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MATHEMATICS

Complex Vector Measure and Integral over Manifolds with Locally Finite Variations

A. V. Potepun
St. Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034 Russia
e-mail: apotepun@pochta.tvoe.tv
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Abstract—It is well known that any compactly supported continuous complex differential n-form can be integrated over real n-dimensional C^1 manifolds in $\mathbb{C}^m$ ($m \geq n$). For $n = 1$, the integral along any locally rectifiable curve is defined. Another generalization is the theory of currents (linear functionals on the space of compactly supported C^∞ differential forms). The topic of the article is the integration of measurable complex differential $(n, 0)$-forms (containing no $d\bar{z}_j$) over real n-dimensional C^0 manifolds in $\mathbb{C}^m$ with locally finite n-dimensional variations (a generalization of locally rectifiable curves to dimensions $n > 1$). The last result is that a real n-dimensional manifold C^1 embedded in $\mathbb{C}^m$ has locally finite variations, and the integral of a measurable complex differential $(n, 0)$-form defined in the article can be calculated by a well-known formula.

Key words: integration of differential form, complex vector measure, n vector, manifold with locally finite variations.

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As is known, the theory of the integration of differential forms on smooth manifolds can be generalized to piecewise smooth manifolds and, in the case of dimension 1, it can be generalized to rectifiable curves (see, e.g., [1, Vol. 3, Chapter 3]). In [2], the theory of integration of 1-forms along rectifiable curves was generalized to higher dimensions. Namely, the notions of an n-dimensional manifold in $\mathbb{R}^m$ with locally finite variations (which, in particular, include all manifolds with locally finite n-dimensional Hausdorff measure) and a certain vector measure on it taking values in the space of n vectors were introduced. The integral of a differential form was defined as integral with respect to this vector measure. The purpose of the present paper is to generalize the theory of integration of complex differential forms, namely, of forms of bidegree $(n, 0)$ (containing no $d\bar{z}_j$) to the case of integration over real n-dimensional manifolds with locally finite variations in $\mathbb{C}^m$. On such manifolds, a vector measure with values in the space of complex n vectors and the integral of a measurable differential form of bidegree $(n, 0)$ with respect to this measure are defined. In the case of a real n-dimensional manifold in $\mathbb{C}^n$, the vector measure can be identified with a measure, the values of which are complex numbers.

As is known, in the theory of functions of several complex variables, the most important are holomorphic differential forms (that is, bidegree forms $(n, 0)$ with holomorphic coefficients). In particular, for the form $f(z_1, \ldots, z_n)dz_1 \wedge \ldots \wedge dz_n$, the Cauchy–Poincaré theorem on the vanishing of the integral along the smooth boundary of an $(n + 1)$-surface is valid. We believe that this result can be generalized to the case of a boundary with finite n-dimensional Hausdorff measure.

At the end of the paper, we prove that, for a manifold smoothly embedded in $\mathbb{C}^m$, the definition of the integral of a differential form coincides with the conventional one.

Consider the mapping from $\mathbb{R}^{2m}$ to $\mathbb{C}^m$ defined by

$$I(x_1, y_1, x_2, y_2, \ldots, x_m, y_m) = (x_1 + iy_1, x_2 + iy_2, \ldots, x_m + iy_m)$$

(which identifies the elements of $\mathbb{R}^{2m}$ with those of $\mathbb{C}^m$). Denoting the basis vectors in $\mathbb{R}^{2m}$ by e_j, we have

$$I(e_j) = \begin{cases} \tilde{e}_k & \text{if } j = 2k - 1; \\ i\tilde{e}_k & \text{if } j = 2k, \end{cases}$$

where the $\tilde{e}_j$ are the basis vectors in $\mathbb{C}^m$.

Let $n \in \mathbb{N}$, $n \le m$. Using the mapping I, we define a mapping from the set of n vectors in $\mathbb{R}^{2m}$ over the field $\mathbb{R}$ to the set of n vectors over $\mathbb{C}$ as follows: given $j_1, \ldots, j_n \in \mathbb{N}$, $1 \le j_1 < \ldots < j_n \le 2m$, let $\alpha = \{j_1, \ldots, j_n\}$, and let $e_\alpha = e_{j_1} \wedge \ldots \wedge e_{j_n}$ be a simple n vector in $\mathbb{R}^{2m}$. We set $P(\alpha) = \beta$, where $\beta = \{k_1, \ldots, k_n\}$, $k_p = [(i_p + 1)/2]$ (the integer part of the fraction $\frac{i_p + 1}{2}$), and

$$J(e_\alpha) = I(e_{j_1}) \wedge \ldots \wedge I(e_{j_n});$$

this is a simple n vector in $\mathbb{C}^m$. It is clear from the definition of $I(e_j)$ that

$$J(e_\alpha) = i^{q(\alpha)}\tilde{e}_\beta, \quad \text{where } q(\alpha) \text{ is the number of even numbers } j_p \text{ in } \alpha. \tag{1}$$

