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MATHEMATICS

Decay Mild Solutions for Elastic Systems with Structural Damping Involving Nonlocal Conditions[1]

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Abstract—This paper deals with a class of elastic systems with structural damping subject to nonlocal conditions. By using a suitable measure of noncompactness on the space of continuous functions on the half-line, we establish the existence of mild solutions with explicit decay rate of exponential type. An example is given to illustrate the abstract results.

Keywords: damping solutions, elastic system with structural damping, nonlocal conditions.

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1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space, we consider the following nonlocal Cauchy problem for the elastic system with structural damping

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = f(t,u(t)), \quad t > 0, \tag{1.1}$$

$$u(0) + g(u) = x_0, \quad u_t(0) + h(u) = y_0, \tag{1.2}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ is a closed linear operator, $\rho \geq 2$ is given constant, $x_0 \in \mathcal{D}(\mathcal{A})$, $y_0 \in X$ and g, h, f are given functions, which will be explained in Section 3.

The elastic systems with structural damping were studied by Chen and Russell [1] in 1981. They considered the elastic system as follows

$$u_{tt}(t) + \mathcal{B}u_t(t) + \mathcal{A}u(t) = 0, \quad t > 0, \tag{1.3}$$

$$u(0) = x_0, \quad u_t(0) = y_0, \tag{1.4}$$

in Hilbert space $\mathbb{H}$, where $\mathcal{A}$ (the elastic operator) and $\mathcal{B}$ (the damping operator) are unbounded positive definite self-adjoint operators in $\mathbb{H}$. If some additional conditions are satisfied, they proved that

$$L_B = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix}$$

generates an analytic semigroup on $\mathbb{W} = \mathbb{H} \oplus \mathbb{H}$. In 1986, Huang [2] discussed above problem, he proposed $\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B)$; then either of the following conditions (a) and (b) implies that L_B generates an analytic semigroup on $\mathbb{W}$:

(a) $\rho_1(\mathcal{A}^{1/2}x, x) \leq (\mathcal{B}x, x) \leq \rho_2(\mathcal{A}^{1/2}x, x)$ for all $x \in \mathcal{D}(A^{1/2})$,

(b) $\rho_1(\mathcal{A}x, x) \leq (\mathcal{B}^2x, x) \leq \rho_2(\mathcal{A}x, x)$ for all $x \in \mathcal{D}(A)$, for some $\rho_1, \rho_2 > 0$ with $\rho_1 \leq \rho_2$.

[1] The article is published in the original.

In 1988, Huang [3] developed problem (1.3)–(1.4) for the damping operator $\mathcal{B}$, and the elastic operator $\mathcal{A}$ is replaced by operator $\mathcal{A}^{\alpha}\left(\frac{1}{2} \le \alpha \le 1\right)$. Fan, Li, and Chen [4] obtained the existence of mild solutions for the elastic system with structural damping in Banach spaces:

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = f(t,u(t)), \quad 0 < t < a, \tag{1.5}$$

$$u(0) = x_0, \quad u_t(0) = y_0, \tag{1.6}$$

where the damping constant $\rho \ge 2$ and the nonlinearity f is Lipschitzian in the second variable. The analyticity and the exponential stability of semigroup generated by the elastic system

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = 0, \quad t > 0, \tag{1.7}$$

$$u(0) = x_0, \quad u_t(0) = y_0, \tag{1.8}$$

were given by Fan and Li [5], where $\rho > 2\cos\alpha$, for a fixed value $\alpha \in \left(0, \frac{\pi}{2}\right)$. In [6] Fan and Gao discussed asymptotic behavior of solutions for the linear elastic system with structural damping

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = h(t), \quad t > 0, \tag{1.9}$$

$$u(0) = x_0, \quad u_t(0) = y_0, \tag{1.10}$$

and the semilinear elastic system with structural damping

$$u_{tt}(t) + \rho\mathcal{A}u_t(t) + \mathcal{A}^2u(t) = f(t,u(t)), \quad 0 < t < a, \tag{1.11}$$

$$u(0) = x_0, \quad u_t(0) = y_0, \tag{1.12}$$

in Banach spaces, where $\rho > 2\cos\alpha$, for a fixed value $\alpha \in \left(0, \frac{\pi}{2}\right)$, $\mathcal{A}$ is a sectorial operator, $-\mathcal{A}$ generates an analytic and exponentially stable semigroup on $X, h : [0,+\infty) \to X$ is continuous, and f is Lipschitz continuous in the second variable. Although problem (1.1)–(1.2) has been an interesting subject, no attempt has made to find its decay solutions with explicit decay rate, up to knowledge. This is the motivation for our study.

Motivated by [6], we deal with elastic systems with structural damping subject to nonlocal conditions. The concept of nonlocal conditions were first used by Byszewski [7]. This notion is more appropriate than the classical one to describe natural phenomena because it allows us to consider additional information, see Deng [8], Byszewski et al. [9]. The purpose of this paper is to use a fixed point principle for condensing maps for measures of noncompactness [10] to prove the existence of decay mild solutions u with $\|u(t)\| = O(e^{-\gamma t})$ as $t \to \infty$.

The rest of our work is organized as follows. In the next section, we recall some notions and facts related to measures of noncompactness and condensing map, which will be used to prove the existence of mild solutions on $[0,T]$, $T > 0$ in Section 3 and the existence of decay mild solutions on $\mathbb{R}^+$ in Section 4. In the last section, we give an example to illustrate the obtained results.

2. PRELIMINARIES

Let E be a Banach space. We denote by $\mathcal{P}_b(E)$ the collection of all nonempty bounded subsets in E.

