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MATHEMATICS

The Strong Continuity of Convex Functions

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Abstract—A convex function defined on an open convex set of a finite-dimensional space is known to be continuous at every point of this set. In fact, a convex function has a strengthened continuity property. The notion of strong continuity is introduced in this study to show that a convex function has this property. The proof is based on only the definition of convexity and Jensen's inequality. The definition of strong continuity involves a constant (the constant of strong continuity). An unimprovable value of this constant is given in the case of convex functions. The constant of strong continuity depends, in particular, on the form of a norm introduced in the space of arguments of a convex function. The polyhedral norm is of particular interest. It is straightforward to calculate the constant of strong continuity when it is used. This requires a finite number of values of the convex function.

Keywords: convex function, strong continuity, constant of strong continuity.

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1. We consider the space $\mathbb{R}^n$ with unit basis vectors $e_1, \dots, e_n$ and an arbitrary norm $\|\cdot\|$. We introduce the notation

$$B(x_0;\beta) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \le \beta\}.$$

Definition. A function $f(x)$ defined on an open set $U \subset \mathbb{R}^n$ is said to be *strongly continuous* at a point $x_0 \in \mathbb{R}^n$ if there is a number $\beta > 0$ and a constant $L \ge 0$ such that $B(x_0; \beta) \subset U$ and

$$|f(x) - f(x_0)| \le L\|x - x_0\| \quad \forall x \in B(x_0;\beta). \tag{1}$$

We show that a convex function has the property of strong continuity.

Due to the equivalence of all norms in the space $\mathbb{R}^n$, we prove inequality (1) for any one norm. In this section, we select the ℓ_1-norm $\|\cdot\|_1$. We introduce the notation

$$B_1(x_0;\beta) = \{x \in \mathbb{R}^n \mid \|x - x_0\|_1 \le \beta\}.$$

Theorem 1. *Let $U \subset \mathbb{R}^n$ be an open convex set and let $f(x)$ be a convex function on U. We take a point $x_0 \in U$ and an arbitrary number $\beta > 0$ such that $x_0 \pm \beta e_k \in U$ for $k \in 1:n$. Then, $B_1(x_0; \beta) \subset U$ and for all $x \in B_1(x_0; \beta)$ the inequality*

$$|f(x) - f(x_0)| \le L\|x - x_0\|_1 \tag{2}$$

holds, where

$$L = \max_{k \in 1:n} \left\{ \frac{f(x_0 \pm \beta e_k) - f(x_0)}{\beta} \right\}. \tag{3}$$

The constant L is unimprovable.

Proof. Let $h_k = \beta e_k$. Any vector $x \in \mathbb{R}^n$ can be represented in the form

$$x = x_0 + \sum_{k=1}^{n} w_k h_k. \tag{4}$$

We find u_k and v_k from the conditions

$$w_k = u_k - v_k,$$
$$|w_k| = u_k + v_k.$$

We obtain

$$u_k = \frac{1}{2}(|w_k| + w_k), \quad v_k = \frac{1}{2}(|w_k| - w_k).$$

It is obvious that $u_k \geq 0$ and $v_k \geq 0$. We have

$$|(x - x_0)_k| = \beta|w_k| = \beta(u_k + v_k).$$

This implies

$$\|x - x_0\|_1 = \beta\sum_{k=1}^{n}(u_k + v_k). \tag{5}$$

We rewrite formula (4) in the form

$$x - x_0 = \sum_{k=1}^{n} u_k h_k + \sum_{k=1}^{n} v_k(-h_k).$$

We introduce the notation $u_{n+k} = v_k$ and $h_{n+k} = -h_k$ to find

$$x - x_0 = \sum_{k=1}^{2n} u_k h_k, \tag{6}$$

where all coefficients u_k are nonnegative.

We fix a point $x \in B_1(x_0; \beta)$. According to (5), the relation

$$\sum_{k=1}^{2n} u_k = \sum_{k=1}^{n}(u_k + v_k) \leq 1 \tag{7}$$

holds true. Representation (6) yields the equality

$$x = \left(1 - \sum_{k=1}^{2n} u_k\right)x_0 + \sum_{k=1}^{2n} u_k(x_0 + h_k).$$

All coefficients in this representation are nonnegative and their sum is 1. By the assumptions of the theorem, the points $x_0, x_0 + h_1, \ldots, x_0 + h_{2n}$ belong to the convex set U. Therefore, $x \in U$. Thus, we have established the inclusion $B_1(x_0; \beta) \subset U$.

