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ASTRONOMY

On the Representation of the Gravitational Potential of Several Model Bodies

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Abstract—A Laplace series of spherical harmonics $Y_n(\theta, \lambda)$ is the most common representation of the gravitational potential for a compact body T in outer space in spherical coordinates r, θ, λ. The Chebyshev norm estimate (the maximum modulus of the function on the sphere) is known for bodies of an irregular structure:$\langle Y_n\rangle \le Cn^{-5/2}$, C = const, $n \ge 1$. In this paper, an explicit expression of $Y_n(\theta, \lambda)$ for several model bodies is obtained. In all cases (except for one), the estimate $\langle Y_n\rangle$ holds under *the exact exponent* 5/2. In one case, where the body T touches the sphere that envelops it,$\langle Y_n\rangle$ decreases much faster, viz.,$\langle Y_n\rangle \le Cn^{-5/2}p^n$, C = const, $n \ge 1$. The quantity $p < 1$ equals the distance from the origin of coordinates to the edge of the surface T expressed in enveloping sphere radii. In the general case, the exactness of the exponent 5/2 is confirmed by examples of bodies that more or less resemble real celestial bodies [16, Fig. 6].

Keywords: gravitational potential, Laplace series, rate of convergence.

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INTRODUCTION

Let us consider the gravitational potential V of the compact body T in outer space in spherical coordinates r, θ, λ. In [1–5], a Laplace series was proposed in order to represent this potential.

$$V(r,\theta,\lambda) = \frac{M}{R}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1} Y_n(\theta,\lambda). \tag{1}$$

Here, M is the mass of T, R is the scale factor, Y_n is the dimensionless spherical harmonic, and the gravitational constant is taken to be unity. In the general case, the spherical harmonic depends on $2n + 1$ parameters (Stokes coefficients). Effective methods for determining the Stokes coefficients by satellite measurements have been developed and put into practice [6–8]. In the case of the axial symmetry, $Y_n(\theta, \lambda) \equiv Y_n(\theta) = c_nP_n(\cos\theta)$ holds, and only one parameter c_n remains. As usual, P_n denotes a Legendre polynomial with standard normalization $P_n(1) = 1$. Equation (1) takes the form

$$V(r,\theta) = \frac{M}{R}\sum_{n=0}^{\infty} c_n\left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta). \tag{2}$$

As is customary in theoretical investigations, let us designate the radius of the enveloping sphere that contains T and that has at least one common point with T as R.

The rate of convergence of series (1) significantly depends on the smoothness of the mass distribution in the body T. The higher the smoothness is, the faster the series converges, as is the case in the theory of functional approximation by truncated series [9–11]. However, in this case it is quite difficult to determine the concept of smoothness [12]. Thus, the density can experience even discontinuities at the intersection of the surfaces of equal density, if only they are sufficiently smooth. Simultaneously, if the surface ∂T of an even homogeneous body T has edges, it dramatically reduces the smoothness of the mass distribution, since ∂T loses smoothness, which can be considered as the surface of equal density.

For applications, one of the most interesting classes of bodies is the class $\mathcal{T}$ of compact bodies with bounded integrable density $\varrho(r, \theta, \lambda)$, which has uniformly bounded variation along any circle centered at the origin of the coordinates. All real celestial bodies belong to this class. For bodies $T \in \mathcal{T}$, the estimate is known [12]

$$\langle Y_n \rangle \le \frac{C}{n^{5/2}}. \tag{3}$$

Here and below, different constants depending on the properties of density ϱ are denoted by C; $\langle \cdot \rangle$ is the Chebyshev norm (the maximum modulus of the function on a sphere). We suppose that $n \ge 1$, since Y_0 is identically equal to one.

We note that such an estimate (with the divisor n^2 instead of $n^{5/2}$) was obtained for the first time by M.S. Yarov–Yarovoi [13].