Definition 2.1. *A function $\Phi : \mathcal{P}_b(E) \to [0,+\infty)$ is called a measure of noncompactness (MNC) in E if*

$$\Phi(\overline{co}\Omega) = \Phi(\Omega), \ \forall\Omega \in \mathcal{P}_b(E),$$

where $\overline{co}\Omega$ is the closure of the convex hull of Ω. An MNC Φ in E is called

(i) *monotone if for $\forall\Omega_1, \Omega_2 \in \mathcal{P}_b(E)$, $\Omega_1 \subset \Omega_2$ implies $\Phi(\Omega_1) \le \Phi(\Omega_2)$;*

(ii) *nonsingular if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for $\forall a \in E$, $\forall\Omega \in \mathcal{P}_b(E)$;*

(iii) *invariant with respect to union with compact set if $\Phi(K \cup \Omega) = \Phi(\Omega)$ for every relatively compact $K \subset E$ and $\Omega \in \mathcal{P}_b(E)$;*

(iv) *algebraically semi-additive if* $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$ *for any* $\Omega_1, \Omega_2 \in \mathcal{P}_b(E)$;

(v) *regular if* $\Phi(\Omega) = 0$ *is equivalent to the relative compactness of* Ω.

An important example of MNC is the Hausdorff MNC $\chi(\cdot)$, which is defined as following:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \text{ has a finite } \varepsilon\text{-net}\} \tag{2.1}$$

for $\Omega \in \mathcal{P}_b(E)$.

Proposition 2.2. *Let* χ *be the Hausdorff MNC on Banach space* $E, \Omega \in \mathcal{P}_b(E)$. *Then for any* $\varepsilon > 0$ *there exists a sequence* $\{x_n\}_{n=1}^{\infty} \subset \Omega$ *such that*

$$\chi(\Omega) \leq 2\chi(\{x_n\}_{n=1}^{\infty}) + \varepsilon. \tag{2.2}$$

Let $C([0,T];X)$ be the space of all continuous functions defined on the interval $[0,T]$ taking values in X, which is a Banach space with the norm $\|u\|_C = \sup_{t\in[0,T]} \|u(t)\|_X$. It is known that, when $X = \mathbb{R}^n$, the Hausdorff MNC on $C([0,T];\mathbb{R}^n)$ is given by (see [11, Example 2.11])

$$\chi_T(D) = \frac{1}{2}\lim_{\delta\to 0}\sup_{u\in D} \max_{t,s\in[0,T],|t-s|<\delta} \|u(t) - u(s)\|_{\mathbb{R}^n}, \ D \subset C([0,T],\mathbb{R}^n). \tag{2.3}$$

The last measure can be seen as the modulus of equicontinuity of a subset in $C([0,T];\mathbb{R}^n)$. In $C([0,T];X)$ with X being of infinite dimensional, there is no such formulation as (2.2). However, if $D \subset C([0,T];X)$ is an equicontinuous set then

$$\chi_T(D) = \sup_{t\in[0,T]} \chi(D(t)). \tag{2.4}$$

Here χ is the Hausdorff MNC in X and $D(t) := \{x(t) : x \in D\}$.

Consider the space $BC(\mathbb{R}^+;X)$ of bounded continuous functions on $\mathbb{R}^+$ taking values on X. Denote by π_T the restriction operator on this space, that is $\pi_T(u)$ is the restriction of u on $[0,T]$. Then

$$\chi_\infty(D) = \sup_{T>0}\chi_T(\pi_T(D)), \quad D \subset BC(\mathbb{R}^+;X) \tag{2.5}$$

is an MNC. We consider other MNCs on this space as following

$$d_T(D) = \sup_{u\in D}\sup_{t\geq T}\|u(t)\|_X, \tag{2.6}$$

$$d_\infty(D) = \lim_{T\to\infty} d_T(D), \tag{2.7}$$

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.8}$$

The regularity of MNC χ^* is proved in [2, Lemma 2.6].

We have the following estimate, whose proof can be found in [10].

Proposition 2.3 [10]. *Let* χ *be the Hausdorff MNC on Banach space* X, *sequence* $\{u_n\}_{n=1}^{\infty} \subset L_1(0,T;X)$ *such that* $\|u_n(t)\|_X \leq v(t)$, *for every* $n \in \mathbb{N}^*$ *and a.e* $t \in [0,T]$, *here* $v \in L_1(0,T)$ *is a nonnegative function. Then we have*

$$\chi\left(\left\{\int_0^t u_n(s)ds\right\}\right) \leq 2\int_0^t \chi(\{u_n(s)\})ds, \tag{2.9}$$

for $t \in [0,T]$.

Using Proposition 2.2 and Proposition 2.3, we get

Proposition 2.4. [12] *Let* χ *be the Hausdorff MNC on Banach space* X *and* $D \subset L_1(0,T;X)$. *If there exist functions* $v, q \in L_1(0,T)$ *which satisfy following conditions*

(i) $\|\theta(t)\|_{L_1(0,T;X)} \leq v(t)$, *for* $\forall\theta \in D$ *and a.e* $t \in [0,T]$,

(ii) $\chi(D(t)) \leq q(t)$, *for a.e* $t \in [0,T]$, $D(t) = \{x(t) : x \in D\}$.

Then

$$\chi\left(\int_0^t D(s)ds\right) \le 4\int_0^t q(s)ds, \tag{2.10}$$

where $\int_0^t D(s)ds = \left\{\int_0^t \theta(s)ds : \theta \in D\right\}$.

We denote by $\left(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)}\right)$ the space of linear bounded operators from X into itself, χ is the Hausdorff MNC on X. For each $T \in \mathcal{L}(X)$, we define the $\chi-$ norm of T (see [11]) as following

$$\|T\|_\chi = \inf\{k > 0 : \chi(T(\Omega)) \le k\chi(\Omega), \Omega \in B_X\}. \tag{2.11}$$

We have following estimate (see [13])

$$\|T\|_\chi \le \|T\|_{\mathcal{L}(X)}. \tag{2.12}$$

We recall the following definition in [14, Definition 6.1.1].