Recall that the function f is convex on U. By Jensen's inequality, we can write

$$f(x) \leq \left(1 - \sum_{k=1}^{2n} u_k\right)f(x_0) + \sum_{k=1}^{2n} u_k f(x_0 + h_k)$$

or

$$f(x) - f(x_0) \leq \sum_{k=1}^{2n} u_k[f(x_0 + h_k) - f(x_0)] \leq L\beta\sum_{k=1}^{2n} u_k,$$

where the constant L is defined by (3). In view of (5), we arrive at

$$\beta\sum_{k=1}^{2n} u_k = \beta\sum_{k=1}^{n}(u_k + v_k) = \|x - x_0\|_1. \tag{8}$$

Thus, we have

$$f(x) - f(x_0) \le L\|x - x_0\|_1. \tag{9}$$

We now estimate the difference $f(x_0) - f(x)$. We introduce the point $y = 2x_0 - x$. We have

$$y - x_0 = x_0 - x = \sum_{k=1}^{n} u_k(-h_k) + \sum_{k=1}^{n} v_k h_k.$$

We set $v_{n+k} = u_k$ and $h_{n+k} = -h_k$ to write

$$y - x_0 = \sum_{k=1}^{2n} v_k h_k.$$

According to (7), we have

$$\sum_{k=1}^{2n} v_k = \sum_{k=1}^{n} (v_k + u_k) \le 1.$$

As before, we use the representation

$$y = \left(1 - \sum_{k=1}^{2n} v_k\right) x_0 + \sum_{k=1}^{2n} v_k (x_0 + h_k).$$

Obviously, $y \in U$. By Jensen's inequality, the inequality

$$f(y) - f(x_0) \le \sum_{k=1}^{2n} v_k [f(x_0 + h_k) - f(x_0)] \le L\beta \sum_{k=1}^{2n} v_k$$

is valid. Due to (8), the equality

$$\beta \sum_{k=1}^{2n} v_k = \beta \sum_{k=1}^{n} (v_k + u_k) = \|x - x_0\|_1,$$

holds, which yields

$$f(y) - f(x_0) \le L\|x - x_0\|_1. \tag{10}$$

We now note that $x_0 = \frac{1}{2}(y + x)$. Owing to the convexity of the function f, we have

$$f(x_0) \le \frac{1}{2}[f(y) + f(x)]$$

or

$$f(y) \ge 2f(x_0) - f(x). \tag{11}$$

Combining (10) and (11), we arrive at the inequality

$$f(x_0) - f(x) \le L\|x - x_0\|_1. \tag{12}$$

Inequalities (9) and (12) imply (2).

We give an example when inequality (2) with a constant L of form (3) is satisfied as the equality. Exactly in this sense the fact that the constant L is unimprovable is understood..

We take the convex function $f(x) = \|x\|_1$ on $U = \mathbb{R}^n$ and calculate L with $x_0 = \mathbb{O}$ and an arbitrary $\beta > 0$. This yields $L = 1$. In this case, inequality (2) is satisfied as the equality for all $x \in \mathbb{R}^n$.

Theorem 1 is thus proved.

Theorem 1 guarantees, in particular, the ordinary continuity of a convex function $f(x)$ on an open convex set $U \subset \mathbb{R}^n$.

2. As noted above, the fact that inequality (1) holds for the ℓ_1-norm implies that it holds for an arbitrary norm in $\mathbb{R}^n$. However, the constant L is rough with this approach. We give a simple independent proof of inequality (1) for an arbitrary norm $\|\cdot\|$ with an unimprovable constant L.

Theorem 2. *Let $U \subset \mathbb{R}^n$ be an open convex set and let $f(x)$ be a convex function on U. We take a point $x_0 \in U$ and an arbitrary $\beta > 0$ such that $B(x_0; \beta) \subset U$. Then inequality* (1) *holds, where*

$$L = \max_{\|g\|=1} \left\{ \frac{f(x_0 + \beta g) - f(x_0)}{\beta} \right\}. \tag{13}$$

The constant L is unimprovable.

