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The Annulation Threshold for Partially Monotonic Automata

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Abstract—A deterministic incomplete automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is partially monotonic if its state set Q admits a linear order such that each partial transformation $\delta(\_, a)$ with $a \in \Sigma$ preserves the restriction of the order to the domain of the transformation. We show that if $\mathscr{A}$ possesses an annihilator word $w \in \Sigma^*$ whose action is nowhere defined, then $\mathscr{A}$ is annihilated by a word of length $|Q| + \lfloor \frac{|Q|-1}{2} \rfloor$ and this bound is tight.

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1. BACKGROUND AND MOTIVATION

A *deterministic incomplete automaton* $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is defined by specifying a finite *state set* Q, a finite *input alphabet* Σ, and a partial *transition function* $\delta : Q \times \Sigma \to Q$. The partial function δ extends in a unique way to a partial action $Q \times \Sigma^* \to Q$ of the free monoid Σ^* over Σ; this extension is still denoted by δ. Thus, each word $w \in \Sigma^*$ defines a partial transformation of the set Q denoted by $\delta(\_, w)$.

Given a deterministic incomplete automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$, it may happen that for some word $w \in \Sigma^*$ the partial transformation $\delta(\_, w)$ is nowhere defined. The intuition is that the automaton $\mathscr{A}$ breaks down when being fed with the input w; we therefore say that the word w *annihilates* $\mathscr{A}$.

There are several rather natural questions concerning the notion of an annihilator word: How *mortal* automata (that is, incomplete automata possessing an annihilator word) can be recognized, how long an annihilator word for a given mortal automaton may be, etc. These questions are tightly related to the synchronization problems for complete automata with 0. Recall that a deterministic automaton is said to be *complete* if its transition function is totally defined. A complete automaton $\mathscr{A} = \langle Q, \Sigma, \zeta\rangle$ is called *synchronizing* if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter at which state in Q it started: $\zeta(q, w) = \zeta(q', w)$ for all $q, q' \in Q$. Any word w with this property is said to be a *reset word* for the automaton.

Given a complete automaton $\mathscr{A} = \langle Q, \Sigma, \zeta\rangle$, we say that $s \in Q$ is a *sink state* if $\zeta(s, a) = s$ for all $a \in \Sigma$. It is clear that any synchronizing automaton may have at most one sink state and any word that resets a synchronizing automaton possessing a sink state brings all states to the sink. In such a situation, we denote the unique sink state by 0 and refer to the automaton as a *synchronizing automaton with* 0.

Every incomplete automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ can be completed in the following obvious way. First one adds a new state 0 to the state set Q; let Q^0 stand for the resulting set $Q \cup \{0\}$. Then one extends the partial function $\delta : Q \times \Sigma \to Q$ to a total function $\zeta : Q^0 \times \Sigma \to Q^0$ setting for all $p \in Q^0$ and all $a \in \Sigma$

$$\zeta(p, a) = \begin{cases} \delta(p, a) & \text{if } \delta(p, a) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

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We call the automaton $\langle Q^0, \Sigma, \zeta\rangle$ the 0-*completion* of the incomplete automaton $\mathscr{A}$ and denote this completion by $\mathscr{A}^0$. It is then clear that the 0-completion of a mortal automaton is a synchronizing automaton with 0, and vice versa, every mortal automaton can be obtained from a synchronizing automaton with 0 by removing the zero state and all arrows leading to it.

Recall that the general problem of determining the *synchronization threshold* (that is, the maximum length of the shortest reset word) for a synchronizing automaton with a given number n of states still remains open. (The famous *Černý conjecture* [1] claiming that this threshold is equal to $(n-1)^2$ is arguably the most longstanding open problem in the combinatorial theory of finite automata.) In contrast, the restriction of this problem to the case of synchronizing automata with 0 admits an easy solution: The synchronization threshold for n-state synchronizing automata with 0 is known to be equal to $\frac{n(n-1)}{2}$ (see, for instance, [2], theorem 6.1). Applying this result to 0-completions of mortal automata, one readily obtains the value $\frac{n(n+1)}{2}$ for the *annulation threshold* (that is, the maximum length of the shortest annihilator word) for mortal automata with n states. However, the situation becomes much more intricate for the important subcase of aperiodic automata.

Recall that a deterministic finite automaton $\mathscr{A}$ (complete or not) is said to be *aperiodic* if all subgroups of its transition monoid are singletons. (In view of a theorem of Schützenberger [3] this amounts to saying that $\mathscr{A}$ can recognize only star-free languages.) It is to be expected that for mortal aperiodic automata with n states the annulation threshold is smaller than in the general case but up to now no bound better than $\frac{n(n+1)}{2}$ has been found. On the other hand, all known examples of mortal aperiodic automata possess short annihilator words so that it even was conjectured that such an automaton always can be annihilated by a word whose length does not exceed the number of states of the automaton (see a discussion in [4], section 3).

Determining the annulation threshold for aperiodic automata is especially important in view of recent results on synchronization of aperiodic automata due to Trahtman [5] and Volkov [6]. Without going into detail, we mention that the problem of finding the synchronization threshold for arbitrary synchronizing aperiodic automata can be easily reduced to the cases of strongly connected automata and synchronizing automata with 0. The currently known upper bound of the synchronization threshold for strongly connected aperiodic automata with n states is $\lfloor\frac{n(n+1)}{6}\rfloor$ (cf. [6]) which is less than $\frac{n(n-1)}{2}$ for all $n > 2$. Therefore any improved upper bound of the synchronization threshold for synchronizing aperiodic automata with 0 will immediately yield a corresponding improvement for arbitrary synchronizing aperiodic automata.