Definition 2.5. *A C_0-semigroup $\{T(t); t \ge 0\}$ is equicontinuous if the function $t \mapsto T(t)$ is continuous from $(0,+\infty)$ to $\mathcal{L}(X)$ endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(X)}$*

To end this section, we recall the fixed point principle for condensing maps that will be used in next section.

Definition 2.6 [13]. *Let β be an MNC on Banach space E, and $\varnothing \ne D \subset E$. A continuous map $F : D \to E$ is said to be condensing with respect to β ($\beta-$ condensing) if for $\forall\Omega \in \mathcal{P}_b(D)$, the relation $\beta(\Omega) \le \beta(F(\Omega))$ implies the relative compactness of Ω.*

Theorem 2.7 [10]. *Let D be a bounded convex closed subset of Banach space E and let $F : D \to D$ be a β condensing map with β being a monotone and nonsingular MNC on E. Then $Fix(F) = \{x \in D : x = F(x)\}$ is a nonempty compact set.*

3. EXISTENCE RESULT

In the formulation of problem (1.1)–(1.2), we assume that

(A) The operator $-\mathcal{A}$ generates a equicontinuous C_0 semigroup $\{T(t)\}_{t\ge 0}$ on Banach space X. By this assumption, $\mathcal{D}(\mathcal{A})$ with the graph norm $\|z\|_{\mathcal{D}(\mathcal{A})} = \|z\| + \|\mathcal{A}z\|$, becomes a Banach space.

(G) The function $g : C([0,T];X) \to \mathcal{D}(\mathcal{A})$ obeys the following conditions:

(i) g is continuous, and

$$\|g(u)\|_{\mathcal{D}(\mathcal{A})} \le \theta_g(\|u\|_C), \tag{3.1}$$

for all $u \in C([0,T];X)$, where $\theta_g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

(ii) There exist non-negative constants η_g, ξ_g such that

$$\chi(g(\Omega)) \le \eta_g\chi_T(\Omega), \tag{3.2}$$

$$\chi(\mathcal{A}g(\Omega)) \le \xi_g\chi_T(\Omega), \tag{3.3}$$

for all bounded set $\Omega \subset C([0,T];X)$.

(H) The function $h : C([0,T];X) \to X$ satisfies following conditions:

(i) There is a continuous and nondecreasing function $\theta_h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(u)\|_X \le \theta_h(\|u\|_C), \tag{3.4}$$

for all $u \in C([0,T];X)$.

(ii) There exists a non-negative constant η_h such that

$$\chi(h(\Omega)) \le \eta_h\chi_T(\Omega), \tag{3.5}$$

for all bounded set $\Omega \subset C([0,T]; X)$.

(F) The nonlinear function $f : \mathbb{R}^+ \times X \to X$ satisfies:

(i) $f(\cdot, v)$ is measurable for each $v \in X$, $f(t, \cdot)$ is continuous for a.e $t \in [0,T]$, and

$$\|f(t,v)\|_X \le m(t)\theta_f(\|v\|_X), \tag{3.6}$$

for all $v \in X$, where $m \in L_1(0,T)$, $\theta_f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing function.

(ii) If the semigroup $T(\cdot)$ is noncompact, there exists $\eta_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\eta_f \in L_1(0,T)$ and

$$\chi(f(t,\Omega)) \le \eta_f(t)\chi(\Omega), \tag{3.7}$$

for all bounded set $\Omega \subset X$.

Remark 3.1. 1) If $f(t,\cdot)$ satisfies the Lipschitz condition, i.e.,

$$\|f(t,v_1) - f(t,v_2)\|_X \le k_f(t)\|v_1 - v_2\|_X,$$

for some $k_f \in L_1(0,T)$, then (3.6) and (3.7) are satisfied.

2) If g is compact, then (3.2) is satisfied.

Set

$$S_1(t) = T(\sigma_1 t), \quad S_2(t) = T(\sigma_2 t), \quad \sigma_1 + \sigma_2 = \rho, \quad \sigma_1\sigma_2 = 1, \quad 0 < \sigma_1 < \sigma_2.$$

We give the definition of mild solution to the problem (1.1)–(1.2) as following

Definition 3.2. A function $u \in C([0, T]; X)$ is said to be a *mild solution* of problem (1.1)–(1.2) on the $[0,T]$ if

$$u(t) = S_2(t)(x_0 - g(u)) + \int_0^t S_2(t-s)S_1(s)v_0 ds + \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau ds, \tag{3.8}$$

for any $t \in [0,T]$, where $v_0 = y_0 - h(u) + \sigma_2\mathcal{A}(x_0 - g(u))$.

We denote $B_R = \{u \in C([0,T];X) : \|u\|_C \le R\}$, where $R > 0$ is given. We define the solution operator $F : B_R \to C([0,T];X)$ by

$$\begin{aligned} F(u)(t) = {} & S_2(t)(x_0 - g(u)) + \int_0^t S_2(t-s)S_1(s)(y_0 - h(u) + \sigma_2\mathcal{A}(x_0 - g(u)))ds \\ & + \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau ds, \end{aligned} \tag{3.9}$$

for $\forall u \in B_R$, $\forall t \in [0,T]$.

