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MATHEMATICS

Stable Periodic Solutions of Periodic Systems of Differential Equations

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Abstract—An infinitely differentiable periodic two-dimensional system of differential equations is considered. It is assumed that there is a hyperbolic periodic solution and there exists a homoclinic solution to the periodic solution. It is shown that, for a certain type of tangency of the stable manifold and unstable manifold, any neighborhood of the nontransversal homoclinic solution contains a countable set of stable periodic solutions such that their characteristic exponents are separated from zero.

Keywords: nontransversal homoclinic solution, stability, characteristic exponents.
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We select a class of infinitely smooth two-dimensional periodic systems of differential equations such that each one has an infinite set of stable periodic solutions in any neighborhood of the nontransversal homoclinic solution such that their characteristic exponents are separated from zero. In [1, 2], diffeomorphisms are studied such that each one has a fixed periodic point and the point homoclinic to it is nontransversal; the conditions of existence in a neighborhood of the homoclinic point are obtained for an infinite set of stable periodic points such that their characteristic exponents are separated from zero.

The aim of the present paper is to find a class of periodic systems such that their Poincaré transforms satisfy the conditions of the theorems of [1, 2].

Consider a system of the kind

$$\frac{dz}{dt} = Z(t, z), \tag{1}$$

where z and Z are two-dimensional vectors and the vector $Z(t, z)$ is infinitely differentiable with respect to all variables and periodic with respect to t with period one, i.e., $Z(t + 1, z) = Z(t, z)$.

Let $z(t, z_0)$ denote the solution with the initial data $t = 0$, $z = z_0$. Assume that $z(t, 0)$ is a hyperbolic periodic solution with period one, and λ and μ are multipliers of that solution. Assume the validity of the inequalities

$$0 < \lambda < 1 < \mu, \quad \lambda\mu < 1. \tag{2}$$

Define the sets

$$W^s(0) = \left\{z_0 \in \mathbb{R}^2 : \lim_{t\to+\infty} \|z(t, z_0) - z(t, 0)\| = 0\right\}$$

and

$$W^u(0) = \left\{z_0 \in \mathbb{R}^2 : \lim_{t\to-\infty} \|z(t, z_0) - z(t, 0)\| = 0\right\}.$$

It is clear that those sets lie in the stable and unstable (respectively) manifolds and, by virtue of conditions (2), they contain nonzero points. The stable and unstable manifolds are defined as follows:

$$W^s(t) = \{(t, z) : z = z(t, z_0), z_0 \in W^s(0)\}$$

and

$$W^u(t) = \{(t, z) : z = z(t, z_0), z_0 \in W^u(0)\}.$$

Let $w \in W^s(0) \cap W^u(0)$, $w \neq 0$. Then the solution $z(t, w)$ of system (1) is said to be *homoclinic* to the solution $z(t, 0)$. It is clear that the following relation holds:

$$\lim_{t\to+\infty} \|z(t,w) - z(t,0)\| = \lim_{t\to-\infty} \|z(t,w) - z(t,0)\| = 0.$$

We say that a homoclinic solution is *transversal* if the stable manifold and unstable manifold transversally intersect each other at points of that solution; otherwise, we say that it is *nontransversal*.

From [3], it is known that infinitely many periodic solutions exist in a neighborhood of any transversal homoclinic solution and all those solutions are unstable. In [4–6], neighborhoods of nontransversal homoclinic solutions are investigated; it is shown that there exists a way of tangency for the stable and unstable manifold such that an arbitrary neighborhood of the nontransversal homoclinic solution contains infinitely many stable solutions, but at least one characteristic exponent of those solutions tends to zero as the period increases. In the present paper, we consider another way of tangency for the stable and unstable manifold (compared with [4–6]) and show that, in the considered case, any arbitrary neighborhood of the nontransversal homoclinic solution contains infinitely many stable solutions such that their characteristic exponents are separated from zero. In [7], an example is provided of a two-dimensional periodic system that has infinitely many stable periodic solutions in a neighborhood of a homoclinic contour such that the characteristic exponents of each solution are separated from zero.

We define the Poincaré transformation of system (1) as follows:

$$T(z_0) = z(1, z_0).$$

It is known that the Poincaré transformation is a diffeomorphism of the same smoothness class as system (1).

Apart from system (1), we consider a two-dimensional system of differential equations of the kind

$$\begin{cases} \dfrac{dx}{dt} = X(t,x,y), \\ \dfrac{dy}{dt} = Y(t,x,y), \end{cases} \tag{3}$$

where $t \in [0, 1]$, x and y are arbitrary, and X and Y are infinitely differentiable scalar functions of three variables. Let $x(t, x_0, y_0)$, $y(t, x_0, y_0)$ denote the solution of system (3) with the initial data $t = 0$, $x = x_0$, $y = y_0$.