Note that if $j_p = 2k - 1$ and $j_{p+1} = 2k$, then $k_p = k_{p+1}$ and $\tilde{e}_\beta = \tilde{e}_{k_1} \wedge \ldots \wedge \tilde{e}_{k_n} = \mathbf{0}$, i.e., $J(e_\alpha)$ vanishes.

In the general case, given an n vector $\sum_\alpha \lambda_\alpha e_\alpha$, we set

$$J\left(\sum_\alpha \lambda_\alpha e_\alpha\right) = \sum_\alpha \lambda_\alpha J(e_\alpha).$$

This is an n vector in $\mathbb{C}^m$.

Let $m \ge n$, and let M be an oriented manifold of real dimension n with locally finite variations embedded in $\mathbb{C}^m$. In [2], the following measure on the manifold M (which is embedded in $\mathbb{R}^{2m}$ in the case under consideration), the values of which are real n vectors was defined as follows:

$$\mu_M(E) = \sum_\alpha \mu_\alpha(E) e_\alpha, \quad E \in \mathfrak{R}_M,$$

where $\mu_\alpha(E)$ is the oriented measure of the α-projection of E and $\mathfrak{R}_M$ is the δ-ring of measurable subsets of M; recall that (see [2, Section 4, definition after Lemma 9])

$$\mathfrak{R}_M = \left\{ E \subset M \,|\, E \in \bigcap_\alpha \mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-, \ \mu_\alpha^+(E) \text{ and } \mu_\alpha^-(E) \text{ are finite for any } \alpha \right\}.$$

Here, $\mathfrak{A}_\alpha^+$ and $\mathfrak{A}_\alpha^-$ are σ-algebras on which the measures μ_α^+ and μ_α^-, respectively, are defined.

Definition 1. We define the complex n vector measure of a set E belonging to the δ-ring $\mathfrak{R}_M$ as the complex n vector

$$\tilde{\mu}_M(E) = J(\mu_M(E)) = \sum_\alpha \mu_\alpha(E) J(e_\alpha).$$

Obviously, in this formula, we can only consider the sum over those α for which $J(e_\alpha) \ne \mathbf{0}$; then, $P(\alpha) = \beta = \{k_1, \ldots k_n\}$, $1 \le k_1 < \ldots < k_n \le m$. Below, we always use the notation

$$\alpha = \{j_1, \ldots j_n\}, \quad \text{where } j_1, \ldots j_n \in \mathbb{N}, \quad 1 \le j_1 < \ldots < j_n \le 2m,$$
$$\beta = \{k_1, \ldots k_n\}, \quad \text{where } k_1, \ldots k_n \in \mathbb{N}, \quad 1 \le k_1 < \ldots < k_n \le m.$$

Consider all α such that $P(\alpha) = \beta$ for fixed β. We have (see (1)) $J(e_\alpha) = i^{q(\alpha)}\tilde{e}_\beta$, and the sum breaks into three groups of summands as follows:

$$\tilde{\mu}_M(E) = \sum_\beta \left(\sum_{P(\alpha) = \beta} i^{q(\alpha)} \mu_\alpha(E) \right) \tilde{e}_\beta = \sum_\beta \bar{\mu}_\beta(E) \tilde{e}_\beta, \tag{2}$$

where $\beta = \{k_1, \ldots, k_n\}$, $1 \le k_1 < \ldots < k_n \le m$, and

$$\bar{\mu}_\beta(E) = \sum_{P(\alpha) = \beta} i^{q(\alpha)} \mu_\alpha(E) \tag{3}$$

is the complex measure of the projection of the set E on the subspace of $\mathbb{C}^m$ generated by the vectors $\tilde{e}_{k_1}, \ldots, \tilde{e}_{k_n}$.

