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On the Law of the Iterated Logarithm for Sequences of Dependent Random Variables

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Abstract—Sufficient conditions for the applicability of the law of the iterated logarithm to sequences of dependent random variables are obtained. As a corollary, a theorem on the law of the iterated logarithm for a sequence of m-orthogonal random variables is proved.

Keywords: law of the iterated logarithm, sequence of m-orthogonal random variables.

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Theorem 1. *Let* $\{Y_n; n = 1, 2, \ldots\}$ *be a sequence of random variables on a probability space, and let* $\{a_n; n = 1, 2, \ldots\}$ *be a sequence of positive numbers such that* $a_n \to \infty$ *as* $n \to \infty$ *and the following condition (Condition A) holds*: *for any* $\varepsilon > 0$,

$$a_{n+r} \geq a_n(1-\varepsilon) \tag{1}$$

for all $r \geq 1$ *and all sufficiently large n. Suppose that*

$$\sum_{n=1}^{\infty} P\left(\max_{[c^n]\leq k<[c^{n+1}]} Y_k > (1+\varepsilon)a_{[c^n]}\right) < \infty \tag{2}$$

for some $c > 1$ *and any* $\varepsilon > 0$. *Then*

$$\limsup \frac{Y_n}{a_n} \leq 1 \quad \textit{almost surely.} \tag{3}$$

Proof. Let δ be any positive number, and let $c > 1$. By virtue of Condition A, we have

$$P(Y_n > (1+\delta)a_n \text{ i.o.})$$
$$\leq P\left(\max_{[c^n]\leq k<[c^{n+1}]} Y_k > (1+\delta)a_{[c^n]}(1-\varepsilon) \text{ i.o.}\right)$$

for any $\varepsilon > 0$. Therefore,

$$P(Y_n > (1+\delta)a_n \text{ i.o.})$$
$$\leq P\left(\max_{[c^n]\leq k<[c^{n+1}]} Y_k > \left(1+\frac{\delta}{2}\right)a_{[c^n]} \text{ i.o.}\right),$$

provided that the positive number ε is so small that $(1 + \delta)(1 - \varepsilon) > 1 + \delta/2$.

Taking c satisfying condition (2) and applying the Borel–Cantelli lemma, we obtain inequality (3).

Condition A is a weakening of the condition that the normalizing number sequence $\{a_n\}$ is nondecreasing; this condition makes it possible to apply Theorem 1 to sequences of random variables when the normalizing sequence is not nondecreasing but satisfies Condition A. Using Theorem 1, we can obtain a sufficient condition for the applicability of the law of the iterated logarithm to sums of m-dependent random variables. Recall that a sequence $\{X_n; n = 1, 2, \ldots\}$ of random variables is referred to as a *sequence of m-dependent random variables*, where m is a nonnegative integer, if the random vectors $(X_p, \ldots, X_q)$ and $(X_r, \ldots, X_s)$ are independent for any integer p, q, r, and s satisfying the conditions $1 \leq p \leq q < r \leq s$ and $r - q > m$.

The notion of a sequence of m-dependent random variables was introduced by Hoeffding and Robbins in a classical paper [1], which also contains conditions for the applicability of the central limit theorem to sequences of m-dependent random variables. At present, the literature on limit theorems for sums of m-dependent random variables has become very extensive.

In [2], the notion of a sequence of m-orthogonal random variables was introduced. Given a nonnegative integer m, we say that a sequence $\{X_n;\ n = 1, 2, \ldots\}$ of random variables defined on some probability space is a *sequence of m-orthogonal random variables* if $EX_n^2 < \infty$ for any n and $E(X_kX_j) = 0$ provided that $|k-j| > m$. In particular, any sequence of 0-orthogonal random variables is a sequence of orthogonal random variables. If $\{X_n\}$ is a sequence of m-dependent random variables with zero expectations and finite variances, then this is a sequence of m-orthogonal random variables. This remains true under the replacement of m-dependence by the weaker condition of pairwise m-dependence. Note that the verification of m-orthogonality is significantly simpler than that of m-dependence or pairwise m-dependence. The study of limit theorems for sequences of m-orthogonal random variables is of certain interest thanks to the great attention given to limit theorems for sums of orthogonal random variables and sums of m-dependent random variables. Below, we present a theorem on the law of the iterated logarithm for sequences of m-orthogonal random variables.