In the axially symmetric case, we have the following

$$\langle Y_n \rangle = |c_n|. \tag{4}$$

Estimate (3) is exact in the following sense. There is a body $T \in \mathcal{T}$ such that for some C inequality (3) holds but the following is valid

$$\sup n^{\sigma} \langle Y_n \rangle = \infty \tag{5}$$

for any fixed $\sigma > 5/2$. Several supporting examples are given in [12]. In this paper, we extend the list of examples, while still limiting ourselves to homogeneous bodies of revolution, for which (4) holds true. In these model examples (for the first four Stokes coefficients, c_n are taken from [12]), the bodies resemble the real planets and moons slightly. However, we then construct more realistic shapes for them as from elements.

1. A HEMISPHERE IN THE REFERENCE FRAME WITH THE ORIGIN AT THE CENTER OF THE SPHERE

For even positive n, we have $c_n = 0$. For odd n, the following holds

$$c_n = 3(-1)^{\frac{n-1}{2}} \frac{(n-2)!!}{(n+3)!!}.$$

Applying the Wallis formula, we obtain

$$|c_n| \sim \frac{3\sqrt{2/\pi}}{n^{5/2}}. \tag{6}$$

2. A SPHERICAL SECTOR IN THE REFERENCE FRAME WITH ITS ORIGIN AT THE TOP OF THE SECTOR

Let us denote the sector half angle by α. Then, we have

$$c_n = \frac{-3}{(1-\cos\alpha)(n+3)} P_{n1}(\cos\alpha).$$

Here, P_{nk} are defined in the APPENDIX together with asymptotics (16), which implies

$$c_n \sim \frac{-3\sqrt{2/\pi}\sqrt{\sin\alpha}}{(1-\cos\alpha)n^{5/2}} \cos\left[\left(n+\frac{1}{2}\right)\alpha + \frac{\pi}{4}\right]. \tag{7}$$

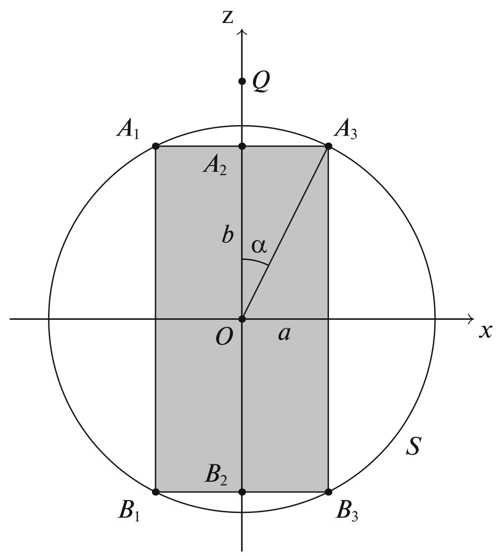

Fig. 1. The cylinder cross section by a plane passing through the axis of symmetry z.

From the cosine sequence, for any α, it is possible to select a sequence separated from zero, which proves the accuracy of estimates (3).

Remark. Formula (6) follows from (7) and is given because of its simplicity.

3. A CYLINDER IN THE REFERENCE FRAME WITH THE ORIGIN AT ITS CENTER AND THE Z AXIS DIRECTED ALONG THE AXIS OF SYMMETRY

Let us denote the base radius by a and the cylinder height by $2b$ (see Fig. 1; $A_2A_3 = a$, $OA_2 = b$, $OA_3 = R = \sqrt{a^2+b^2}$, $\angle A_2OA_3 = \alpha$). Then, for odd n, we will have $c_n = 0$, and for even n:

$$c_n = -\frac{2R^3}{a^2b(n+1)}P_{n+2,1}\left(\frac{b}{R}\right).$$

Taking the designation of the angle A_2OA_3 by α into account let us represent the last formula in the form

$$c_n = -\frac{2}{(n+1)\sin^2\alpha\cos\alpha}P_{n+2,1}(\cos\alpha).$$

Taking the asymptotics (16) into account we obtain

$$c_n = -\frac{2\sqrt{2/\pi}}{n^{5/2}\sin^{3/2}\alpha\cos\alpha}\cos\left[\left(n+\frac{5}{2}\right)\alpha+\frac{\pi}{4}\right].$$