In the present paper we study incomplete automata of a special kind which we call partially monotonic. A deterministic incomplete automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is *partially monotonic* if its state set Q admits a linear order such that each partial transformation $\delta(\_, a)$ with $a \in \Sigma$ preserves the restriction of the order to the domain of the transformation. This means that for all $q, q' \in Q$ such that $q \le q'$ and both $\delta(q, a)$ and $\delta(q', a)$ are defined, one has $\delta(q, a) \le \delta(q', a)$. It is well known and easy to verify that each partially monotonic automaton is aperiodic (but the converse, generally speaking, is not true). Our main result gives a linear upper bound for the annulation threshold for partially monotonic automata:

Theorem 1. *If a partially monotonic automaton with n states is mortal, then it has an annihilating word of length at most $n + \lfloor\dfrac{n-1}{2}\rfloor$.*

We also present a series of examples of mortal partially monotonic automata showing that this bound is tight for $n \ge 6$.

Our proof of Theorem 1 is based on a careful analysis of certain properties of **complete** monotonic automata which for brevity will be called *monotonic* in the sequel. (The term "monotonic automaton" was also used in [7] but in a different sense.) Basic synchronization properties of monotonic automata have been already studied in [8] but here we need some refinements of the results of that paper.

Throughout the paper we assume that the state set Q of automata under consideration is the set $\{1, 2, 3, \dots, n\}$ of the first n positive integers with the usual order $1 < 2 < 3 < \dots < n$.

2. STRONGLY CONNECTED MONOTONIC AUTOMATA

We call a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ *strongly connected* if the graph of $\mathscr{A}$ is strongly connected, that is, for any states $p, q \in Q$ there is a word $w \in \Sigma^*$ such that $\delta(p, w) = q$. We need the following property of strongly connected monotonic automata.

Lemma 1. *Let $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ be a strongly connected monotonic automaton. Then for any state $q \in Q \setminus \{1\}$ there is a letter $a \in \Sigma$ such that $\delta(q, a) < q$ and for any state $q \in Q \setminus \{n\}$ there is a letter $b \in \Sigma$ such that $\delta(q, b) > q$.*

Proof. Consider a word w such that $\delta(q, w) = 1 < q$. (Such a word w exists because the automaton $\mathscr{A}$ is strongly connected.) We find the shortest prefix u of the word w such that $\delta(q, u) < q$ and take the last letter of u as a.

By the choice of the letter a we see that $u = va$, $p = \delta(q, v) \geq q$ and $\delta(p, a) < q$. The automaton $\mathscr{A}$ is monotonic whence $\delta(q, a) \leq \delta(p, a) < q$.

A letter b with $\delta(q, b) > q$ can be found in a similar way. □

Let X be a subset of the state set of the automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$. We define the *full preimage of degree* k of the set X as

$$P^k(X) = \{q \in Q \mid (\exists w \in \Sigma^*)\ |w| \leq k \text{ and } \delta(q, w) \in X\}.$$

Observe that $P^k(X) \subseteq P^{k+1}(X)$ for any k. Moreover, if the automaton $\mathscr{A}$ is strongly connected, then the equality $P^k(X) = P^{k+1}(X)$ implies that $P^k(X) = Q$. Therefore, the inequality $P^k(X) \neq Q$ implies that $|P^k(X)| \geq |X| + k$.

Let $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ be a strongly connected monotonic automaton. We say that a subset X of the set Q is a *special set* if $|P^1(X) \setminus X| = 1$ and $X \cap \{1, n\} = \varnothing$.

Our next lemma points out a useful property of some special sets.

Lemma 2. *Let $p \in Q$ and let $X = P^k(\{p\})$ be a special set in a strongly connected monotonic automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$. Then there is a word $w \in \Sigma^*$ of length at most $n - 2 - k$ such that either $\delta(1, w) = n$ or $\delta(n, w) = 1$.*

Proof. Let q denote the unique state of the difference $P^1(X) \setminus X$. We construct two sequences $q = q_0 > q_1 > \cdots > q_\alpha = 1$ and $q = p_0 < p_1 < \cdots < p_\beta = n$ in the following way. Using Lemma 1 for each $i \in \{0, 1, 2, \ldots, \alpha - 1\}$ we find a letter $a_i \in \Sigma$ such that $\delta(q_i, a_i) < q_i$. We denote $\delta(q_i, a_i)$ by q_{i+1}. Also for each $j \in \{0, 1, 2, \ldots, \beta - 1\}$ we find a letter $b_j \in \Sigma$ such that $p_{j+1} = \delta(p_j, b_j) > p_j$.

Let

$$f = \begin{cases} \min\{i \mid q_i \notin P^1(X)\} & \text{if } q_0 \neq 1,\\ 0 & \text{if } q_0 = 1,\end{cases}$$

and

$$g = \begin{cases} \min\{i \mid p_i \notin P^1(X)\} & \text{if } p_0 \neq n,\\ 0 & \text{if } p_0 = n.\end{cases}$$

Then $\{p_1, p_2, \ldots, p_{g-1}, q_1, q_2, \ldots, q_{f-1}\} \subseteq X$, whence $|X| \geq f + g - 2$.

Now we assume that $f \leq g$ and using Lemma 1 we find a word w such that $\delta(n, w) = 1$. Then we show that the length of this word is at most $n - 2 - k$. By symmetry, in case $g \geq f$ we find a word w of length at most $n - 2 - k$ such that $\delta(1, w) = n$.