From the assumptions imposed on g, h, f, we see that F is a continuous map on B_R. Set

$$M = \sup_{t\in[0,T]} \|S_2(t)\|_{\mathcal{L}(X)}, \quad \Lambda_T = \sup_{t\in[0,T]} \int_0^t \|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)} ds,$$

$$\Upsilon_T = \sup_{t\in[0,T]} \int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} m(\tau)d\tau ds,$$

$$\Theta_T = \begin{cases} 0, & \text{if the semigroup } T(\cdot) \text{ is compact,} \\ \sup\limits_{t\in[0,T]} \int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} \eta_f(\tau)d\tau ds, & \text{otherwise.} \end{cases}$$

Lemma 3.3. Let **(A)**, **(G)**, **(H)**, **(F)** *hold. If*

$$\liminf_{n\to\infty} \frac{1}{n}[M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T] < 1 \tag{3.10}$$

then there exists $R > 0$ *such that* $F(B_R) \subset B_R$.

Proof. Assume to the contrary that for each $n \in \mathbb{N}$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset B_R$ with $\|u_n\|_C \le n$ but $\|F(u_n)\|_C > n$. From the formulation of F, we have

$$F(u_n)(t) = F_1(u_n)(t) + F_2(u_n)(t) + F_3(u_n)(t),$$

where

$$F_1(v)(t) = S_2(t)(x_0 - g(v));$$

$$F_2(v)(t) = \int_0^t S_2(t-s)S_1(s)(y_0 - h(v) + \sigma_2\mathcal{A}(x_0 - g(v)))ds;$$

$$F_3(v)(t) = \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau, v(\tau))d\tau ds.$$

We have

$$\|F(u_n)(t)\|_X \le \|F_1(u_n)(t)\|_X + \|F_2(u_n)(t)\|_X + \|F_3(u_n)(t)\|_X, \tag{3.11}$$

$$\|F_1(u_n)(t)\|_X \le \|S_2(t)\|_{\mathcal{L}(X)}(\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|g(u_n)\|_{\mathcal{D}(\mathcal{A})}) \le M(\|x_0\|_{\mathcal{D}(\mathcal{A})} + \theta_g(n)), \tag{3.12}$$

$$\begin{aligned}\|F_2(u_n)(t)\|_X &\le \int_0^t \|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)}(\|y_0\|_X + \|h(u_n)\|_X + \sigma_2\|\mathcal{A}x_0\|_X \\ &+ \sigma_2\|\mathcal{A}g(u_n)\|_X)ds \le (\|y_0\|_X + \|h(u_n)\|_X + \sigma_2\|\mathcal{A}x_0\|_X + \sigma_2\|\mathcal{A}g(u_n)\|_{\mathcal{D}(\mathcal{A})}\Lambda_T \\ &\le (\|y_0\|_X + \theta_h(n) + \sigma_2\|\mathcal{A}x_0\|_X + \sigma_2\theta_g(n))\Lambda_T.\end{aligned} \tag{3.13}$$

$$\begin{aligned}\|F_3(u_n)(t)\|_X &\le \int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}\|f(\tau, u_n(\tau))\|_X d\tau ds \\ &\le \int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)} m(\tau)\theta_f(\|u_n(t)\|_X)d\tau ds \le \theta_f(n)\Upsilon_T.\end{aligned} \tag{3.14}$$

From (3.11)–(3.14), we have

$$\|F(u_n)(t)\|_X \le (M\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|y_0\|_X + \sigma_2\|\mathcal{A}x_0\|_X) + (M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T),$$

which implies

$$\|F(u_n)\|_C \le (M\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|y_0\|_X + \sigma_2\|\mathcal{A}x_0\|_X) + (M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T).$$

Therefore

$$1 < \frac{1}{n}\|F(u_n)\|_C \le \frac{1}{n}[M\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|y_0\|_X + \sigma_2\|\mathcal{A}x_0\|_X] + \frac{1}{n}[M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T]. \tag{3.15}$$

Passing to the limit in the last inequality, one gets a contradiction. Lemma 3.3. is proved. □

Lemma 3.4. *Let the assumptions of Lemma* 3.3 *hold. Then*

$$\chi_T(F(D)) \le [M\eta_g + 4(\eta_h + \sigma_2\xi_g)\Lambda_T + 8\Theta_T]\chi_T(D), \tag{3.16}$$

for all bounded sets $D \subset B_R$.

Proof. From the algebraically semi-additive property of χ_T, we have

$$\chi_T(F(D)) \le \chi_T(F_1(D)) + \chi_T(F_2(D)) + \chi_T(F_3(D)). \tag{3.17}$$

1. For every $z_1, z_2 \in F_1(D)$, there exist $u_1, u_2 \in D$ such that for $t \in [0, T]$,

$$z_i(t) = F_1(u_i)(t) = S_2(t)(x_0 - g(u_i)) \quad (i = 1, 2).$$

We have

$$\|z_1(t) - z_2(t)\|_X = \|S_2(t)(g(u_2) - g(u_1))\|_X.$$

Then

$$\|z_1 - z_2\|_C \le M\,\|g(u_2) - g(u_1)\|_X.$$

Hence

$$\chi_T(F_1(D)) \le M\chi(g(D)) \le M\eta_g\chi_T(D). \tag{3.18}$$

2. Applying Proposition 2.2., for every $\varepsilon > 0$ there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset D$ such that

$$\chi_T(F_2(D)) \le 2\chi_T(\{F_2(u_n)\}_{n=1}^{\infty}) + \varepsilon, \tag{3.19}$$

$$\chi(\{F_2(u_n)(t)\}) \le 2\int_0^t \chi(S_2(t-s)S_1(s)(y_0 - h(\{u_n\}) + \sigma_2\mathcal{A}(x_0 - g(\{u_n\}))))ds$$

$$= 2\int_0^t \chi[S_2(t-s)S_1(s)(h(\{u_n\}) + \sigma_2\mathcal{A}g(\{u_n\}))]ds$$

$$\le 2\int_0^t \|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)}(\chi(h(\{u_n\})) + \sigma_2\chi(\mathcal{A}g(\{u_n\})))ds.$$

