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MATHEMATICS

Stability of Periodic Points of a Diffeomorphism of a Plane in a Homoclinic Orbit

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Abstract—We considered a self-diffeomorphism of the plane with a fixed hyperbolic point at the origin and a non-transverse point homoclinic to it. Periodic points located in a sufficiently small neighborhood of the homoclinic point are divided into single-pass and multi-pass points depending on the location of the orbit of the periodic point with respect to the orbit of the homoclinic point. It follows from the works of W. Newhouse, L.P. Shil'nikov, B.F. Ivanov and other authors that for a certain method of tangency of the stable and unstable manifolds there can be an infinite set of stable periodic points in a neighborhood of a non-transverse homoclinic point, but at least one of the characteristic exponents of these points tends to zero with increasing period. Previous works of the author imply that for a different method of tangency of the stable and unstable manifolds there can be an infinite set of stable single-pass periodic points, the characteristic exponents of which are bounded away from zero in the neighborhood of a non-transverse homoclinic point. It is shown in this paper that under certain conditions imposed primarily on the method of tangency of the stable and unstable manifolds there can be a countable set of two-pass stable periodic points, the characteristic exponents of which are bounded away from zero in any neighborhood of a non-transverse homoclinic point.

Keywords: plane diffeomorphism, hyperbolic point, non-transverse homoclinic point, stability.

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In this paper, we have studied a C^1-self-diffeomorphism with a fixed hyperbolic (saddle) point at the origin. It is assumed that there is a non-transverse point homoclinic to it. It is known that there can be an infinite set of periodic points of the original diffeomorphism in an arbitrary neighborhood of a non-transverse homoclinic point. Periodic points located in a sufficiently small neighborhood of the homoclinic point are divided into single-pass and multi-pass points depending on the location of the orbit of the periodic point with respect to the orbit of the homoclinic point.

The main purpose of this paper is to show that there can be an infinite set of multi-pass (to be more specific, two-pass) stable periodic points, the characteristic exponents of which are bounded away from zero in an arbitrary neighborhood of a non-transverse homoclinic point. The structure of the neighborhood of a non-transverse homoclinic point was previously studied in the works of W. Newhouse, L.P. Shil'nikov, B.F. Ivanov, and other authors [1–4]. It follows from the listed works that under certain conditions imposed primarily on the method of tangency of the stable and unstable manifolds there can be an infinite set of stable periodic points in a neighborhood of a non-transverse homoclinic point, but at least one of the characteristic exponents of these points tends to zero with increasing period.

This work is a continuation of [5, 6]. In the referenced papers, diffeomorphisms with a fixed hyperbolic point at the origin and a non-transverse point homoclinic to it were considered. The method of tangency of the stable and unstable manifolds was slightly different than in [1–4]. It is shown that there can be an infinite set of stable single-pass periodic points, the characteristic exponents of which are bounded away from zero, in an arbitrary neighborhood of a non-transverse homoclinic point. An example of a diffeomorphism with such properties was given in [7].

Let f be a self-diffeomorphism with a hyperbolic fixed point at the origin, i.e., $f(0) = 0$. We assume that f is linear in some bounded neighborhood of V at the origin, to be more specific, if $(x, y) \in V$, then

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix}, \tag{1}$$

where

$$0 < \lambda < 1 < \mu.$$

Assume that

$$\lambda\mu^2 < 1. \tag{2}$$

It is clear that there is such $\overline{\gamma} > 2$ such that $\lambda\mu^{\overline{\gamma}} = 1$.

We denote the stable and unstable manifolds of zero point by $W^s(0)$, $W^u(0)$, as usual. It is known that *the stable and unstable manifolds* of the hyperbolic point of diffeomorphism f are defined as follows

$$W^s(0) = \left\{z \in \mathbb{R}^2 : \lim_{k\to+\infty} \left\|f^k(z)\right\| = 0\right\},$$

$$W^u(0) = \left\{z \in \mathbb{R}^2 : \lim_{k\to+\infty} \left\|f^{-k}(z)\right\| = 0\right\},$$

where f^k, f^{-k} are the degrees of diffeomorphisms f and f^{-1}.

It is assumed that *is a non-transverse homoclinic point*. Namely, it is assumed that there is a non-zero point w at the intersection of the stable and unstable manifolds, and this point is the tangent point of these manifolds.