In the case $n = m$, the space of complex n vectors is one-dimensional ($\beta = \{1, 2, \ldots, m\}$ is the only possibility) and, therefore,

$$\tilde{\mu}_M(E) = \left(\sum_{P(\alpha) = \{1, 2, \ldots, m\}} i^{q(\alpha)} \mu_\alpha(E) \right) \tilde{e}_1 \wedge \ldots \wedge \tilde{e}_m.$$

In this case, instead of the n vector measure $\tilde{\mu}_M$, we can consider the measure on $\mathfrak{R}_M$, the values of which are complex numbers defined as

$$\bar{\mu}_M(E) = \bar{\mu}_{\{1, 2, \ldots, m\}}(E) = \sum_{P(\alpha) = \{1, 2, \ldots, m\}} i^{q(\alpha)} \mu_\alpha(E).$$

As is known [4, Chapter 1, Section 3], given an additive vector function m defined on the ring $\mathfrak{R}$ of subsets of a set T and taking values in a normed space X, its variation is defined as

$$|m|(A) = \sup\left\{ \sum_{k=1}^{p} \|m(A_k)\| \,\middle|\, \bigcup_{k=1}^{p} A_k \subset A, \text{ the } A_k \text{ belong to } \mathfrak{R} \text{ and are disjointed} \right\}$$

(here, $\|m(A_k)\|$ is the norm of $m(A_k)$ in the space X). A vector function m is said to have finite variations if $|m|(A) < \infty$ for any $A \in \mathfrak{R}$.

Lemma 1. *The measures* $\bar{\mu}_\beta$ *and* $\tilde{\mu}_M$ *have finite variations.*

Proof. (1) Suppose that $A \in \mathfrak{R}_M$, $\bigcup_{k=1}^{p} A_k \subset A$, and the A_k belong to $\mathfrak{R}_M$ and are disjoint. Then (see (3)),

$$\sum_{k=1}^{p} |\bar{\mu}_\beta(A_k)| = \sum_{k=1}^{p} \left| \sum_{P(\alpha) = \beta} i^{q(\alpha)} \mu_\alpha(A_k) \right| \le \sum_{k=1}^{p} \sum_{P(\alpha) = \beta} \left| i^{q(\alpha)} \mu_\alpha(A_k) \right|$$

$$= \sum_{k=1}^{p} \sum_{P(\alpha) = \beta} \left| \mu_\alpha^+(A_k) - \mu_\alpha^-(A_k) \right| \le \sum_{k=1}^{p} \sum_{P(\alpha) = \beta} \left| \mu_\alpha^+(A_k) + \mu_\alpha^-(A_k) \right|$$

$$= \sum_{P(\alpha) = \beta} \left(\sum_{k=1}^{p} \mu_\alpha^+(A_k) + \sum_{k=1}^{p} \mu_\alpha^-(A_k) \right) \le \sum_{P(\alpha) = \beta} (\mu_\alpha^+(A) + \mu_\alpha^-(A)).$$

Since $A \in \mathfrak{R}_M$, it follows that the numbers $\mu_\alpha^+(A)$ and $\mu_\alpha^-(A)$ are finite and

$$|\bar{\mu}_\beta|(A) = \sup\left\{ \sum_{k=1}^{p} |\bar{\mu}_\beta(A_k)| \right\} \le \sum_{P(\alpha) = \beta} (\mu_\alpha^+(A) + \mu_\alpha^-(A)) < \infty.$$

(2) The space of n vectors in $\mathbb{C}^m$ is finite-dimensional; therefore, all norms in this space are equivalent, and we can consider an arbitrary norm

$$\tilde{\mu}_M(A_k) = \sum_\beta \bar{\mu}_\beta(A_k) \tilde{e}_\beta.$$

The properties of the norm imply

$$\sum_{k=1}^{p} \|\tilde{\mu}_M(A_k)\| \le \sum_{k=1}^{p} \sum_\beta |\bar{\mu}_\beta(A_k)| \cdot \|\tilde{e}_\beta\| \le \sum_{k=1}^{p} \max_\beta \|\tilde{e}_\beta\| \sum_\beta |\bar{\mu}_\beta(A_k)|$$

$$\le \max_\beta \|\tilde{e}_\beta\| \sum_\beta \left(\sum_{k=1}^{p} |\bar{\mu}_\beta(A_k)| \right) \le \max_\beta \|\tilde{e}_\beta\| \sum_\beta \sum_{P(\alpha) = \beta} (\mu_\alpha^+(A) + \mu_\alpha^-(A)) \quad \text{(see item (1))}.$$