We introduce

$$f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x(1, x_0, y_0) \\ y(1, x_0, y_0) \end{pmatrix}. \tag{4}$$

We assume that the partial derivatives $\dfrac{\partial X}{\partial x}, \dfrac{\partial X}{\partial y}, \dfrac{\partial Y}{\partial x}, \dfrac{\partial Y}{\partial y}$ are bounded for any x and y and any t from $[0, 1]$. Then f is an infinitely smooth diffeomorphism of the plane C^∞ into itself.

It follows from [7] that there exists a two-dimensional periodic system of kind (1) with an infinitely differentiable right-hand part such that its Poincaré transform coincides with f. Then we show that there exists a class of systems of kind (3) such that the corresponding diffeomorphism f satisfies the conditions of theorems from [1, 2]. Thus, a class of systems of kind (1) is selected for which an arbitrary neighborhood of the nontransversal homoclinic solution contains infinitely many stable periodic solutions such that their characteristic exponents are separated from zero.

Thus, the structure of the neighborhood of a nontransversal homoclinic solution depends on the way of tangency for the stable manifold and unstable manifold. Let us determine that way.

Let

$$h(t) = \frac{1}{|t|} - \sin\frac{1}{|t|}, \quad \sigma_k = (2\pi k)^{-1}.$$

It is clear that the function $h(t)$ is defined for any nonzero t.

Let $y^0 > 0$, $\mu > 1$, and γ be such that $\gamma^{2\pi} = \mu$. We define the function $g(t)$ as follows:

$$\begin{cases} g(t) = \gamma^{-h(t)}[y^0 + (h(t))^{-1}], \quad t \neq 0, \\ g(0) = 0. \end{cases} \tag{5}$$

Theorem 1. *Let $g(t)$ be given by conditions* (5). *Then it is infinitely smooth on the real line*,

$$\frac{d^m g(0)}{dt^m} = 0, \quad m = 1, 2, \ldots, \qquad g(\sigma_k) = \mu^{-k}(y^0 + \sigma_k), \tag{6}$$

and for any positive α, *there exists* k_0 *such that if* $k > k_0$ *and* $t \in (\sigma_k - \mu^{-\alpha k}, \sigma_k + \mu^{-\alpha k})$, *then the inequalities*

$$\left|\frac{dg(t)}{dt}\right| < \mu^{-(\alpha+1)k} \tag{7}$$

hold.

Proof. It is clear that the relation

$$\lim_{t\to 0} |t|^{-m} \mu^{-|t|^{-1}} = 0$$

holds for any positive integer m.

Then the function g has the derivatives of all orders at the origin and they all are equal to zero. It is obvious that relations (6) hold for any k.

The following relations are obvious provided that $t > 0$:

$$\frac{dh(t)}{dt} = -\frac{1}{t^2}\left(1 - \cos\frac{1}{t}\right) = -\frac{2}{t^2}\sin^2\left(\frac{1}{2t}\right)$$

and

$$\frac{dg(t)}{dt} = -\gamma^{-h(t)}[\ln\gamma(y^0 + h^{-1}(t)) + h^{-2}(t)]\frac{dh(t)}{dt}.$$

We fix a positive number α. Let $t \in (\sigma_k - \mu^{-\alpha k}, \sigma_k + \mu^{-\alpha k})$. We define $u = t - \sigma_k$. It is clear that $|u| < \mu^{-\alpha k}$. We obtain

$$h(t) = \frac{2\pi k}{1 + 2\pi k u} - \sin\left(\frac{2\pi k}{1 + 2\pi k u}\right).$$

Taking into account that the sine is a periodic function, we have the relation

$$\left|\frac{dh(t)}{dt}\right| = \frac{2(2\pi k)^2}{(1 + 2\pi k u)^2}\sin^2\left(\frac{\pi k}{1 + 2\pi k u} - \pi k + \pi k\right) = \frac{8(\pi k)^2}{(1 + 2\pi k u)^2}\sin^2\left(\frac{2(\pi k)^2 u}{1 + 2(\pi k u)}\right).$$

It is clear that, if k is sufficiently large, then the following inequalities hold:

$$\left(1 - \frac{\alpha}{2}\right) \le \frac{1}{1 + 2\pi k u} \le \left(1 + \frac{\alpha}{2}\right).$$

