l_n \rangle$. Let us strengthen the main condition on A_n: Let us consider strong schemes such that for each $n \in \mathbb{N}$ every $u \in A_{n+1}$ has the form $u = v_1 v_2 \dots v_k$, where $v_i \in A_n$, and for every $w_1, w_2 \in A_n$ there exists $i < k$ such that $v_i v_{i+1} = w_1 w_2$. Note that such schemes exist and can generate non-periodic sequences, e.g., $\langle \{a_n, \overline{a}_n\}, 2^n \rangle$ generates the Thue–Morse sequence ω_T.

Let $\langle A_n, l_n \rangle$ be a strong scheme satisfying the strengthened condition from the previous paragraph, generating $\omega \notin \mathcal{P}$. Let $p_n = \omega[0, l_n]$. Thus $p_n \in A_n$ and $\lim\limits_{n\to\infty} p_n = \omega$. Assume $\omega_n = p_n p_n p_n \ldots \in \mathcal{P}$. Then $\omega_n \to \omega$ and it only remains to find some common regulator f for all ω_n.

Let $v = \omega_n[i, j]$, $|v| = k$; since $\omega_n \in \mathcal{P}$, it follows that v occurs in ω_n infinitely many times. The inequality $k \geqslant |p_n| = l_n$ holds only for finite number of different n, and this yields only finitely many conditions on $f(k)$. Now we can assume that $k < l_n$. Let us take t such that $l_{t-1} < k \leqslant l_t$ (it is important that t does not depend on n and is uniquely determined by k). Then $t < n$. There exists m such that $ml_t \leqslant i$ and $j \leqslant (m+2)l_t$, i.e., v occurs in some ab, where $a, b \in A_t$. Then by the scheme property v occurs in every $c \in A_{t+1}$. But in every $2l_{t+1}$-length factor of ω_n, there exists an occurrence of some $c \in A_{t+1}$ (completely contained in some p_n). Therefore it is sufficient to take $f(k) \geqslant 2l_{t+1}$. □

By the argument of Theorem 4.3, there exists an infinite set of periodic sequences with common regulator (while the period tends to infinity). This construction can be used in the following theorem: after adding one symbol to an almost periodic sequence, we can not check whether it is still almost periodic.

Theorem 4.4. *Given $\omega \in \mathcal{EAP}$, its regulator f and some $l \geqslant \mathrm{pr}(\omega)$, it is impossible to find algorithmically* $\mathrm{pr}(\omega)$.

Lemma 4.1. *If $a\omega \in \mathcal{AP}$ for $a \in \Sigma^*$ and ω is periodic with a period l, then $a\omega$ is periodic with the period l.*

Proof. It suffices to prove the lemma for a single-letter a. Let $\alpha = 012\ldots(l-1)012\ldots(l-1)012\ldots(l-1)\ldots$ be a periodic sequence over the alphabet $\Sigma_l = \{0, 1, 2, \ldots, l-1\}$. Then by Corollary 3.1, $a\omega \times \alpha \in \mathcal{AP}$. In this sequence the symbol $\langle a, 0 \rangle$ occurs infinitely many times, hence $a = \omega(l)$. □

Proof of Theorem 4.4. It suffices to construct $\omega_n \in \mathcal{EAP}$, $\omega \in \mathcal{AP}$ with a common regulator f for all ω_n such that $\omega_n \to \omega$ and $\mathrm{pr}(\omega_n) = 1$ ($\omega \in \mathcal{AP}$ means $\mathrm{pr}(\omega) = 0$).

Note that $1\omega_T \in \mathcal{AP}$. Indeed, for each n the words $a_n a_n$ and $\overline{a}_n a_n$ occur in ω_T, and hence $1a_n$ occurs in ω_T too. Similarly, $0\omega_T \in \mathcal{AP}$.