It is easy to see that $\{F_2(u_n)\}$ is an equicontinuous set. Therefore

$$\chi_T(\{F_2(u_n)\}) \le 2(\eta_h\chi_T(\{u_n\}) + \sigma_2\xi_g\chi_T(\{u_n\}))\Lambda_T \le 2(\eta_h + \sigma_2\xi_g)\Lambda_T\chi_T(D), \tag{3.20}$$

thanks to the assumptions (G) and (H). Since $\varepsilon > 0$ is arbitrary, from (3.19) and (3.20) we have

$$\chi_T(F_2(D)) \le 4(\eta_h + \sigma_2\xi_g)\Lambda_T\chi_T(D). \tag{3.21}$$

3. Applying Proposition 2.2. again, for every $\varepsilon > 0$, there exists $\{u_n\}_{n=1}^{\infty} \subset D$ such that

$$\begin{gathered}\chi_T(F_3(D)) \le 2\chi_T(\{F_3(u_n)\}_{n=1}^{\infty}) + \varepsilon,\\ \chi(\{F_3(u_n)(t)\}) \le 4\int_0^t\int_0^s \chi(S_2(t-s)S_1(s-\tau)f(\tau,\{u_n(\tau)\}))d\tau ds.\end{gathered} \tag{3.22}$$

If $T(\cdot)$ is a compact semigroup, so is $S_2(\cdot)$. Then

$$\chi(S_2(t-s)S_1(s-\tau)f(\tau,\{u_n(\tau)\})) = 0 \quad \text{for a.e.} \quad \tau \in [0,T].$$

In this case we have $\chi(\{F_3(u_n)(t)\}) = 0$. In the opposite case, we get

$$\chi(\{F_3(u_n)(t)\}) \le 4\int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}\chi(f(\tau,\{u_n(\tau)\}))d\tau ds$$

$$\le 4\int_0^t\int_0^s \|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}\eta_f(\tau)\chi(\{u_n(\tau)\})d\tau ds.$$

It is easily seen that $\{F_3(u_n)\}$ is an equicontinuous set, which implies

$$\chi_T(\{F_3(u_n)\}) \le 4\Theta_T\chi_T(D). \tag{3.23}$$

From (3.22) and (3.23), we have

$$\chi_T(F_3(D)) \le 8\Theta_T\chi_T(D). \tag{3.24}$$

Combining (3.17), (3.18), (3.21) and (3.24) yields

$$\chi_T(F(D)) \le [M\eta_g + 4(\eta_h + \sigma_2\xi_g)\Lambda_T + 8\Theta_T]\chi_T(D). \tag{3.25}$$

Lemma 3.4. is proved. □

Theorem 3.5. *Let the assumptions of Lemma* 3.4. *hold. Then problem* (1.1)–(1.2) *has at least one mild solution on* $[0,T]$ *provided that*

$$l := M\eta_g + 4(\eta_h + \sigma_2\xi_g)\Lambda_T + 8\Theta_T < 1, \tag{3.26}$$

$$\lim_{n\to\infty}\frac{1}{n}[M\theta_g(n) + (\theta_h(n) + \sigma_2\theta_g(n))\Lambda_T + \theta_f(n)\Upsilon_T] < 1. \tag{3.27}$$

Proof. From inequality (3.26), the solution operator F is a χ_T – condensing. Indeed, let be $D \subset B_R$ is a bounded set such that $\chi_T(D) \le \chi_T(F(D))$. Applying Lemma 3.4, we obtain

$$\chi_T(D) \le \chi_T(F(D)) \le l\chi_T(D).$$

Therefore $\chi_T(D) = 0$, and then D is relatively compact.

By assumption (3.27), applying Lemma 3.3, we have $F(B_R) \subset B_R$. Next we apply Theorem 2.7, the χ_T – condensing map F defined by (3.9) has set $Fix(F) \subset B_R$ which is compact and nonempty set. This shows that the problem (1.1)–(1.2) has at least one mild solution $u(t)$, $t \in [0,T]$ given by (3.8). □

4. EXISTENCE OF DECAY MILD SOLUTIONS

In this section, we consider solution operator F on the following set:

$$BC_R^\gamma(\beta) = B_R \cap \left\{ v \in BC(\mathbb{R}^+; X) : \sup_{t\in\mathbb{R}^+} e^{\gamma t}\|v(t)\|_X \le \beta \right\},$$

where B_R is the closed ball in $BC(\mathbb{R}^+; X)$ centered at origin with radius R; β and γ are positive numbers which are chosen later. This is a bounded convex closed subset in $BC(\mathbb{R}^+; X)$.

On $BC_R^\gamma(\beta)$ we make use of MNC χ^* given by (2.8). We will prove that F keeps $BC_R^\gamma(\beta)$ invariant, i.e., $F(BC_R^\gamma(\beta)) \subset BC_R^\gamma(\beta)$, and F is χ^* condensing on $BC_R^\gamma(\beta)$. To this end, we have to replace **(A)**, **(G)**, **(H)**, and **(F)** by stronger ones:

(A*) Operator $-\mathcal{A}$ is the infinitesimal generator of a equicontinuous C_0 semigroup $\{T(t)\}_{t\ge 0}$ such that

$$\|T(t)\|_{\mathcal{L}(X)} \le Ce^{-\delta t}, \quad t \ge 0,$$

where $C, \delta > 0$ are positive constants.

(G*) The assumption **(G)** is satisfied with any $T > 0$.

(H*) The assumption **(H)** is satisfied with any $T > 0$.