Theorem 2. *Let $\{X_n;\ n = 1, 2, \ldots\}$ be a sequence of m-orthogonal random variables with zero expectations. Put*

$$S_n = \sum_{k=1}^{n} X_k, \qquad B_n = ES_n^2, \qquad \chi(n) = (2B_n \log\log B_n)^{1/2}.$$

Suppose that

$$B_n \to \infty, \tag{4}$$

$$\frac{EX_n^2}{B_n} \to 0 \tag{5}$$

as $n \to \infty$, and

$$\sum_{n=1}^{\infty} P\left(\max_{[c^n]\le k<[c^{n+1}]} S_k > (1+\varepsilon)\chi([c^n])\right) < \infty \tag{6}$$

for some $c > 1$ and any $\varepsilon > 0$. Then

$$\limsup \frac{S_n}{\chi(n)} \le 1 \quad \textit{almost surely}. \tag{7}$$

Proof. Let us show that the sequence $\{B_n\}$ satisfies Condition A. By virtue of the Cauchy–Bunyakovskii inequality, we have

$$B_n = E(S_{n+1} - X_{n+1})^2 = B_{n+1} + EX_{n+1}^2 - 2E(S_{n+1}X_{n+1}) = B_{n+1} + EX_{n+1}^2 + 2\theta(ES_{n+1}^2 EX_{n+1}^2)^{1/2},$$

where $|\theta| \le 1$. Therefore, we can write

$$\frac{B_n}{B_{n+1}} = 1 + \frac{EX_{n+1}^2}{B_{n+1}} + 2\theta\left(\frac{EX_{n+1}^2}{B_{n+1}}\right)^{1/2}.$$

Condition (5) implies

$$\frac{B_n}{B_{n+1}} \to 1 \quad (n \to \infty). \tag{8}$$

For any integer $p \ge 1$, we have

$$B_{n+p} = B_n + E(X_{n+1} + \ldots + X_{n+p})^2 + 2E[S_n(X_{n+1} + \ldots + X_{n+p})] \ge B_n + 2E[S_n(X_{n+1} + \ldots + X_{n+p})],$$

$$E[S_n(X_{n+1} + \ldots + X_{n+p})] = E[(X_{n-m+1} + \ldots + X_n)(X_{n+1} + \ldots + X_{n+p})]$$

for $n > m$ by virtue of the m-orthogonality condition. Again, using this condition, we can easily show that

$$E\left|S_n(X_{n+1} + \ldots + X_{n+p})\right| \le \sum_{j=0}^{m}\sum_{i=0}^{m-j}\left|E(X_{n-j}X_{n+i})\right|.$$

Hence

$$B_{n+p} \ge B_n - 2\sum_{j=0}^{m}\sum_{i=0}^{m-j}\left|E(X_{n-j}X_{n+i})\right| \ge B_n - 2\sum_{j=0}^{m}\sum_{i=0}^{m-j}(EX_{n-j}^2 EX_{n+i}^2)^{1/2}. \tag{9}$$

It follows from (5) and (8) that, for any fixed integer p, we have

$$\frac{EX_{n+p}^2}{B_n} = \frac{EX_{n+p}^2}{B_{n+p}}\frac{B_{n+p}}{B_{n+p-1}}\ldots\frac{B_{n+1}}{B_n} \to 0 \quad (n \to \infty). \tag{10}$$

Relations (9) and (10) imply

$$B_{n+s} \ge B_n\left(1 - 2\sum_{j=0}^{m}\sum_{i=0}^{m-j}\left(\frac{EX_{n-j}^2}{B_n}\frac{EX_{n+i}^2}{B_n}\right)^{1/2}\right), \tag{11}$$
$$B_{n+s} \ge B_n(1-\varepsilon)$$

for any $\varepsilon > 0$ and $s \ge 1$ and all sufficiently large n. Thus, the sequence $\{B_n\}$ satisfies Condition A.

Inequality (11) remains valid for B_n replaced by $\chi(n) = (2B_n \log\log B_n)^{1/2}$. Thus, the sequence of $a_n = \chi(n)$ satisfies Condition A. Condition (2) of Theorem 1 also holds for $Y_n = S_n$ and $a_n = \chi(n)$ by virtue of (6). According to Theorem 1, relation (7) holds.

Theorem 2 remains true under the replacement of m-orthogonality by m-dependence (provided that the other conditions in the theorem are satisfied) or by the even weaker condition of pairwise m-dependence. The resulting theorem generalizes Theorem 2 of [3], in which conditions (4) and (5) are assumed to hold and, instead of (6), it is required that, for any $b > 1$, there exist positive constants C and δ such that

$$P\left(\max_{1\le k\le n} S_k \ge b\chi(n)\right) \le C(\log B_n)^{-1-\delta}$$

for all sufficiently large n. The theorem of [2] differs from this theorem in that the m-dependence condition is replaced by the m-orthogonality condition.

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