4. A CONE IN THE REFERENCE FRAME WITH THE ORIGIN AT ITS APEX

Let us denote the cone half angle by α. Then, we have

$$c_n = -\frac{6}{(n+3)\sin^2\alpha}P_{n+1,1}(\cos\alpha),$$

whence it follows

$$c_n \sim -\frac{6}{n^{5/2}\sin^{3/2}\alpha}\cos\left[\left(n+\frac{3}{2}\right)\alpha+\frac{\pi}{4}\right].$$

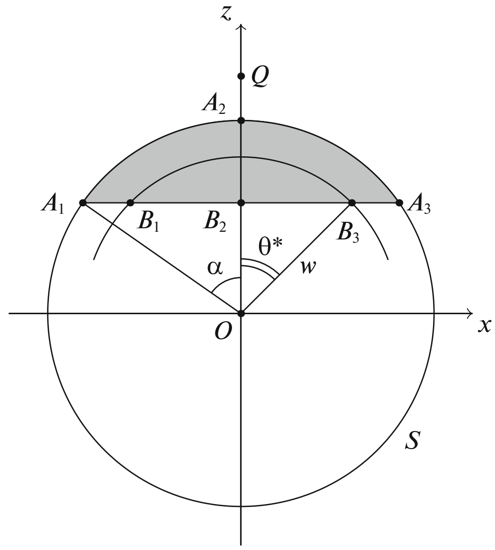

Fig. 2. The cross section of the spherical segment by a plane passing through the axis of symmetry z; $OA_1 = OA_2 = OA_3 = \alpha$, $\angle A_1OA_2 = \alpha$; $OB_2 = a\cos\alpha$.

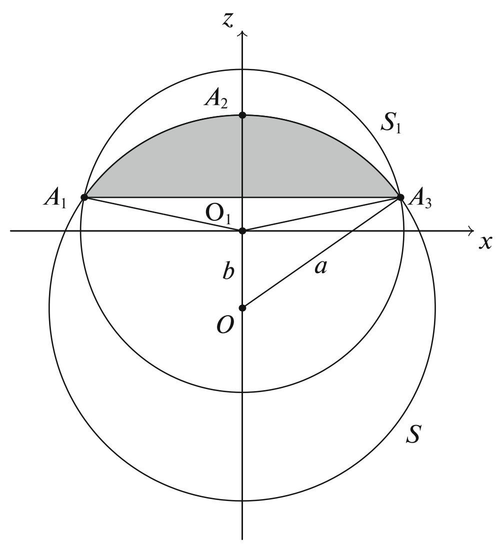

Fig. 3. The cross section of the spherical segment in the reference frame O_1.

5. A SPHERICAL SEGMENT IN DIFFERENT REFERENCE FRAMES

Let us consider a spherical segment T with a radius a and a half-angle α, $0 < \alpha \le \pi/2$. Let us investigate the Laplace series T in coordinate systems with the z axis along the symmetry axis but with different positions of the point of origin.

5.1. A Reference Frame with the Origin at the Center of the Corresponding Sphere

Figure 2 shows the section T in the form of the plane passing through the axis of symmetry; for fixed $w = OB_1 = OB_3$, the angle θ varies from 0 to θ^*, $\cos\theta^* = (a/w)\cos\alpha$.