The following equations follow from the definition of a homoclinic point

$$\lim_{k\to+\infty} \left\|f^k(w)\right\| = \lim_{k\to+\infty} \left\|f^{-k}(w)\right\| = 0.$$

Let w_1 and w_2 be two such points of the orbit of the homoclinic point w such that $w_1 \in V$, $w_2 \in V$, and their coordinates are $w_1 = (0, y^0)$, $w_2 = (x^0, 0)$. It follows from the definition of a homoclinic point that a natural number exists ω such that $f^{\omega}(w_1) = w_2$.

Assume that the following inclusion holds true for some $\overline{\lambda}$, $\overline{\mu}$ such that $\lambda < \overline{\lambda} < 1$, $1 < \overline{\mu} < \mu$

$$V_1 = \{(x, y) : |x| < \overline{\lambda}^{-1}\left|x^0\right|, |y| < \overline{\mu}\left|y^0\right|\} \subset V. \tag{3}$$

Assume that

$$x^0 > 0, \quad y^0 > 0. \tag{4}$$

Let

$$U = \{(x, y) : |x| < \beta, \left|y - y^0\right| < \beta\}$$

be such a neighborhood of the point w_1 that $U \subset V_1, f^{\omega}(U) \subset V_1, f(U) \cap V_1 = \emptyset$, $f^{\omega-1}(U) \cap V_1 = \emptyset$ and the sets $U, f(U), \ldots, f^{\omega}(U)$ do not intersect pairwise. Let us denote the restriction $f^{\omega}|_U$ by L. It is clear that the L is a mapping of the class C^1, and the matrix $DL(0)$ is nondegenerate.

A periodic point of the original diffeomorphism $u \in U$ is called a *single-pass* periodic point if a natural number exists j such that $f^jL(u) = u$, and $(\omega + j)$ is the smallest period, and for any $k = 1, 2, \ldots, j - 1$ $f^kL(u) \in V_1$.

The periodic point $u \in U$ of the original diffeomorphism is called the *s-pass or multi-pass* periodic point $(s > 1)$ if there are natural numbers $j_1, j_2, \ldots, j_s$ such that $f^{j_s}L\ldots f^{j_2}Lf^{j_1}L(u) = u$, and $(s\omega + j_1 + j_2 + \ldots + j_s)$ is the smallest period, and for any $k = 1, 2, \ldots, s - 1$ we have $f^{j_k}L\ldots f^{j_2}Lf^{j_1}L(u) \in U$, and for any $k = 1, 2, \ldots, s$, $l = 1, 2, \ldots, j_k - f^lLf^{j_{k-1}}L\ldots f^{j_1}L(u) \in V_1$.

In this paper, we study multi-pass periodic points with $s = 2$. Such points are called two-pass points.

Let us write the mapping L in the following coordinates:

$$L(x, y) = \begin{pmatrix} x^0 + F_1(x, y - y^0) \\ F_2(x, y - y^0) \end{pmatrix},$$

where $F_1(x, y - y^0)$, $F_2(x, y - y^0)$ are C^1 functions defined in U. It is clear that $F_1(0, 0) = F_2(0, 0) = 0$.

In [1–4], a neighborhood of the homoclinic point was studied, and it was assumed that there is $r > 1$ such that

$$\frac{\partial F_2(0,0)}{\partial y} = \ldots = \frac{\partial^{r-1} F_2(0,0)}{\partial y^{r-1}} = 0, \quad \frac{\partial^r F_2(0,0)}{\partial y^r} \neq 0. \tag{5}$$

It follows from the specified works that if conditions (5), which determine the method of tangency of the stable and unstable manifolds, are fulfilled, there can be an infinite set of stable periodic points in a neighborhood of the homoclinic point, but at least one of the characteristic exponents at these points tends to zero with increasing period. In [5, 6], it was shown that for a different method of tangency of the stable and unstable manifolds there can be an infinite set of single-pass stable periodic points, characteristic exponents of which are bounded away from zero, in the neighborhood of the homoclinic point. The problem of the stability of multi-pass periodic points was not considered in these papers. The main purpose of this work is to show that if conditions (5) are not fulfilled, there can be an infinite set of two-pass stable periodic points, characteristic exponents of which are bounded away from zero, in an arbitrary neighborhood of the homoclinic point.