(F*) The assumption **(F)**, is satisfied with $\theta_f(r) = r$, $\eta_f \in L^\infty(\mathbb{R}^+)$, and $m \in L^1_{\text{loc}}(\mathbb{R}^+)$, such that

$$K = \sup_{s\ge 0}\int_0^s e^{-(\delta\sigma_1-\gamma)(s-\tau)}m(\tau)d\tau < +\infty.$$

Set

$$M_\infty = \sup_{t\in\mathbb{R}^+}\|S_2(t)\|_{\mathcal{L}(X)}, \quad \Lambda_\infty = \sup_{t\in\mathbb{R}^+}\int_0^t\|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)}ds,$$

$$\Upsilon_\infty = \sup_{t\in\mathbb{R}^+}\int_0^t\int_0^s\|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}m(\tau)d\tau ds,$$

$$\Theta_\infty = \begin{cases} 0, & \text{if the semigroup } T(\cdot) \text{ is compact,} \\ \sup_{t\in\mathbb{R}^+}\int_0^t\int_0^s\|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}\eta_f(\tau)d\tau ds, & \text{otherwise.} \end{cases}$$

In what follows, let $\gamma \in (0, \delta\sigma_1]$ be fixed.

Lemma 4.1. *Let* (**A***), (**G***), (**H***), *and* (**F***) *hold. If*

$$\lim_{n\to\infty}\frac{1}{n}[M_\infty\theta_g(n)+(\theta_h(n)+\sigma_2\theta_g(n))\Lambda_\infty]+\Upsilon_\infty<1, \tag{4.1}$$

$$\text{and}\quad \frac{KC^2}{\delta\sigma_2-\gamma}<1, \tag{4.2}$$

then there exist positive numbers R, β *such that* $F(BC_R^\gamma(\beta))\subset BC_R^\gamma(\beta)$.

Proof. 1. Firstly, we prove the existence of B_R which is invariant under the solution operator F. Indeed, assume to the contrary that for each $n\in\mathbb{N}$, there exists $u_n\in B_n$ satisfying $\|u_n\|_\infty\le n$ but $\|F(u_n)\|_\infty>n$. By the same arguments as in Lemma 3.3, we have

$$\|F_1(u_n)(t)\|_X\le M_\infty(\|x_0\|_{\mathcal{D}(\mathcal{A})}+\theta_g(n)), \tag{4.3}$$

$$\|F_2(u_n)(t)\|_X\le(\|y_0\|_X+\theta_h(n)+\sigma_2\|\mathcal{A}x_0\|_X+\sigma_2\theta_g(n))\Lambda_\infty, \tag{4.4}$$

$$\|F_3(u_n)(t)\|_X\le n\Upsilon_\infty. \tag{4.5}$$

From (4.3), (4.4), and (4.5), we obtain

$$\|F(u_n)(t)\|_X\le(M_\infty\|x_0\|_{\mathcal{D}(\mathcal{A})}+\|y_0\|_X+\sigma_2\|\mathcal{A}x_0\|_X)+(M_\infty\theta_g(n)+(\theta_h(n)+\sigma_2\theta_g(n))\Lambda_\infty+n\Upsilon_\infty),$$

for each $t\ge0$. Therefore

$$\begin{aligned}1<\frac{1}{n}\|F(u_n)\|_\infty&\le\frac{1}{n}[M_\infty\|x_0\|_{\mathcal{D}(\mathcal{A})}+\|y_0\|_X+\sigma_2\|\mathcal{A}x_0\|_X]\\&+\frac{1}{n}[M_\infty\theta_g(n)+(\theta_h(n)+\sigma_2\theta_g(n))\Lambda_\infty+n\Upsilon_\infty].\end{aligned} \tag{4.6}$$

Passing to the limit as $n\to\infty$ in the last relation, we get a contradiction to (4.1).

2. Next, we prove that there is a positive number β such that $F(BC_R^\gamma(\beta))\subset BC_R^\gamma(\beta)$. Indeed, assume to the contrary that for each $n\in\mathbb{N}$ there exists $u_n\in BC_R^\gamma(n)$ (this means $\sup_{t\in\mathbb{R}^+}e^{\gamma t}\|u_n(t)\|_X\le n$) such that $\sup_{t\in\mathbb{R}^+}e^{\gamma t}\|F(u_n)(t)\|_X>n$. Then

$$\begin{aligned}e^{\gamma t}\|F_1(u_n)(t)\|_X&\le e^{\gamma t}\|S_2(t)\|_{\mathcal{L}(X)}(\|x_0\|_{\mathcal{D}(\mathcal{A})}+\|g(u_n)\|_{\mathcal{D}(\mathcal{A})})\le e^{\gamma t}\|S_2(t)\|_{\mathcal{L}(X)}(\|x_0\|_{\mathcal{D}(\mathcal{A})}+\theta_g(\|u_n\|_\infty))\\&\le\frac{C}{e^{(\delta\sigma_2-\gamma)t}}[\|x_0\|_{\mathcal{D}(\mathcal{A})}+\theta_g(R)]\le C[\|x_0\|_{\mathcal{D}(\mathcal{A})}+\theta_g(R)],\end{aligned} \tag{4.7}$$

$$\begin{aligned}e^{\gamma t}\|F_2(u_n)(t)\|_X&\le e^{\gamma t}\int_0^t\|S_2(t-s)S_1(s)\|_{\mathcal{L}(X)}(\|y_0\|_X+\|h(u_n)\|_X+\sigma_2\|\mathcal{A}x_0\|_X+\sigma_2\|\mathcal{A}g(u_n)\|_X)ds\\&\le C[\|y_0\|_X+\theta_h(\|u_n\|_\infty)+\sigma_2\|\mathcal{A}x_0\|_X+\sigma_2\theta_g(\|u_n\|_\infty)]e^{(\gamma-\delta\sigma_2)t}\int_0^t e^{\delta(\sigma_2-\sigma_1)s}ds\\&\le C_1[\|y_0\|_X+\theta_h(R)+\sigma_2\|\mathcal{A}x_0\|_X+\sigma_2\theta_g(R)],\end{aligned} \tag{4.8}$$

for every $t\ge0$, thank to $u_n\in BC_R^\gamma(n)$. Here C_1 is a positive constant which is independent of u_n.