Then the following relations hold for any t from $(\sigma_k - \mu^{-\alpha k}, \sigma_k + \mu^{-\alpha k})$:

$$|h(t)|^{-1} \le 1,$$

$$\left|\frac{dh(t)}{dt}\right| \le \frac{32(\pi k)^6 u^2}{(1 + 2\pi k u)^4} \le 32\left(1 + \frac{\alpha}{2}\right)^4 (\pi k)^6 \mu^{-2\alpha k},$$

and

$$\left|\frac{dg(t)}{dt}\right| \le 32\left(1 + \frac{\alpha}{2}\right)^4 \gamma(\ln\gamma(1 + y^0) + 1)(\pi k)^6 \mu^{-(1+1.5\alpha)k}.$$

Those inequalities imply the validity of conditions (7). The theorem is proved.

Further reasoning shows that properties of the function g determine the way of the touching of the stable manifold and unstable manifold at the homoclinic point of the diffeomorphism f.

Let A, A_1, M, x^0, y^0, and ε be positive constants such that

$$\begin{gathered}\mu^{-1}A_1 < y^0 \le x^0 < A < A_1,\\ 2\varepsilon < \min[A - x^0, y^0 - \mu^{-1}A_1, 0.4y^0],\\ M > \max[2\mu^2 A_1, x^0 - y^0 - A_1 \ln\lambda].\end{gathered} \tag{8}$$

We introduce the notation

$$s(t) = (x^0 - y^0 - M)t + M + y^0.$$

We define the sets

$$F_1 = \{(t,x,y) : t \in [0,1], \left|\lambda^{-t}x\right| \le A, \left|\mu^{-t}y\right| \le A\},$$

$$\tilde{F}_1 = \{(t,x,y) : t \in [0,1], \left|\lambda^{-t}x\right| \le A_1, \left|\mu^{-t}y\right| \le A_1\},$$

$$F_2 = \{(t,x,y) : t \in [0,1], \left|\lambda^{t-1}(x - Mt)\right| \le \varepsilon, \left|\mu^{t-1}(y - Mt) - y^0\right| \le \varepsilon\},$$

$$\tilde{F}_2 = \{(t,x,y) : t \in [0,1], \left|\lambda^{t-1}(x - Mt)\right| \le 2\varepsilon, \left|\mu^{t-1}(y - Mt) - y^0\right| \le 2\varepsilon\},$$

$$F_3 = \left\{\begin{gathered}(t,x,y) : t \in [0,1],\\ \left|y^0 + (x - (y^0 + M))\cos\left(\frac{3}{2}\pi t\right) - (y - (y^0 + M))\sin\left(\frac{3}{2}\pi t\right)\right| \le \varepsilon,\\ \left|(x - (y^0 + M))\sin\left(\frac{3}{2}\pi t\right) + (y - (y^0 + M))\cos\left(\frac{3}{2}\pi t\right)\right| \le \varepsilon\end{gathered}\right\},$$

$$\tilde{F}_3 = \left\{\begin{gathered}(t,x,y) : t \in [0,1],\\ \left|y^0 + (x - (y^0 + M))\cos\left(\frac{3}{2}\pi t\right) - (y - (y^0 + M))\sin\left(\frac{3}{2}\pi t\right)\right| \le 2\varepsilon,\\ \left|(x - (y^0 + M))\sin\left(\frac{3}{2}\pi t\right) + (y - (y^0 + M))\cos\left(\frac{3}{2}\pi t\right)\right| \le 2\varepsilon\end{gathered}\right\},$$

$$F_4 = \left\{\begin{gathered}(t,x,y) : t \in [0,1],\\ \left|x - s(t)\right| \le \varepsilon,\\ \left|y - g(-x + s(t))t + Mt - M\right| \le \varepsilon\end{gathered}\right\},$$

and

$$\tilde{F}_4 = \left\{\begin{gathered}(t,x,y) : t \in [0,1],\\ \left|x - s(t)\right| \le 2\varepsilon,\\ \left|y - g(-x + s(t))t + Mt - M\right| \le 2\varepsilon\end{gathered}\right\},$$

where λ and μ satisfy inequalities (2), while the function g is given by conditions (5). It is clear that $F_i \subset \tilde{F}_i$, and, by virtue of conditions (8), the sets $\tilde{F}_i$, $i = 1, 2, 3, 4$ are pairwise disjoint.