As it can be seen in the proof of Theorem 4.3, we can choose a sequence $k_n \to \infty$ such that all periodic sequences of the form $\omega(0)\ldots\omega(k_n)\omega(0)\ldots\omega(k_n)\omega(0)\ldots$ have a common regulator f. Let us take a subsequence m_n of the sequence k_n such that all the symbols $\omega(m_n)$ are the same. Without loss of generality, suppose these symbols are 0.

Let $\omega_n = 1\omega(0)\ldots\omega(m_n)\omega(0)\ldots\omega(m_n)\omega(0)\ldots$ and $\omega = 1\omega_T$. There exists a common regulator g for these sequences. Indeed, it suffices to satisfy $g(k) \geqslant f(k) + 1$ (from considering the words occurring infinitely many times) and $g(k) \geqslant k$ (from considering the words occurring only finitely many times: This may happen only for prefixes occurring exactly once).

If $\omega_n \in \mathcal{AP}$, then by Lemma 4.1 the sequence ω_n is periodic with the period m_n. But $\omega_n(0) = 1 \neq \omega_n(m_n) = 0$. Therefore $\mathrm{pr}(\omega_n) = 1$.

The case where all the symbols $\omega(m_n)$ are 1 is analogous (then ω_n begins with 0). □

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REFERENCES

1. J.-P. Allouche and J. Shallit, “The Ubiquitous Prouhet–Thue–Morse Sequence,” in: *Sequences and Their Applications* (Proceedings of SETA’98, Springer Verlag, 1999), pp. 1–16.
2. J. Cassaigne, “Recurrence in Infinite Words,” in *Proceedings of the 18th Symposium on Theoretical Aspects of Computer Science (STACS 2001)* (Springer Verlag, 2001), pp. 1–11.
3. K. Jacobs, *Maschinenerzeugte* 0-1-*Folgen* (Selecta Mathematica II. Springer Verlag: Berlin, Heidelberg, New York, 1970).
4. M. Morse and G. A. Hedlund, “Symbolic Dynamics,” American Journal of Mathematics **60**, 815–866 (1938).
5. M. Morse and G. A. Hedlund, “Symbolic Dynamics II: Sturmian Rrajectories,” American Journal of Mathematics **62**, 1–42 (1940).
6. M. Keane, “Generalized Morse sequences,” Z. Wahrscheinlichkeitstheorie verw. Geb. **10**, 335–353 (1968).
7. An. Muchnik, A. Semenov, and M. Ushakov, “Almost Periodic Sequences,” Theoretical Computer Science **304**, 1–33, (2003).
8. Yu. L. Pritykin, “Finite-Automaton Transformations of Strictly Almost-Periodic Sequences,” Mat. Zametki **80** (5), 751–756, (2006).
9. Yu. L. Pritykin, “Finite Automata Mappings of Strongly Almost Periodic Sequences and Algorithmic Undecidability,” in: *Proceedings of XXVIII Conference of Young Scientists (Moscow State University, Faculty of Mechanics and Mathematics, 2006)*, pp. 177–181.
10. M. A. Raskin, “On the Estimate of the Regulator for Automaton Mapping of Almost Periodic Sequence,” in: *Proceedings of XXVIII Conference of Young Scientists (Moscow State University, Faculty of Mechanics and Mathematics, 2006)*, pp. 181–185.
11. A. L. Semenov, “On Certain Extensions of the Arithmetic of Addition of Natural Numbers,” Izv. Akad. Nauk, Ser. Matem. **15**, pp. 401–418 (1980).
12. A. L. Semenov, “Logical Theories of One-Place Functions on the Set of Natural Numbers,” Izv. Akad. Nauk, Ser. Matem. **22**, 587–618 (1983).
13. A. Thue, “Über unendliche Zeichenreihen,” Norske vid. Selsk. Skr. Mat. Nat. Kl. **7**, 1–22 (1906).
14. A. Weber, “On the Valuedness of Finite Transducers,” Acta Informatica **27**, 749–780 (1989).

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