By Lemma 1 there are a chain $n = r_1 > r_2 > \cdots > r_{s+1} = 1$ and a word $w = c_1c_2\cdots c_s$ such that $\delta(r_i, c_i) = r_{i+1}$ for each $i \in \{1, 2, \ldots, s\}$. If the intersection $\{r_1, r_2, \ldots, r_{s+1}\} \cap X$ is empty, then

$$|w| = s \leq n - 1 - |X| = n - |P^k(\{p\})| - 1 \leq n - k - 2.$$

In the opposite case we consider the first element r_h of this chain that lies in $P^1(X)$. Observe that $r_h = q$. Indeed, if $r_h \neq q$, then $r_h \in X$ whence $r_{h-1} \in P^1(X)$.

The ways of constructing the chains $r_1 > r_2 > \cdots > r_{s+1}$ and $q_0 > q_1 > \cdots > q_\alpha$ are identical and $r_h = q_0$. Therefore we can take $c_h = a_0$ and obtain $r_{h+1} = q_1$, take $c_{h+1} = a_1$ and obtain $r_{h+2} = q_2$ and so on. Finally, we take $c_{h+f-1} = a_{f-1}$ and obtain $r_{h+f} = q_f$. Observe that $r_i \notin P^1(X)$ for any $i \geq h+f$. Indeed, if we consider the first element $r_i \in P^1(X)$ for $i \geq h+f$ we come to a contradiction because as above $r_i = q = q_0 > q_f = r_{h+f}$ but $i \geq h+f$ implies $r_i \leq r_{h+f}$. Summarizing, we have obtained that $\{r_1, r_2, \ldots, r_{s+1}\} \cap P^1(X) = \{q_0, q_1, \ldots, q_{f-1}\}$ whence $|w| = s \leq n-1-|P^1(X)| + f = n-2+(f-|X|)$.

To complete the proof, it suffices to show that $f - |X| \leq -k$. Observe that if $p_i \in P^u(\{p\})$ for some $i > 0$ and $u \leq k$, then $p_{i-1} \in P^{u+1}(\{p\})$. Therefore $p_i \in P^{k-f+1}(\{p\})$ for some $i \in \{1, 2, \ldots, f-1\}$ contradicts $p_0 \notin P^k(\{p\})$, whence $p_i \notin P^{k-f+1}(\{p\})$ for $i \in \{1, 2, \ldots, f-1\}$ and, similarly, $q_i \notin P^{k-f+1}(\{p\})$ for $i \in \{1, 2, \ldots, f-1\}$. It means that $\{p_1, p_2, \ldots, p_{f-1}, q_1, q_2, \ldots, q_{f-1}\} \subseteq X \setminus P^{k-f+1}(\{p\})$. Hence $2f-2 \leq |X| - |P^{k-f+1}(\{p\})| \leq |X| - (k-f+2)$, and therefore, $f - |X| \leq -k$. □

The next fact is crucial for the proof of our main result.

Proposition 1. *Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a strongly connected monotonic automaton. Given a state $p \in Q$, there is a word w of length at most $\frac{3}{2}(n-1)$ such that $\delta(q, w) = p$ for all $q \in Q$.*

Proof. Consider the chain of the full preimages of the set $\{p\}$:

$$\{p\} = P^0(\{p\}) \subseteq P^1(\{p\}) \subseteq P^2(\{p\}) \subseteq \cdots \subseteq P^t(\{p\}). \tag{1}$$

Let $Y = P^s(\{p\})$ be the least set of this chain containing either 1 or n. We suppose that $1 \in Y$. The case $n \in Y$ is symmetric.

If there is no special sets in the chain (1), then $|P^i(\{p\}) \setminus P^{i-1}(\{p\})| \geq 2$ for each $i \in \{1, \ldots, s\}$ whence $2s + 1 \leq |Y| \leq |Q| = n$. By Lemma 1 there is a word u of length at most $n-1$ such that $\delta(n, u) = 1$. Since the automaton $\mathscr{A}$ is monotonic, $\delta(q, u) = 1$ for any $q \in Q$. The inclusion $1 \in Y = P^s(\{p\})$ implies that there is a word v of length at most $s \leq \frac{1}{2}(n-1)$ such that $\delta(1, v) = p$. If $w = uv$ then $|w| \leq \frac{3}{2}(n-1)$ and $\delta(q, w) = p$ for any $q \in Q$.

Now let $X = P^k(\{p\})$ be the greatest special set in the chain (1). By the choice of k and s we obtain that $|Y| - |P^1(X)| \geq 2(s-k-1)$, hence $|Y| \geq |P^1(X)| + 2(s-k-1) = |P^{k+1}(\{p\})| + 2(s-k-1) \geq k+2+2(s-k-1) = 2s-k$.

By Lemma 2 there is a word $u \in \Sigma^*$ of length at most $n-2-k$ such that either $\delta(1, u) = n$ or $\delta(n, u) = 1$.

Case 1: $\delta(1, u) = n$.

Since the automaton $\mathscr{A}$ is monotonic, $\delta(q, u) = n$ for any $q \in Q$. Since the automaton $\mathscr{A}$ is strongly connected, there is a word u_1 of length at most $n - |Y|$ such that $\delta(n, u_1) \in Y$. By the definition of $Y = P^s(\{p\})$ there is a word v_1 of length at most s such that $\delta(\delta(n, u_1), v_1) = p$. Denote uu_1v_1 by w_1. Observe that $|w_1| = |u| + |u_1| + |v_1| \leq (n-2-k) + (n-|Y|) + s \leq (n-2-k) + (n-2s+k) + s = 2(n-1) - s$ and $\delta(q, w_1) = p$ for any $q \in Q$.