$$\begin{aligned}e^{\gamma t}\|F_3(u_n)(t)\|_X&\le e^{\gamma t}\int_0^t\int_0^s\|S_2(t-s)S_1(s-\tau)\|_{\mathcal{L}(X)}\|f(\tau,u_n(\tau))\|_X\,d\tau ds\\&\le C^2e^{\gamma t}\int_0^t\int_0^s e^{-\delta\sigma_2(t-s)}e^{-\delta\sigma_1(s-\tau)}m(\tau)\|u_n(\tau)\|_X\,d\tau ds\end{aligned}$$

$$= C^2 e^{\gamma t} \int_0^t e^{-\delta\sigma_2(t-s)} \left(\int_0^s e^{-\delta\sigma_1(s-\tau)-\gamma\tau} m(\tau) e^{\gamma\tau} \|u_n(\tau)\|_X \, d\tau \right) ds$$

$$\leq n C^2 e^{\gamma t} \int_0^t e^{-\delta\sigma_2(t-s)} \left(\int_0^s e^{-\delta\sigma_1(s-\tau)-\gamma\tau} m(\tau) d\tau \right) ds,$$

we see that

$$\int_0^s e^{-\delta\sigma_1(s-\tau)-\gamma\tau} m(\tau) d\tau = \int_0^s e^{-\delta\sigma_1(s-\tau)+\gamma(s-\tau)} e^{-\gamma s} m(\tau) d\tau$$

$$= e^{-\gamma s} \int_0^s e^{-(\delta\sigma_1-\gamma)(s-\tau)} m(\tau) d\tau \leq K e^{-\gamma s},$$

thank to $\gamma \leq \delta\sigma_1$. Therefore

$$e^{\gamma t} \|F_3(u_n)(t)\|_X \leq n C^2 K e^{\gamma t} \int_0^t e^{-\delta\sigma_2(t-s)-\gamma s} ds \leq \frac{nKC^2}{\delta\sigma_2 - \gamma}. \tag{4.9}$$

From (4.7)–(4.9), we have

$$e^{\gamma t} \|F(u_n)(t)\|_X \leq C_2 [\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|y_0\|_X + \sigma_2 \|\mathcal{A} x_0\|_X + (1+\sigma_2)\theta_g(R) + \theta_h(R)] + \frac{nKC^2}{\delta\sigma_2 - \gamma},$$

for $\forall u_n \in BC_R^\gamma(n)$ and $\forall t \geq 0$, where $C_2 = \max\{C, C_1\}$. This implies that

$$1 < \frac{1}{n} \sup_{t\geq 0} e^{\gamma t} \|F(u_n)(t)\|_X \leq \frac{C_2}{n} [\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|y_0\|_X + \sigma_2 \|\mathcal{A} x_0\|_X + (1+\sigma_2)\theta_g(R) + \theta_h(R)] + \frac{KC^2}{\delta\sigma_2 - \gamma}.$$

Passing to the limit as $n \to \infty$ in the last relation, we get a contradiction to (4.2). □

Lemma 4.2. *Let* (**A***), (**G***), (**H***), *and* (**F***) *hold. Then we have*

$$\chi^*(F(D)) \leq [M_\infty \eta_g + 4(\eta_h + \sigma_2 \xi_g)\Lambda_\infty + 8\Theta_\infty] \chi^*(D), \tag{4.10}$$

for all bounded set $D \subset BC_R^\gamma(\beta)$.

Proof. Let $D \subset BC_R^\gamma(\beta)$ be a bounded set. We have

$$\chi^*(F(D)) = \chi_\infty(F(D)) + d_\infty(F(D)). \tag{4.11}$$

1. Thanks to Lemma 3.4, we obtain following estimates

$$\chi_\infty(F(D)) \leq \chi_\infty(F_1(D)) + \chi_\infty(F_2(D)) + \chi_\infty(F_3(D)), \tag{4.12}$$

and

$$\chi_\infty(F_1(D)) \leq M_\infty \eta_g \chi_\infty(D), \tag{4.13}$$

$$\chi_\infty(F_2(D)) \leq 4(\eta_h + \sigma_2 \xi_g)\Lambda_\infty \chi_\infty(D), \tag{4.14}$$

$$\chi_\infty(F_3(D)) \leq 8\Theta_\infty \chi_\infty(D). \tag{4.15}$$

From (4.12)–(4.15), we have

$$\chi_\infty(F(D)) \leq [M_\infty \eta_g + 4(\eta_h + \sigma_2 \xi_g)\Lambda_\infty + 8\Theta_\infty] \chi_\infty(D). \tag{4.16}$$

2. Now let $D \subset BC_R^\gamma(\beta)$ be a bounded set. Then, for all $u \in D$, we have

$$e^{\gamma t} \|F(u)(t)\|_X \leq \beta \quad \text{as} \quad t \to \infty.$$

This means $\|F(u)(t)\|_X \leq \beta e^{-\gamma t}$, $\forall u \in D$, for all large t. Equivalently, for a large T, one has $d_T(F(D)) \leq \beta e^{-\gamma T}$.

Therefore

$$d_\infty(F(D)) = \lim_{T\to\infty} d_T(F(D)) = 0. \tag{4.17}$$

Taking into account (4.11), (4.16), and (4.17), one gets the conclusion of the lemma. □

Combining Lemma 4.1 and Lemma 4.2 we get the following theorem.

Theorem 4.3. *If the assumptions of Lemma* 4.1 *are satisfied and the inequality*

$$l_\infty := M_\infty \eta_g + 4(\eta_h + \sigma_2 \xi_g)\Lambda_\infty) + 8\Theta_\infty < 1 \tag{4.18}$$

takes place, then the problem (1.1)–(1.2) *has at least one mild solution on* $\mathbb{R}^+$ *such that* $e^{\gamma t}\|u(t)\| = O(1)$ *as* $t \to +\infty$.