Stokes constants were calculated in [14]:

$$c_n = \frac{3}{2(2+\cos\alpha)\sin^4(\alpha/2)} P_{n+1,2}(\cos\alpha). \tag{8}$$

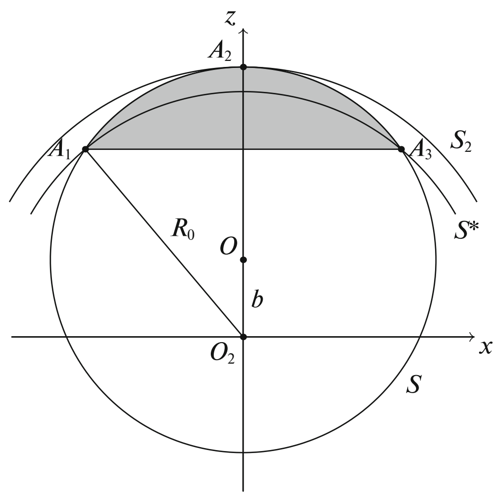

Fig. 4. The cross section of the spherical segment in the reference frame O_2.

Using (16), we find the asymptotics

$$c_n \sim \frac{6}{(2+\cos\alpha)n^{5/2}}\sqrt{\frac{\cos^3(\alpha/2)}{\pi\sin^5(\alpha/2)}}\cos\left[\left(n+\frac{3}{2}\right)\alpha+\frac{3\pi}{4}\right].$$

5.2. A Reference Frame with the Origin Displaced Upward

Let the origin be displaced upward by a distance of $b > 0$ (see Fig. 3; $OO_1 = b$, $R = O_1A_1 = \sqrt{a^2 + 2ab\cos\alpha + b^2}$. The enveloping sphere $\boldsymbol{S}_1$ passes through the points A_1 and A_3). The quantity c_n cannot be expressed explicitly in terms of n, α, a, b. However, in [14], formulas (3) and (5) were proved for T, which is sufficient for our purposes.

5.3. A Reference Frame with the Origin Displaced Downward

Let the origin be displaced downward by a distance of $b > 0$ (see Fig. 4). In this case, $O_2O = b$, $R_0 = O_2A_1 = \sqrt{a^2 + 2ab\cos\alpha + b^2}$, $R = a + b$, $R > R_0$. The enveloping sphere $\boldsymbol{S}_2$ passes through the point A_2. The circle with the center at O_2 that passes through A_1 and A_3 represents a cross section of the boundary of the convergence region of the Laplace series $\boldsymbol{S}^*$. In [14], an estimate significantly stronger than (3) was obtained:

$$\langle Y_n \rangle \le \frac{C}{n^{5/2}} p^n, \tag{9}$$

where

$$p = \frac{\sqrt{a^2 + 2ab\cos\alpha + b^2}}{a+b} < 1.$$

A sphere with its center at O_2 that passes through the points A_1 and A_3 is the convergence boundary of the Laplace series $\boldsymbol{S}^*$. Note that the cross sections of segments and spheres are shown in the figures. In the space, segment edge corresponds to the points A_1 and A_3, so that the sphere and the segment have a common circle.

6. A SEGMENTED SPHERE

Bodies that slightly resemble the real planets and moons were investigated above. Let us now construct more realistic shapes of the segments of the same radius a. At this point, the reference frame with the origin at the center of the generating sphere is used.

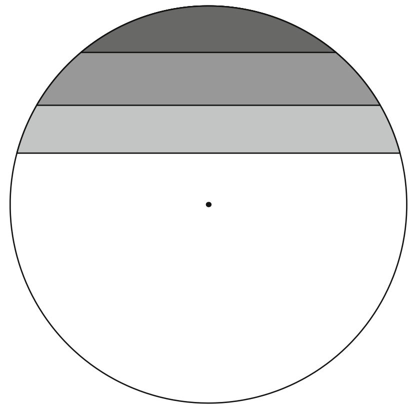

Fig. 5. The cross section of the body T_2 at $K = 3$, $K' = 0$.

Let T_1 be the reflection of T with respect to the equatorial plane, the southern segment. The gravitational potential T_1 at the point with Cartesian coordinates x, y, z coincides with the potential T at the point $x, y, -z$. Therefore, Stokes coefficients c_n for T_1 coincide with those for T multiplied by $(-1)^n$. Instead of (8), for T_1 at $n \geq 2$ we obtain

$$Mc_n = 2(-1)^n \pi a^3 \varrho P_{n+1,2}(\cos\alpha). \tag{10}$$

Here, we take the fact into account that in the case of the merger of bodies located inside the sphere $r \leq R$ harmonics MY_n are combined instead of Y_n.