Furthermore, by Lemma 1 there is a word u_2 of length at most $n-1$ such that $\delta(n, u_2) = 1$. Since the automaton $\mathscr{A}$ is monotonic, $\delta(q, u_2) = 1$ for any $q \in Q$. The inclusion $1 \in Y = P^s(\{p\})$ implies that there is a word v_2 of length at most s such that $\delta(1, v_2) = p$. Denote u_2v_2 by w_2. It is easy to see that $|w_2| \leq n-1+s$ and $\delta(q, w_2) = p$ for any $q \in Q$.

Let w be the shortest word in the pair w_1, w_2. The inequality $n-1+s \leq 2(n-1) - s$ implies that $s \leq \frac{1}{2}(n-1)$, therefore $|w| \leq \frac{3}{2}(n-1)$.

Case 2: $\delta(n, u) = 1$.

The inclusion $1 \in Y = P^s(\{p\})$ implies that there is a word v of length at most s such that $\delta(1, v) = p$. Let $w = uv$. It is easy to see that $|w| \leq n - k - 2 + s$ and $\delta(q, w) = p$ for any $q \in Q$. Observe that $-k \leq 0$ and $s - k \leq |Y| - s \leq n - s$, therefore $|w| \leq \min\{n + s - 2, 2n - s - 2\} \leq \frac{3}{2}n - 2 < \frac{3}{2}(n - 1)$. □

Propositions 2 and 3 below describe the structure of monotonic automata. We use these statements to generalize Proposition 1 to the set of all synchronizing monotonic automata and to prove Theorem 1. We need some definitions to formulate these propositions.

Given a word $w \in \Sigma^*$ and non-empty subset $X \subseteq Q$, we write $X.w$ for the set $\{\delta(x, w) \mid x \in X\}$ Given an automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$, we define the *rank* of a word $w \in \Sigma^*$ with respect to $\mathscr{A}$ as the cardinality of the image of the transformation $\delta(\_\_, w)$ of the set Q, that is $|Q.w|$. (Thus, reset words are precisely words of rank 1.) Define the *rank* $r(\mathscr{A})$ of an automaton $\mathscr{A}$ as the minimum rank of words with respect to $\mathscr{A}$. (Thus, synchronizing automata are precisely automata of rank 1.)

A subset X of a set Q is said to be *invariant with respect to a transformation* $\varphi : Q \to Q$ if $X\varphi \subseteq X$. A subset of the state set of an automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is called *invariant* if it is invariant with respect to all the transformations $\delta(\_\_, a)$ with $a \in \Sigma$. If X is an invariant subset, we define the *restriction* of $\mathscr{A}$ to X as the automaton $\mathscr{A}_X = \langle X, \Sigma, \delta_X\rangle$ where δ_X is the restriction of the transition function δ to the set $X \times \Sigma$.

To prove Propositions 2 and 3 we use some constructions and arguments taken from the proof of theorem 1 in [8]. In particular, the next two lemmas coincide with respectively lemmas 1 and 2 in [8].

Lemma 3. *Let X be a non-empty subset of Q such that* $\max(X.w) \leq \max(X)$ *for some* $w \in \Sigma^*$. *Then for each* $p \in [\max(X.w), \max(X)]$ *there exists a word* $\mathcal{D}(X, w, p)$ *of length at most* $\max(X) - p$ *such that* $\max(X.\mathcal{D}(X, w, p)) \leq p$.

The dual statement is

Lemma 4. *Let X be a non-empty subset of Q such that* $\min(X.w) \geq \min(X)$ *for some* $w \in \Sigma^*$. *Then for each* $p \in [\min(X), \min(X.w)]$ *there exists a word* $\mathcal{U}(X, w, p)$ *of length at most* $p - \min(X)$ *such that* $\min(X.\mathcal{U}(X, w, p)) \geq p$.

For $x, y \in Q$ with $x \leq y$ we denote by $[x, y]$ the interval $\{x, x + 1, x + 2, \ldots, y\}$.

Proposition 2. *Let $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ be a synchronizing monotonic automaton. Then there is an invariant subset P of the set Q such that the restriction $\mathscr{A}_P$ is a strongly connected automaton, and there is a word $w_{\mathscr{A}}$ of length at most $|Q| - |P|$ such that $\delta(q, w_{\mathscr{A}}) \in P$ for any $q \in Q$.*

Proof. Let $P = \{q \mid \{q\} = Q.w$ for some $w \in \Sigma^*\}$. In other words, P is the set of all states to which the automaton $\mathscr{A}$ can be reset. Observe that the set P is invariant. Indeed, arguing by contradiction, suppose that there are $q \in P$ and $w \in \Sigma^*$ such that $\delta(q, w) \notin P$. By the definition of P there is a word u such that $Q.u = \{q\}$. Hence $Q.uw$ is an one-element set $\{\delta(q, w)\}$. It contradicts the definition of P. The automaton $\mathscr{A}_P$ is strongly connected because for any states $p, q \in P$ there is a word $w \in \Sigma^*$ such that $Q.w = \{q\}$, in particular, $\delta(p, w) = q$.