Assume that the coordinate functions of the mapping L are as follows

$$\begin{gathered} F_1(x, y - y^0) = ax + b(y - y^0) + \varphi_1(x, y - y^0), \\ F_2(x, y - y^0) = cx + g(y - y^0) + \varphi_2(x), \end{gathered} \tag{6}$$

where a, b, and c are the real numbers such that

$$b < 0, \quad c > 0, \tag{7}$$

and g, φ_1, φ_2 are continuously differentiable functions of one or two variables defined in a neighborhood of the origin and equal to zero along with their first-order derivatives at the origin. Assume that the first order derivatives of these functions are bounded by $\frac{1}{2}$ in the neighborhood of U.

The nature of tangency of the stable and unstable manifolds at the point w_2 is determined by the properties of the function g. Let us describe the properties of this function using sequences. Let σ_k, ε_k be positive vanishing sequences. The sequence σ_k decreases. Let Δ_k be a negative zero sequence.

Assume that

$$\sigma_k - \varepsilon_k > \sigma_{k+1} + \varepsilon_{k+1} \tag{8}$$

for any k.

Let γ and p be positive constants independent of k with $1 \le \gamma < \overline{\gamma}$. Let m_k be a strictly increasing sequence of positive integers and n_k be a sequence of positive integers such that

$$n_k = \gamma_k m_k + p_k, \tag{9}$$

where $1 \le \gamma_k \le \gamma$, $|p_k| \le p$.

Assume that the following inequalities are true for any k

$$10x^0 \max[\lambda^{n_k}\mu^{(n_k+m_k)}, \lambda^{m_k}\mu^{n_k}] \le \varepsilon_k. \tag{10}$$

Denote

$$x_k = [\lambda^{n_k}(x^0 + b\Delta_k) + \lambda^{(m_k+n_k)}a(x^0 + b\sigma_k)](1 - a^2\lambda^{(m_k+n_k)})^{-1},$$

$$\overline{x}_k = [\lambda^{m_k}(x^0 + b\sigma_k) + \lambda^{(m_k+n_k)}a(x^0 + b\Delta_k)](1 - a^2\lambda^{(m_k+n_k)})^{-1}.$$

Let the following inequalities hold for any k

$$\begin{gathered} \left|\mu^{m_k}(cx_k + g(\sigma_k)) - (y^0 + \Delta_k)\right| < 0.1\varepsilon_k\mu^{-n_k}, \\ \left|\mu^{n_k}(c\overline{x}_k + g(\Delta_k)) - (y^0 + \sigma_k)\right| < 0.1\varepsilon_k. \end{gathered} \tag{11}$$

Assume that there is a constant $\alpha > 1 + 0.5(\overline{\gamma} - \gamma)$ such that the following inequality holds for any k and $t \in (\sigma_k - \varepsilon_k, \sigma_k + \varepsilon_k)$

$$\left|\frac{dg(t)}{dt}\right| < \mu^{-\alpha(m_k+n_k)}. \tag{12}$$

If the function g satisfies inequalities (11), (12), then conditions (5) are not fulfilled.

It follows from (11), (12) that

$$\lim_{k\to+\infty}\mu^{-m_k}(\sigma_k)^{-1}=\lim_{k\to+\infty}\mu^{-n_k}(\Delta_k)^{-1}=0.$$

Define $\delta_k = 10\lambda^{n_k}|\Delta_k|$.

Assume that

$$B_k=\{(x,y):|x-x_k|<\delta_k,\left|y-(y^0+\sigma_k)\right|<\varepsilon_k\}.$$

It is clear that we will have $B_k\subset U$ for sufficiently large numbers of k.

Lemma 1. *Let conditions* (1)–(4), (6)–(12) *be fulfilled, then there is a number k_0 such that the following inclusions hold true for $k>k_0$*

$$f^{n_k}Lf^{m_k}L(\mathrm{cl}(B_k))\subset B_k, \tag{13}$$

where $\mathrm{cl}(B_k)$ *is the closure* B_k.

Proof. Assume that $(x, y)\in \mathrm{cl}(B_k)$, it is clear that $x=x_k+u$, $y=y^0+\sigma_k+v$, where $|u|\le\delta_k$, $|v|\le\varepsilon_k$.