Let system (3) be such that

$$X(t,x,y) = (\ln\lambda)x, \quad Y(t,x,y) = (\ln\mu)y \tag{9}$$

for all (t, x, y) from F_1,

$$X(t,x,y) = -(\ln\lambda)x + M((\ln\lambda)t + 1), \quad Y(t,x,y) = -(\ln\mu)y + M((\ln\mu)t + 1) \tag{10}$$

for all (t, x, y) from F_2,

$$X(t,x,y) = \frac{3}{2}\pi(y - y^0 - M), \quad Y(t,x,y) = -\frac{3}{2}\pi(x - y^0 - M) \tag{11}$$

for all (t, x, y) from F_3,

$$X(t,x,y) = (x^0 - y^0 - M), \quad Y(t,x,y) = g(-x + s(t)) - M \tag{12}$$

for all (t, x, y) from F_4, and

$$X(t,x,y) = 0, \quad Y(t,x,y) = 0 \tag{13}$$

for all $(t,x,y) \notin \bigcup_{i=1}^{4} \tilde{F}_i$, $t \in [0, 1]$.

It is clear that the diffeomorphism f defined by (4) is an infinitely smooth diffeomorphism of the plane C^∞ into itself such that the origin is its fixed hyperbolic point.

We introduce the sets

$$V = \{(x,y) : |x| \le A, |y| \le A\},$$

$$U_1 = \{(x,y) : |x| \le \varepsilon, |y - y^0| \le \varepsilon\},$$

$$U_2 = \{(x,y) : |x| \le \lambda\varepsilon, |y - \mu y^0| \le \mu\varepsilon\},$$

$$U_3 = \{(x,y) : |x - M| \le \varepsilon, |y - y^0 - M| \le \varepsilon\},$$

$$U_4 = \{(x,y) : |x - M - y^0| \le \varepsilon, |y - M| \le \varepsilon\},$$

and

$$U_5 = \{(x,y) : |x - x^0| \le \varepsilon, |y - g(x^0 - x)| \le \varepsilon\}.$$

It is clear that $U_1 \subset V$, $U_i \not\subset V$, $i = 2, 3, 4$, $U_5 \subset V$ and the set U_i, $i = 1, 2, 3, 4, 5$, are pairwise disjoint. Conditions (9)–(13) imply that $f(U_i) = U_{i+1}$, $i = 1, 2, 3, 4$.

Let $L = f^4|U_1$. Then

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^0 - (y - y^0) \\ x + g(y - y^0) \end{pmatrix}.$$

The last relation follows from conditions (9)–(13). It is obvious that the diffeomorphism f has a fixed hyperbolic point and a nontransversal homoclinic point $(x^0, 0)$ such that the way of touching for the stable manifold and unstable manifold at that point is determined by properties of the function g.

In [4–6], it is assumed that the function g is satisfies the following conditions:

$$g(0) = \frac{dg(0)}{dt} = \ldots = \frac{d^{m-1}g(0)}{dt^{m-1}} = 0, \quad \frac{d^m g(0)}{dt^m} \ne 0, \quad m \ge 2.$$

It follows from the cited papers that if the way of touching for the stable manifold and unstable manifold at a homoclinic point is determined by the indicated conditions, then any arbitrary neighborhood of the nontransversal homoclinic solution can contain infinitely many stable periodic solutions, but at least one characteristic exponent of those solutions tends to zero as the period increases.

The following theorem holds.

Theorem 2. *Let system* (3) *satisfy condition* (2) *and conditions* (9)–(13). *Then any arbitrary neighborhood of the nontransversal homoclinic point* $(x^0, 0)$ *of the diffeomorphism* f *defined by relation* (4) *contains a denumerable set of stable periodic points such that their characteristic exponents are separated from zero*.

Proof. By virtue of Theorem 1, the function g defined by relations (5) satisfies conditions (6) and (7).

Let the inequality

$$0 < \alpha < -\frac{\ln\lambda}{\ln\mu} - 1$$

be satisfied. Then for any positive S there exists a positive integer k_0 such that the following relations hold for any k exceeding k_0:

$$\left|g(\sigma_k) + \lambda^k(x^0 + \sigma_k) - \mu^{-k}(y^0 - \sigma_k)\right| < S\mu^{-(\alpha+1)k}.$$

Taking into account the above inequalities, one can easily see that the diffeomorphism f satisfies the conditions of the theorem from [2]. Therefore, any arbitrary neighborhood of the point $(x^0, 0)$ contains a denumerable set of stable periodic points such that their characteristic exponents are separated from zero.

The theorem is proved.

Corollary. *Let the conditions of Theorem 2 be satisfied. Assume that the diffeomorphism f is the Poincaré transform of the two-dimensional infinitely smooth periodic system of differential equations given by* (1). *Then any arbitrary neighborhood of the nontransversal homoclinic trajectory of system* (1) *contains an infinite set of stable periodic solutions such that their characteristic exponents are separated from zero.*

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