Consider the interval $I_1 = [\min(P), n]$. It is invariant. Indeed, arguing by contradiction, suppose that there are $q \in I$ and $w \in \Sigma^*$ such that $\delta(q, w) \notin I$. Since the transformation $\delta(\_\_, w)$ is order preserving, $\delta(\min(P), w) \leq \delta(q, w) < \min(P)$. That is $\delta(\min(P), w) \notin P$. It contradicts the fact that P is an invariant set. Similarly, we obtain the dual fact that the interval $I_2 = [1, \max(P)]$ is invariant. Therefore the intersection $I = I_1 \cap I_2$ is also invariant. We can apply Lemma 4 to the set I_1, the state 1 and a reset word $u \in \Sigma^*$. Let $w_1 = \mathcal{U}(I_1, u, 1)$; then the length of w_1 is at most $|Q| - |I_1|$ and $\delta(1, w_1) \in I_1$. Since the transformation $\delta(\_\_, w_1)$ is order preserving, it means that $Q.w_1 \subseteq I_1$. By symmetry, we can find a word w_2 of length at most $|Q| - |I_2|$ such that $Q.w_2 \subseteq I_2$. The concatenation w_1w_2 has the length at most $|Q| - |I|$. Since the interval $|I_1|$ is invariant, $Q.w_1w_2 \subseteq I_1 \cap I_2 = I$.

Now fix a reset word v such that $Q.v = \max(P)$. Let $|I \setminus P| = s$. We enumerate the elements $q_1, q_2, \ldots, q_s$ of the difference $I \setminus P$ so that $q_1 < q_2 < q_3 < \cdots < q_s$.

We prove that for each $k \in \{0, 1, \dots, s\}$ there is a word u_k of length at most k such that $I.u_k \subseteq \{q_{k+1}, q_{k+2}, \dots, q_s\} \cup P$. We induct on k with the obvious base $k = 0$ (u_0 is the empty word). Let $k > 0$. We find the last letter a in v such that $v = v_1 a v_2$, $\delta(\max(P), v_1) \le q_k$ and $\delta(\max(P), v_1 a) \ge q_k$. Observe that $\delta(\max(P), v_1 a) \ne q_k$, because the set P is invariant. Since the transformation $\delta(\_\_, a)$ is order preserving, it means that $\delta(q_i, a) > q_k$ for all $i \in \{k, k+1, \dots, s\}$. Let $u_k = u_{k-1}a$. By the induction assumption $I.u_{k-1} \subseteq \{q_k, q_{k+1}, \dots, q_s\} \cup P$. Hence $I.u_k \subseteq (\{q_k, q_{k+1}, \dots, q_s\} \cup P).a \subseteq \{q_{k+1}, q_{k+2}, \dots, q_s\} \cup P$ as required.

We obtain that $Q.w_1w_2u_s \subseteq I.u_s \subseteq P$ and the length of the word $w_1w_2u_s$ is at most $|Q| - |I| + |I \setminus P| = |Q| - |P|$. Thus, the word $w_1w_2u_s$ can be chosen to play the role of $w_{\mathscr{A}}$ from the formulation of the proposition. $\square$

Now we generalize Proposition 2 to arbitrary monotonic automaton.

Proposition 3. *Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a monotonic automaton of rank k. Then there are pairwise disjoint invariant subsets $P_1, P_2, \dots, P_k$ of the set Q such that all restrictions $\mathscr{A}_{P_i}$ for $i \in \{1, \dots, k\}$ are strongly connected automata, and there is a word $w_{\mathscr{A}}$ of length at most $|Q| - \left|\bigcup_{i=1}^{k} P_i\right|$ such that $\delta(q, w_{\mathscr{A}}) \in \bigcup_{i=1}^{k} P_i$ for any $q \in Q$.*

Proof. Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a monotonic automaton of rank k. We induct on k.

If $k = 1$, then the automaton $\mathscr{A}$ is synchronizing and we can apply Proposition 2.

Let $k > 1$. Consider the set $X = \{\min(Q.w) \mid w \in \Sigma^*,\ |Q.w| = k\}$. (This set is not empty because by the condition of the proposition there exists a word of rank k with respect to $\mathscr{A}$.)

Let $M = \max(X)$ and let $v \in \Sigma^*$ be such that $\min(Q.v) = M$ and $|Q.v| = k$. Observe that the interval $Y = [1, M]$ is invariant. Indeed, arguing by contradiction, suppose that there are $q \in Y$ and $w \in \Sigma^*$ such that $\delta(q, w) > M$. Since the transformation $\delta(\_\_, w)$ is order preserving, $\min(Q.vw) = \delta(M, w) \ge \delta(q, w) > M$. At the same time, $|Q.vw| \le |Q.v| \le k$ whence $\min(Q.vw)$ belongs to the set X (since the rank of $\mathscr{A}$ is k, the inequality $|Q.wv| \le k$ means that $|Q.wv| = k$). It contradicts the choice of M. Observe that the restriction $\mathscr{A}_Y$ is a synchronizing automaton. Indeed, there is a word $v \in \Sigma^*$ such that $\min(Q.v) = M$ (and $|Q.v| = k$) by the choice of M but the interval $Y = [1, M]$ is invariant whence $Y.v = \{M\}$.

Consider the set $Z = \{q \in Q \mid \delta(q, w) \le M \text{ for some } w \in \Sigma^*\}$. Observe that Z is an interval and that $Y \subseteq Z$ since for $q \in Y$ the empty word can serve as w with $\delta(q, w) \le M$. Therefore $\max(Z) \ge M$. Fix a word $u \in \Sigma^*$ such that $\delta(\max(Z), u) \le M$. Then $\delta(q, u) \le M$ for each $q \in Z$ as the transformation $\delta(\_\_, u)$ is order preserving.

Now consider the interval $T = [\max(Z) + 1, n] = Q \setminus Z$. It is invariant. Indeed, suppose that there exist $q \in T$ and $w \in \Sigma^*$ such that $\delta(q, w) \le \max(Z)$. This means that $\delta(q, wu) \le M$ whence $q \in Z$, in a contradiction to the choice of q.