Denote

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=f^{m_k}L\begin{pmatrix}x\\ y\end{pmatrix}$$

it is clear that

$$\bar{x}=\bar{x}_k+\lambda^{m_k}[au+bv+\varphi_1(x_k+u,\sigma_k+v)],$$

$$\bar{y}=\mu^{m_k}(cx_k+g(\sigma_k))+\mu^{m_k}[cu+g(\sigma_k+v)-g(\sigma_k)+\varphi_2(x_k+u)].$$

We can apply the mean value theorem to the functions g, φ_1, and φ_2 for any fixed k, and as a result we obtain the following taking into account (12)

$$|\varphi_1(x_k+u,\sigma_k+v)|\le 0.5(|x_k|+|u|+|\sigma_k|+|v|)<(\lambda^{n_k}x^0+\delta_k+\sigma_k+\varepsilon_k),$$

$$|\varphi_2(x_k+u)|\le 0.5(|x_k|+|u|)<(\lambda^{n_k}x^0+\delta_k),$$

$$|g(\sigma_k+v)-g(\sigma_k)|\le\varepsilon_k\mu^{-\alpha(m_k+n_k)}.$$

Denote

$$\bar{\delta}_k=\lambda^{m_k}[(|a|+1)\delta_k+(|b|+1)\varepsilon_k+\lambda^{n_k}x^0+\sigma_k],$$

$$\bar{\varepsilon}_k=0.1\varepsilon_k\mu^{-n_k}+\mu^{m_k}[(|c|+1)\delta_k+\varepsilon_k\mu^{-\alpha(m_k+n_k)}+\lambda^{n_k}x^0].$$

Define the following sequence of sets

$$\bar{B}_k=\{(x,y):|x-\bar{x}_k|<\bar{\delta}_k,\left|y-(y^0+\Delta_k)\right|<\bar{\varepsilon}_k\}. \tag{14}$$

It is clear that we will have $\bar{B}_k\subset U$ for sufficiently large numbers of k.

It is evident that

$$|\bar{x}-\bar{x}_k|<\bar{\delta}_k,$$
$$\left|\bar{y}-(y^0+\Delta_k)\right|<\bar{\varepsilon}_k,$$

consequently $f^{m_k}L(\mathrm{cl}(B_k))\subset\bar{B}_k$.

Let $(x, y)\in\bar{B}_k$. It is clear that $x=\bar{x}_k+\bar{u}$, $y=y^0+\Delta_k+\bar{v}$, where $|\bar{u}|<\bar{\delta}_k$, $|\bar{v}|<\bar{\varepsilon}_k$.

Denote

$$\begin{pmatrix}\bar{\bar{x}}\\ \bar{\bar{y}}\end{pmatrix}=f^{n_k}L\begin{pmatrix}x\\ y\end{pmatrix},$$

it is clear that

$$\bar{\bar{x}} = x_k + \lambda^{n_k}[a\bar{u} + b\bar{v} + \varphi_1(\bar{x}_k + \bar{u}, \Delta_k + \bar{v})],$$

$$\bar{\bar{y}} = \mu^{n_k}(c\bar{x}_k + g(\Delta_k)) + \mu^{n_k}[c\bar{u} + g(\Delta_k + \bar{v}) - g(\Delta_k) + \varphi_2(\bar{x}_k + \bar{u})].$$

We obtain the following from the last equalities taking into account conditions (9)–(12)

$$|\bar{\bar{x}} - x_k| < \lambda^{n_k}[(|a| + 1)\bar{\delta}_k + (|b| + 1)\bar{\varepsilon}_k + \lambda^{m_k}x^0 + |\Delta_k|] \le \delta_k,$$

$$|\bar{\bar{y}} - (y^0 + \sigma_k)| < 0.1\varepsilon_k + \mu^{n_k}[(|c| + 1)\bar{\delta}_k + \bar{\varepsilon}_k + \lambda^{m_k}x^0] \le \varepsilon_k.$$

The last inequalities hold for sufficiently large numbers k. Inclusions (13) are proved. The lemma is proved.

It is evident that for any k $B_k \cap \bar{B}_k = \emptyset$, where $\bar{B}_k$ is defined in (14).

Theorem 1. *Let f be a self-diffeomorphism with a fixed hyperbolic point at the origin and a non-transverse homoclinic to it. Let conditions* (1)–(4), (6)–(12) *be fulfilled, then there is a countable set of two-pass stable periodic points, the characteristic exponents of which are bounded away from zero, in any neighborhood of the homoclinic point* w_1.

PROOF. It follows from inclusions (13) that there is a fixed point u_k of the mapping $f^{n_k}Lf^{m_k}L$, which is a periodic point of the diffeomorphism f, in the set B_k in the case of sufficiently large numbers k.