Let T_2 be the combination of the northern segments with the parameters α_i, ϱ_i and southern segments with the parameters $\alpha'_{i'}$, $\varrho'_{i'}$, $i = 1, ..., K$, $i' = 1, ..., K'$. One of the numbers K, K' can be zero; then the corresponding sum in the expression c_n (see below) is considered to be zero. Without loss of generality, we consider the sequences α_i and $\alpha'_{i'}$ to be increasing ones. Figure 5 shows the body T_2 for $K = 3$, $K' = 0$.

Since the segments are nested within each other, in fact in the northern hemisphere there is one segment half angle α_K and in the southern, with the angle $\alpha'_{K'}$. The density in the northern hemisphere as one moves towards the equator successively takes the values $\varrho_1 + ... + \varrho_K$, $\varrho_2 + ... + \varrho_K$, ..., ϱ_K. The situation in the southern hemisphere is similar when moving toward the equator. Therefore, some of the values of ϱ_i, $\varrho'_{i'}$ can be negative, while maintaining the positive density of the body T_2. Note that T_2 is a nonuniform sphere, if $\alpha_K = \alpha'_{K'} = \pi/2$ is true.

In terms of potential additivity, the harmonic coefficients T_2 at $n \geq 2$ are

$$c_n = \frac{2\pi a^3}{M}\left[\sum_{i=1}^{K} \varrho_i P_{n+1,2}(\cos\alpha_i) + (-1)^n \sum_{i'=1}^{K'} \varrho'_i P_{n+1,2}(\cos\alpha'_{i'})\right]. \tag{11}$$

Let us consider some special cases of the above construction.

Let T_3 be T_2 in the case of symmetry of the northern and southern hemispheres, i.e., $K = K'$, $\alpha_i = \alpha'_{i'}$, $\varrho_i = \varrho'_{i'}$ holds true. Then, both sums in (11) are identical. Therefore, for odd n, we will have $c_n = 0$, and for even $n \geq 2$

$$c_n = \frac{4\pi a^3}{M}\sum_{i=0}^{K} \varrho_i P_{n+1,2}(\cos\alpha_i). \tag{12}$$

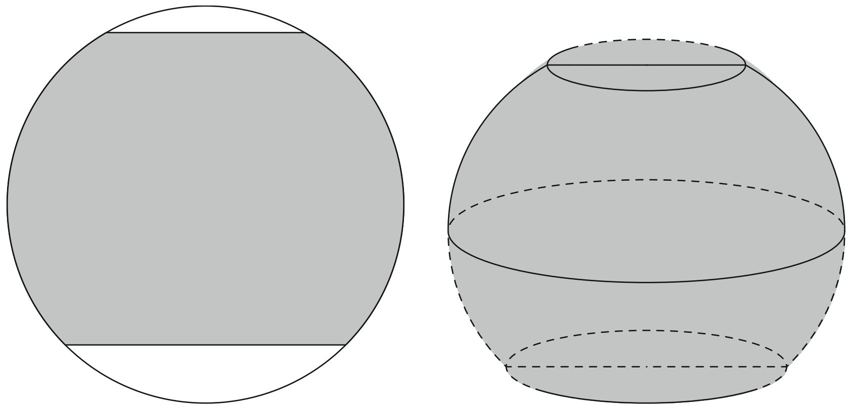

Fig. 6. The body T_4 (left) and its sectional view (left).

Let T_4 be a barrel-shaped homogeneous body with a density of ϱ obtained by cutting of the northern α and southern α' segment half angles from the sphere (Fig. 6).