Let v_1 be a word of minimal rank (k) with respect to the automaton $\mathscr{A}$, v_2 is a word of minimal rank with respect to the automaton $\mathscr{A}_T$, v_3 is a reset word for the automaton $\mathscr{A}_Y$. Denote $v_1v_2v_3$ by v. Then the word uv also has rank k with respect to $\mathscr{A}$, has a minimal rank with respect to $\mathscr{A}_T$ and the word v resets $\mathscr{A}_Y$. We have $Q.uv = Z.uv \cup T.uv$ and $Z.uv \subseteq Y.v$. Since $Y \subseteq Z$ and $|Y.v| = 1$ we obtain that $Z.uv = Y.v$. The sets Y and T are invariant, therefore $k = |Q.uv| = |Y.v| + |T.uv| = 1 + |T.uv|$. Hence the rank of the restriction $\mathscr{A}_T$ is $k - 1$.

By the induction assumption there are pairwise disjoint invariant subsets $P_1, P_2, \dots, P_{k-1}$ of the set T and an invariant subset P_k of the set Y such that all automata $\mathscr{A}_{P_i}$ for $i \in \{1, \dots, k\}$ are strongly connected. There is a word $w_1 = w_{\mathscr{A}_T}$ of length at most $|T| - \left|\bigcup_{i=1}^{k-1} P_i\right|$ such that $T.w_1 \subseteq \bigcup_{i=1}^{k-1} P_i$. There is a word $w_2 = w_{\mathscr{A}_Y}$ of length at most $|Y| - |P_k|$ such that $Y.w_2 \subseteq P_k$.

We apply Lemma 3 to the set Y, the state $\max(Z)$ and the word u that was fixed above. Let $w_3 = \mathcal{D}(Y, u, \max(Z))$; then the length of w_3 is at most $|Z| - |Y| = |Q| - |T| - |Y|$ and $Z.w_3 \subseteq Y$. Since T is an invariant set we obtain that $Q.w_3 \subseteq Y \cup T$.

We obtain that $Q.w_3w_1w_2 \subseteq (Y \cup T)w_1w_2 \subseteq \bigcup_{i=1}^{k} P_i$ the length of the word $w_3w_1w_2$ is at most $|Q| - \Big|\bigcup_{i=1}^{k} P_i\Big|$. Thus, the word $w_3w_1w_2$ can be chosen to play the role of $w_{\mathscr{A}}$ from the formulation of the proposition. □

3. PROOF OF THEOREM 1

Let $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ be a mortal partially monotonic automaton. Then any $\delta(\_\_, a)$ with $a \in \Sigma$ is an order preserving partial transformation on the set Q. Each order preserving partial transformation can be extended to the whole set Q so that the resulting total transformation remains order preserving. Do such completions (in an arbitrary way) for all transformations $\delta(\_\_, a)$ with $a \in \Sigma$. Then we obtain a monotonic automaton $\mathscr{B} = \langle Q, \Sigma, \zeta\rangle$.

We mark every element $q \in Q$ such that $\delta(q, a)$ is not defined for some letter $a \in \Sigma$. Observe that each invariant set of the automaton $\mathscr{B}$ contains a marked element. Indeed, in the opposite case we can find an invariant set I without marked elements. It means that the restriction $\mathscr{A}_I$ is a complete automaton, hence $\delta(q, w)$ is defined for any $q \in I$ and $w \in \Sigma^*$. It contradicts the assumption that the automaton $\mathscr{A}$ is mortal.

Let k be the rank of the automaton $\mathscr{B}$. We apply Proposition 3 to this automaton. We find pairwise disjoint invariant subsets $P_1, P_2, \dots, P_k$ of the set Q such that all automata $\mathscr{B}_{P_i}$ for $i \in \{1, \dots, k\}$ are strongly connected and a word $w_{\mathscr{B}}$ of length at most $|Q| - \Big|\bigcup_{i=1}^{k} P_i\Big|$ such that $\zeta(q, w_{\mathscr{B}}) \in \bigcup_{i=1}^{k} P_i$ for any $q \in Q$. Observe that all sets $P_1, P_2, \dots, P_k$ are "half-invariant" in the automaton $\mathscr{A}$ in the following sense: There are no elements $x \in P_i$ and $y \notin P_i$ such that $\delta(x, w) = y$ for some word $w \in \Sigma^*$.

For each $i \in \{1, \dots, k\}$ we apply Proposition 1 to the automaton $\mathscr{B}_{P_i}$ and a marked element $p_i \in P_i$. We find a word w_i of length at most $\frac{3}{2}(|P_i| - 1)$ such that $\zeta(q, w_i) = p_i$ for any $q \in P_i$. Since the state p_i is marked, there is a letter $a_i \in \Sigma$ such that $\delta(p_i, a_i)$ is not defined.

Consider the word $v = w_{\mathscr{B}}w_1a_1w_2a_2 \cdots w_ka_k$. For any $q \in Q$ we then have $\zeta(q, w_{\mathscr{B}}) \in P_i$ for some $i \in \{1, \dots, k\}$. Hence $\zeta(q, w_{\mathscr{B}}w_1a_1 \dots w_i) = p_i$. It means that either $\delta(q, w_{\mathscr{B}}w_1a_1 \dots w_i) = p_i$ or $\delta(q, w_{\mathscr{B}}w_1a_1 \dots w_i)$ is not defined. In both cases $\delta(q, w_{\mathscr{B}}w_1a_1 \dots w_ia_i)$ is not defined. Therefore $\delta(q, v)$ is not defined for any $q \in Q$. Thus, v is an annihilator word for the automaton $\mathscr{A}$.