Let $\bar{u}_k = f^{m_k}L(u_k)$. It is clear that $\bar{u}_k \in \bar{B}_k$.

Denote

$$\Theta_k = D(f^{n_k}Lf^{m_k}L(u_k)) = Df^{n_k}L(\bar{u}_k)Df^{m_k}L(u_k).$$

Next, $\mathrm{Det}\Theta_k$ is the determinant of the matrix Θ_k, and $\mathrm{Tr}\Theta_k$ is its trace.

It is obvious that

$$\Theta_k = \begin{pmatrix} \lambda^{n_k}a_2(k) & \lambda^{n_k}b_2(k) \\ \mu^{n_k}c_2(k) & \mu^{n_k}g_2(k) \end{pmatrix} \begin{pmatrix} \lambda^{m_k}a_1(k) & \lambda^{m_k}b_1(k) \\ \mu^{m_k}c_1(k) & \mu^{m_k}g_1(k) \end{pmatrix},$$

where $\lim\limits_{k\to+\infty} a_i(k) = a$, $\lim\limits_{k\to+\infty} b_i(k) = b$, $\lim\limits_{k\to+\infty} c_i(k) = c$, $\lim\limits_{k\to+\infty} g_i(k) = 0$, $i = 1, 2$, $|g_1(k)| < \mu^{-\alpha(m_k+n_k)}$.

It is evident that

$$\mathrm{Det}\,\Theta_k = (\lambda\mu)^{(m_k+n_k)}(a_2(k)g_2(k) - b_2(k)c_2(k))(a_1(k)g_1(k) - b_1(k)c_1(k)),$$

$$\mathrm{Tr}\,\Theta_k = \lambda^{(m_k+n_k)}a_1(k)a_2(k) + \lambda^{m_k}\mu^{n_k}b_1(k)c_2(k) + \lambda^{n_k}\mu^{m_k}b_2(k)c_1(k) + \mu^{(m_k+n_k)}g_1(k)g_2(k),$$

or we obtain the following taking into account conditions (12)

$$\mathrm{Det}\,\Theta_k = (\lambda\mu)^{(m_k+n_k)}(bc)^2(1 + \theta_k),$$

$$\mathrm{Tr}\,\Theta_k = \lambda^{m_k}\mu^{n_k}bc[1 + (\lambda\mu^{-1})^{(n_k-m_k)} + \eta_k],$$

where $\lim\limits_{k\to+\infty} \theta_i = \lim\limits_{k\to+\infty} \eta_k = 0$.

Let $\rho_i(k)$, $i = 1, 2$, be the eigenvalues of the matrix Θ_k.

It is obvious that

$$\rho_i(k) = 0.5\,\mathrm{Tr}\,\Theta_k \mp 0.5((\mathrm{Tr}\,\Theta_k)^2 - 4\,\mathrm{Det}\,\Theta_k)^{0.5}.$$

It is evident that

$$|\rho_1(k)| = 0.5\lambda^{m_k}\mu^{n_k}|bc|\left|1 + (\lambda\mu^{-1})^{(n_k-m_k)} + \eta_k + [(1 - (\lambda\mu^{-1})^{(n_k-m_k)})^2 + \psi_k]^{0.5}\right|,$$

where $\lim\limits_{k\to+\infty} \psi_k = 0$.

It is obvious that

$$|\rho_1(k)| = \lambda^{m_k}\mu^{n_k}|bc|q_k,$$

where q_k are determined by the last relations. From conditions (9) it follows that q_k are positive, bounded, and bounded away from zero. It is evident that

$$|\rho_2(k)| = |\mathrm{Det}\,\Theta_k||\rho_1(k)|^{-1},$$

from which

$$|\rho_2(k)| = \lambda^{n_k}\mu^{m_k}|bc|Q_k,$$

where Q_k are positive, bounded, and bounded away from zero.

It is known that the characteristic exponents $\nu_i(k)$, $i = 1, 2$, of the points u_k are defined as follows

$$\nu_i(k) = (m_k + n_k + 2\omega)^{-1}\ln|\rho_i(k)|.$$

As a result, the following inequalities hold for sufficiently large numbers of k taking into account conditions (2), (9):

$$\nu_1(k) \le -0.25(\overline{\gamma} - \gamma)(1 + \gamma)^{-1}\ln\mu,$$
$$\nu_2(k) \le 0.25(1 + \gamma)^{-1}\ln(\lambda\mu).$$

The last inequalities prove the theorem.

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