Proof. By inequality (4.18), the solution operator F is a χ^* condensing, thanks to Lemma 4.2. Indeed, if $D \subset BC_R^\gamma(\beta)$ is a bounded such that $\chi^*(D) \le \chi^*(F(D))$. Applying Lemma 4.2, we obtain

$$\chi^*(D) \le \chi^*(F(D)) \le l_\infty \chi^*(D).$$

Therefore $\chi^*(D) = 0$, and so D is relatively compact.

By assumption (4.1), (4.2) and Lemma 4.1, we have $F(BC_R^\gamma(\beta)) \subset BC_R^\gamma(\beta)$. So applying Theorem 2.7, the solution operator F defined by (3.9) has a compact and nonempty fixed point set in $BC_R^\gamma(\beta)$, which contains decay solutions of the problem (1.1)–(1.2). □

5. AN EXAMPLE

Let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$. We consider the following problem:

$$u_{tt}(t,x) - \rho\Delta_x u_t(t,x) + \Delta_x^2 u(t,x) = f(t,x,u(t,x)), \quad t > 0, \quad x \in \Omega, \tag{5.1}$$

$$u(0,x) + \int_\Omega k(x,y)u(0,y)dy = \varphi(x), \quad u_t(0,x) + \sum_{i=1}^N C_i u(t_i,x) = \psi(x), \tag{5.2}$$

$$u\big|_{\partial\Omega} = 0, \tag{5.3}$$

where $f : \mathbb{R}^+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$, $\psi \in L^2(\Omega)$, and

$$k \in L^2(\Omega\times\Omega) \text{ such that } \Delta_x k \in L^2(\Omega\times\Omega).$$

Let $X = L^2(\Omega)$ and $\mathcal{A} = -\Delta_x$, with the domain $\mathcal{D}(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega)$. It is known that (see, e.g. [15]), $-\mathcal{A}$ generates a compact (and hence equicontinuous) semigroup $T(\cdot)$, which is exponential stable, i.e., $\|T(t)\| \le e^{-\lambda_1 t}$, with $\lambda_1 > 0$ being the first eigenvalue of $\mathcal{A}$. Writing

$$u(t) = u(t,\cdot), \quad f(t,v) = f(\cdot, v(\cdot)),$$

$$g(u) = \int_\Omega k(\cdot,y)u(0,y)dy, \quad h(u) = \sum_{i=1}^N C_i u(t_i,\cdot),$$

we can transform the problem (5.1), (5.3) to the abstract form (1.1)–(1.2). Concerning the nonlinear function f, we assume that, there exists a function $m \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ such that

$$|f(t,x,z)| \le m(t)|z|, \quad \forall (t,x,z) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}.$$

Then we have

$$\|f(t,v)\|_X \le m(t)\|v\|_X, \quad \forall v \in X.$$

Noting that the operator

$$G(v) = \int_\Omega k(\cdot,y)v(y)dy$$

is of Hilbert–Schmidt type as an operator on $L^2(\Omega)$, then it is compact. This implies the map g defined by $g(u) = G(u(0,\cdot)$ is also compact. Similarly, $\mathcal{A}g$ is a compact mapping. So the condition (**G***)(ii) is satisfied with $\eta_g = \xi_g = 0$. In addition, we see that

$$\|g(u)\|_X \le \|k\|_{L^2(\Omega\times\Omega}\|u(0,\cdot)\|_X \le \|k\|_{L^2(\Omega\times\Omega)}\|u\|_C.$$

Thus (**G***)(i) is testified.

Regarding the function h, we have

$$\|h(u_1) - h(u_2)\|_X \le \sum_{i=1}^N |C_i|\,\|u_1(t_i,\cdot) - u_2(t_i,\cdot)\|_X \le \left(\sum_{i=1}^N |C_i|\right)\|u_1 - u_2\|_C.$$

Then

$$\chi(h(\Omega)) \le \left(\sum_{i=1}^N |C_i|\right)\chi_\infty(\Omega).$$

The assumption (**H***)(ii) is satisfied with $\eta_h = \sum_{i=1}^N |C_i|$. On the other hand, it is easily seen that

$$\|h(u)\| \le \left(\sum_{i=1}^N |C_i|\right)\|u\|_\infty,$$

which implies (**H***)(i).

By the above settings, by simple computations one gets

$$M_\infty = 1, \quad \Theta_\infty = 0, \quad \Lambda_\infty \le [\lambda_1(\sigma_2 - \sigma_1)]^{-1},$$

$$K = \sup_{t\ge 0}\int_0^t e^{-(\lambda_1\sigma_1-\gamma)(t-\tau)}m(\tau)d\tau,$$

$$\theta_g(r) = \|k\|_{L^2(\Omega\times\Omega)}\cdot r, \quad \theta_h(r) = \left(\sum_{i=1}^N |C_i|\right)\cdot r.$$

Applying Theorem 4.3 we can conclude that, the problem (5.1)–(5.3) has at least on solution $u \in BC(\mathbb{R}^+; L^2(\Omega))$ satisfying $e^{\gamma t}\|u(t)\|_{L^2(\Omega)} = O(1)$ as $t \to +\infty$ provided that

$$K < \lambda_1\sigma_2 - \gamma, \tag{5.4}$$

$$\|k\|_{L^2(\Omega\times\Omega)} + \left(\sum_{i=1}^N |C_i| + \sigma_2\|k\|_{L^2(\Omega\times\Omega)}\right)[\lambda_1(\sigma_2 - \sigma_1)]^{-1} + \Upsilon_\infty < 1. \tag{5.5}$$

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