The length of the word v is at most

$$|Q| - \Big|\bigcup_{i=1}^{k} P_i\Big| + \sum_{i=1}^{k}\left(\frac{3}{2}(|P_i| - 1) + 1\right) = |Q| - \Big|\bigcup_{i=1}^{k} P_i\Big| + \sum_{i=1}^{k}|P_i| + \sum_{i=1}^{k}\left(\frac{1}{2}(|P_i| - 1)\right)$$
$$= |Q| + \sum_{i=1}^{k}\left(\frac{1}{2}(|P_i|-1)\right) = |Q| - \frac{k}{2} + \frac{1}{2}\sum_{i=1}^{k}|P_i| \le |Q| - \frac{1}{2} + \frac{1}{2}|Q| = |Q| + \frac{1}{2}(|Q| - 1),$$

whence the length of the word v is at most $|Q| + \lfloor(|Q| - 1)/2\rfloor$ as required. The theorem is proved.

4. THE TIGHTNESS OF THE BOUND

We present a series of examples of partially monotonic automata $\mathscr{A}_n = \langle Q, \Sigma, \delta\rangle$, where $n = 6, 7, \dots,$ such that the automaton $\mathscr{A}_n$ is annihilated by a word of length $|Q| + \lfloor(|Q| - 1)/2\rfloor$ but is not annihilated by any shorter word. The state set Q_n of the automaton $\mathscr{A}_n$ is the chain $1 < 2 < 3 < \dots < n$. We denote $\lfloor(n + 1)/2\rfloor$ by ℓ. The input alphabet Σ of $\mathscr{A}_n$ contains two letters a and b.

The action of the letter a on the set Q_n is defined as follows:

$$\delta(j, a) = \begin{cases} 1 & \text{if } j = 1, \\ \ell & \text{if } j = \ell + 2, \\ \text{is not defined} & \text{if } j = \ell + 1, \\ j - 1 & \text{in all other cases.} \end{cases}$$

The action of the letter b is defined as follows:

$$\delta(j,b) = \begin{cases} n & \text{if } j = n, \\ j+1 & \text{in all other cases.} \end{cases}$$

The following figure shows the automaton $\mathscr{A}_9$.

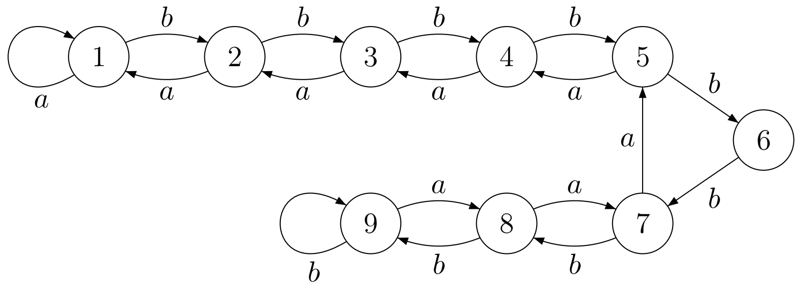

Figure.

It is easy to verify that the automata $\mathscr{A}_n$ are partially monotonic and that, for each n, the word $w_n = a^{n-2}b^{\ell}a$ annihilates the automaton $\mathscr{A}_n$.

Now we prove that the automaton $\mathscr{A}_n$ has no annihilator word of length less than $n+\ell-1$.

Theorem 2. *The length of any annihilator word for the automaton* $\mathscr{A}_n$ *is at least* $n+\ell-1$.

Proof. Let w be an annihilator word for $\mathscr{A}_n$, let u be the longest prefix of the word w with the property

$$Q.u \cap \{1,n\} \neq \varnothing,$$

and let $|Q.u| = k$. Let v be the suffix of the word w such that $w = uv$.

We denote by u_i and v_i the prefixes of the length i of the words u and v accordingly. It is easy to calculate the differences $|Q.u_{i-1}| - |Q.u_i|$ and $|Q.uv_{i-1}| - |Q.uv_i|$.

If $i = 1$ then $|Q.u_0| - |Q.u_1| = \begin{cases} 2 & \text{if } u_1 = a, \\ 1 & \text{if } u_1 = b. \end{cases}$

Let $i > 1$. The difference $|Q.u_{i-1}| - |Q.u_i|$ does not exceed 1 and $|Q.u_{i-1}| - |Q.u_i| = 0$ if the word u_i has a suffix ab.

Let $i > 0$. The difference $|Q.uv_{i-1}| - |Q.uv_i|$ does not exceed 1 and $|Q.uv_{i-1}| - |Q.uv_i| = 1$ only if the state $\ell+1$ is in the set $Q.uv_{i-1}$ and the last letter of the word v_i is a (in particular, the word uv_i has a suffix ba).

First consider the case $u = u_{k-1}a$. (The last letter of the word u is a.) It means that $n \notin Q.u$ and $1 \in Q.u$ by the choice of the word u. Using the calculation of the differences $|Q.u_{i-1}| - |Q.u_i|$, we obtain that the length of the word u is at least $n-k-1$.

The above observation concerning the value of the differences

$$|Q.uv_{i-1}| - |Q.uv_i|$$

with $|Q.u| = k$ implies that the word v contains at least k factors ba. But $1 \in Q.u$ and $\delta(q, ba) = q$ for any state $q \in \{1, 2, \ldots, \ell-1\}$, therefore the word v contains at least $\ell-1$ extra letters b. Hence the length of v is at least $2k+\ell-1 \geq k+\ell$. Therefore the length of $w = uv$ is at least $n-k-1+k+\ell = n+\ell-1$.