Proof. We first note that the maximum in (13) is attainable. This follows from the ordinary continuity of the convex function $f(x)$ on the open convex set U and the fact that the argument $x = x_0 + \beta g$ of the function f when $\|g\| = 1$ runs over the compact sphere

$$S(x_0;\beta) = \{x \in \mathbb{R}^n \,|\, \|x - x_0\| = \beta\},$$

which is contained in the set U by the assumptions of the theorem.

We now turn to proving inequality (1). We fix $x \in B(x_0; \beta)$, $x \neq x_0$, and put $g = \dfrac{x - x_0}{\|x - x_0\|}$. It is evident that $\|g\| = 1$. We introduce the function

$$\varphi(t) = f(x_0 + t\beta g).$$

This function is convex on the interval $[-1, 1]$.

The point $x \in B(x_0; \beta)$, $x \neq x_0$, can be represented in the form

$$x = x_0 + \beta \frac{\|x - x_0\|}{\beta} \frac{x - x_0}{\|x - x_0\|}.$$

We introduce the notation $\alpha = \|x - x_0\|/\beta$. Then, $x = x_0 + \alpha\beta g$, where $\alpha \in (0, 1]$.

Due to the convexity of the function φ, we have

$$\begin{aligned} f(x) &= f(x_0 + \alpha\beta g) = \varphi(\alpha) = \varphi(\alpha \cdot 1 + (1 - \alpha) \cdot 0) \\ &\le \alpha\varphi(1) + (1 - \alpha)\varphi(0) = \alpha f(x_0 + \beta g) + (1 - \alpha) f(x_0). \end{aligned}$$

It follows that

$$\begin{aligned} f(x) - f(x_0) &\le \alpha[f(x_0 + \beta g) - f(x_0)] \\ &= \|x - x_0\| \frac{f(x_0 + \beta g) - f(x_0)}{\beta} \le L\|x - x_0\|, \end{aligned} \tag{14}$$

where the constant L is defined by formula (13).

Now we take the point $y = x_0 - \alpha\beta g$. We have

$$\begin{aligned} f(y) &= \varphi(-\alpha) = \varphi(\alpha \cdot (-1) + (1 - \alpha) \cdot 0) \le \alpha\varphi(-1) + (1 - \alpha)\varphi(0) \\ &= \alpha f(x_0 + \beta(-g)) + (1 - \alpha) f(x_0). \end{aligned}$$

It follows that

$$\begin{aligned} f(y) - f(x_0) &\le \alpha[f(x_0 + \beta(-g)) - f(x_0)] \\ &= \|x - x_0\| \frac{f(x_0 + \beta(-g)) - f(x_0)}{\beta} \le L\|x - x_0\|. \end{aligned} \tag{15}$$

Further, since $x_0 = \frac{1}{2}(x + y)$, we have $f(x_0) \le \frac{1}{2}[f(x) + f(y)]$ or

$$f(y) \ge 2f(x_0) - f(x). \tag{16}$$

Combining (15) and (16), we arrive at the inequality

$$f(x_0) - f(x) \le L\|x - x_0\|. \tag{17}$$

To derive the required inequality (1), we combine (14) and (17).

We give an example when inequality (1) with a constant L of form (13) is satisfied as the equality. We take the convex function $f(x) = \|x\|$ on $U = \mathbb{R}^n$ and calculate the constant L for it with $x_0 = \mathbb{0}$ and an arbitrary $\beta > 0$. We have $L = 1$. In this case, inequality (1) is satisfied as the equality for all $x \in \mathbb{R}^n$. This gives us reason to regard the constant L as being unimprovable.

Theorem 2 is proved.

3. Theorem 1 is of a constructive nature: we can find an appropriate number $\beta > 0$ in a finite number of attempts; to calculate a constant L of form (3), a finite number of values of the function $f(x)$ is needed. Only the definition of a convex function is used when proving the theorem. The ordinary continuity of a convex function on an open convex set is a corollary of the theorem.

Theorem 2 establishes the strong continuity of a convex function with a sharp constant L in the case when the space $\mathbb{R}^n$ is endowed with an arbitrary norm. The proof uses the ordinary continuity of a convex function. Calculating a constant L of form (13) is related to solving the maximization problem for a convex function on the unit sphere of the space $\mathbb{R}^n$.

The problem of whether inequality (1) holds in the case of the Euclidean norm was previously studied in [1–3] without analyzing the sharpness of the constant L.

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