The body T_4 is obtained by adding the body T_2 at $K = K' = 1$ and $\varrho_1 = \varrho_1' = -\varrho$ to the sphere with a density of ϱ. If one of the segments is missing we obtain a dome instead of a barrel. Since all the harmonics of the sphere (except for the zero one) disappear, for $n \geq 2$ we arrive at the equalities

$$c_n = \frac{2\pi a^3 \varrho}{M}[P_{n+1,2}(\cos\alpha) + (-1)^n P_{n+1,2}(\cos\alpha')]. \tag{13}$$

If the barrel is symmetrical, i.e., we have $\alpha = \alpha'$, odd harmonics disappear, and even harmonics are

$$c_n = -\frac{4\pi a^3 \varrho}{M} P_{n+1,2}(\cos\alpha). \tag{14}$$

CONCLUSIONS

We investigated the rate of convergence of the Laplace series of several model bodies, viz., a hemisphere, spherical sector, cylinder, cone, spherical segment, and segmented sphere. In all cases (with one exception), the decrease rate of $\langle Y_n \rangle$ is described by the *best possible* estimate (3). An important role is played here by the shape S, the intersection of the boundary ∂T of the body T and the enveloping sphere $\boldsymbol{S}$.

In Examples 1, 2, and 5.1, the shape S consists of the part of the sphere $\boldsymbol{S}$ with a positive area. The boundary S is the edge of the surface ∂T. The segmented sphere can be attributed to this class of bodies. Boundaries between segments with different densities can be considered as edges.

In Examples 3, 4, and 5.2, S consists of curves that lie on the sphere $\boldsymbol{S}$ that represent the edges of the surface ∂T.

An exception is Example 5.3, where $\langle Y_n \rangle$ decreases much faster according to (9). In this case, the shape S is the point at which ∂T touches $\boldsymbol{S}$; in its vicinity the surface ∂T is analytic. Interestingly, the region of convergence of series (1) is the sphere $\boldsymbol{S^*}$, whose radius equals to the distance to the edge of the surface ∂T that lies within the sphere $\boldsymbol{S}$. Thus, in this case too the sphere of convergence $\boldsymbol{S^*}$ is determined by the edge of the surface ∂T, as in the previous cases.

The case where S consists of a finite number of points A_k, in whose neighborhood ∂T lies inside a cone with the vertex A_k with an axis passing through the point of origin and the half angle less than $\pi/2$ was not explored.

The model bodies (a hemisphere, spherical sector, cylinder, cone, and spherical segment) slightly resemble the real planets and moons. A segmented sphere is a more realistic shape for the representation of the gravitational potential of a celestial body.

APPENDIX

1. Generating functions. On the product of the segment $-1 \le x \le 1$ and the circle $|z| < 1$ the following expansions hold true

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n,$$

$$(1-2xz+z^2)^{1/2} = 1 - xz - \sum_{n=1}^{\infty} P_{n1}(x)z^{n+1}, \tag{A.1}$$

$$(1-2xz+z^2)^{3/2} = 1 - 3xz + \frac{3}{2}(x^2+1)z^2 + \left(\frac{1}{2}x^3 - \frac{3}{2}x\right)z^3 + 3\sum_{n=2}^{\infty} P_{n2}(x)z^{n+2}.$$

Here, $P_{n0} = P_n$ is the Legendre polynomial and P_{nk} are successive integrals

$$P_{nk}(x) = \int_{-1}^{x} P_{n,k-1}(y)dy.$$

2. The asymptotics P_{nk} is given by the expression

$$P_{nk}(\cos\theta) = \sqrt{\frac{2}{\pi}}\frac{\sin^{k-1/2}\theta}{n^{k+1/2}}\left\{\cos\left[\left(n+\frac{1}{2}\right)\theta + \left(k-\frac{1}{2}\right)\frac{\pi}{2}\right] + \frac{r_k(n,\theta)}{n\sin^{k+1}\theta}\right\}, \tag{A.2}$$

where $r_k(n, \theta)$ are bounded at $0 \le \theta \le \pi$, $n \ge 2$. Formulas (A.1) and (A.2) can be found in [15, 16].

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