Now consider the case $u = u_{k-1}b$. (The last letter of the word u is b.) This means that $1 \notin Q.u$ and $n \in Q.u$ by the choice of the word u. Either the word u is a power of the letter b or u contains a letter a, and therefore, a factor ab. Using the above calculation of the differences $|Q.u_{i-1}| - |Q.u_i|$, we obtain in both cases that the length of the word u is at least $n-k$.

Let $\ell+1 \notin Q.u$. Again, the above observation concerning the value of the differences $|Q.uv_{i-1}| - |Q.uv_i|$ with $|Q.u| = k$ implies that the word v contains at least k factors ba. But $n \in Q.u$ and

$\delta(q, ba) = q$ for any state $q \in \{\ell+2, \ell+3, \ldots, n\}$, therefore the word v contains at least $n - \ell - 1$ extra letters a. Hence the length of v is at least

$$2k + n - \ell - 1 \geq 2k + \ell - 2 \geq k + \ell - 1.$$

Therefore the length of $w = uv$ is at least $n - k + k + \ell - 1 = n + \ell - 1$.

Let $\ell + 1 \in Q.u$. Since $n \in Q.u$ we have $k \geq 2$. Now the word v has at least $k - 1$ factors ba and at least $n - \ell - 1$ extra letters a. Hence the length of v is at least $2(k-1) + n - \ell - 1 \geq 2(k-1) + \ell - 2$.

If $k \geq 3$, then the length of v is at least $k + \ell - 1$ and the length of $w = uv$ is at least $n - k + k + \ell - 1 = n + \ell - 1$.

Let $k = 2$. It means that $Q.u = \{\ell + 1, n\}$. We already have proved that the length of v is at least ℓ. Our aim is to show that the length of the word u is at least $n - 1$. Then the length of $w = uv$ is at least $n - 1 + \ell$.

Let the word u have the word ba as a factor. If the first letter of u is a, then u contains at least two factors ab. If the first letter of u is b, then u contains at least one factor ab. Using the above calculation of the differences $|Q.u_{i-1}| - |Q.u_i|$, we obtain in both cases that the length of the word u is at least $n - 1$. and the length of $w = uv$ is at least $n - 1 + \ell$.

Suppose that the word u has no factors ba. This means that $u = a^s b^t$ for some non-negative integers s and t. Observe that $\delta(q, u)$ is defined for any $q \neq \ell + 1$. The assumption $\delta(1, a^s b^t) = n$ implies that $t = n - 1$ whence $Q.u = \{n\}$. This contradicts the assumption that $k = 2$. Therefore $\delta(1, b^t) = \delta(1, a^s b^t) = \ell + 1$ whence $t = \ell$. Suppose that $s \leq \ell - 2$. This means that $\delta(s + 2, u)$ is defined but $\delta(s + 2, u) = \delta(2, b^\ell) = \ell + 2$. This contradicts the assumption that $Q.u = \{\ell + 1, n\}$. Therefore, $s \geq \ell - 1$ whence $|u| = s + t \geq 2\ell - 1 \geq n - 1$. □

We point out that each automaton $\mathscr{A}_n$ is extreme in the class of all mortal partially monotonic automata with n states.

Using the series $\mathscr{A}_n$ we can obtain a series of examples of synchronizing aperiodic automata such that their shortest reset words have lengths exceeding the cardinalities of their state sets. Indeed, consider the 0-completion $\mathscr{A}_n^0$ of the incomplete automaton $\mathscr{A}_n$ for each $n = 6, 7, \ldots$. If we denote the number of states of the automaton $\mathscr{A}_n^0$ by s ($s = n + 1$), then we obtain that the automaton $\mathscr{A}_n$ is synchronized by a word of length $s + \lfloor s/2 \rfloor - 2 > s$ but is not synchronized by any shorter word. Thus, we have disproved a conjecture discussed in [4].

We do not know whether or not the automata $\mathscr{A}_n^0$ are extreme in the class of all synchronizing aperiodic automata.

REFERENCES

1. J. Černý, "Poznámka k Homogénnym Eksperimentom s Konečnými Automatami," Mat.-Fyz. Cas. Slovensk. Akad. Vied. **14** (3), 208–216 (1964).
2. I. Rystsov, "Reset Words for Commutative and Solvable Automata," Theoret. Comput. Sci. **172** (1–2), 273–279 (1997).
3. M. P. Schützenberger, "On Finite Monoids Having Only Trivial Subgroups," Inf. Control **8**, 190–194 (1965).
4. D. S. Ananichev and M. V. Volkov, "Synchronizing Generalized Monotonic Automata," Theoret. Comput. Sci. **330** (1), 3–13 (2005).
5. A. N. Trahtman, "The Černý Conjecture for Aperiodic Automata," Discrete Math. Theoret. Comput. Sci. **9** (2), 3–10 (2007).
6. M. V. Volkov, "Synchronizing Automata Preserving a Chain of Partial Orders," in: *Implementation and Application of Automata*, Ed. by J. Holub and J. Ždárek (Lect. Notes Comput. Sci., Springer, Berlin, 2007), Vol. 4783, pp. 27–37.
7. D. Eppstein, "Reset Sequences for Monotonic Automata," SIAM J. Comput. **19** (3), 500–510 (1990).
8. D. S. Ananichev and M. V. Volkov, "Synchronizing Monotonic Automata," Theoret. Comput. Sci. **327** (3), 225–239 (2004).

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