Hence,

$$|\tilde{\mu}_M|(A) = \sup\left\{\sum_{k=1}^{p}|\tilde{\mu}_M(A_k)|\right\} \le \max_\beta\|\tilde{e}_\beta\|\sum_\beta\sum_{P(\alpha)=\beta}(\mu_\alpha^+(A)+\mu_\alpha^-(A)) < \infty. \quad \bullet$$

Theorem 1. *Let M be an oriented manifold of real dimension n smoothly embedded in $\mathbb{C}^m$ $(m \ge n)$, and let $E \subset U = f(\mathbb{R}^n)$, where f is a positive parameterization (a C^1 diffeomorphism) of a neighborhood $U \subset M$ and $E \in \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-)$. Then, the set $f^{-1}(E)$ is Lebesgue measurable and, for each $\beta = \{k_1, \ldots, k_n\}$,*

$$E \in \mathfrak{R}_M \text{ implies } \bar{\mu}_\beta(E) = \int_{f^{-1}(E)} \det f'_\beta d\lambda_n,$$

$$E \in \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-) \text{ implies } |\bar{\mu}_\beta|(E) = \int_{f^{-1}(E)} |\det f'_\beta| d\lambda_n.$$

Here, if

$$f: \begin{cases} z_1 = x_1 + iy_1 = f_1(t_1, \ldots, t_n), \\ \vdots \\ z_m = x_m + iy_m = f_m(t_1, \ldots, t_n), \end{cases} \quad \text{and} \quad f' = \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \cdots & \frac{\partial f_1}{\partial t_n} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} & \cdots & \frac{\partial f_2}{\partial t_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial t_1} & \frac{\partial f_m}{\partial t_2} & \cdots & \frac{\partial f_m}{\partial t_n} \end{pmatrix},$$

then $\det f'_\beta$ *is the determinant of matrix composed of the rows of f' with numbers $k_1, \ldots, k_n$.*

Proof. (1) Let us write the coordinate functions of the mapping f in real form:

$$f: \begin{cases} x_1 = g_1(t_1, \ldots, t_n), \\ y_1 = g_2(t_1, \ldots, t_n), \\ \vdots \\ x_m = g_{2m-1}(t_1, \ldots, t_n), \\ y_m = g_{2m}(t_1, \ldots, t_n), \end{cases} \quad \text{i.e., } f_k = g_{2k-1} + ig_{2k}.$$

According to Theorem 2 of [3], the set $f^{-1}(E)$ is Lebesgue measurable and, as shown in the proof of the same theorem (the end of item 3.3), we have

$$\mu_{j_1 \ldots j_n}(E) = \mu_\alpha(E) = \int_{f^{-1}(E)} \det g'_\alpha d\lambda_n.$$

Thus, it follows from (3) that

$$|\bar{\mu}_\beta|(E) = \sum_{P(\alpha)=\beta} i^{q(\alpha)}\mu_\alpha(E) = \int_{f^{-1}(E)} \sum_{P(\alpha)=\beta} i^{q(\alpha)} \det g'_\alpha d\lambda_n. \tag{4}$$

We have

$$\det f'_\beta = \begin{vmatrix} \frac{\partial f_{k_1}}{\partial t_1} & \frac{\partial f_{k_1}}{\partial t_2} & \dots & \frac{\partial f_{k_1}}{\partial t_n} \\ \frac{\partial f_{k_2}}{\partial t_1} & \frac{\partial f_{k_2}}{\partial t_2} & \dots & \frac{\partial f_{k_2}}{\partial t_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{k_n}}{\partial t_1} & \frac{\partial f_{k_n}}{\partial t_2} & \dots & \frac{\partial f_{k_n}}{\partial t_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial g_{2k_1-1}}{\partial t_1} + i\frac{\partial g_{2k_1}}{\partial t_1} & \dots & \frac{\partial g_{2k_1-1}}{\partial t_n} + i\frac{\partial g_{2k_1}}{\partial t_n} \\ \frac{\partial g_{2k_2-1}}{\partial t_1} + i\frac{\partial g_{2k_2}}{\partial t_1} & \dots & \frac{\partial g_{2k_2-1}}{\partial t_n} + i\frac{\partial g_{2k_2}}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{2k_n-1}}{\partial t_1} + i\frac{\partial g_{2k_n}}{\partial t_1} & \dots & \frac{\partial g_{2k_n-1}}{\partial t_n} + i\frac{\partial g_{2k_n}}{\partial t_n} \end{vmatrix}$$

$$= \begin{pmatrix} \frac{\partial g_{2k_1-1}}{\partial t_1} & \dots & \frac{\partial g_{2k_1-1}}{\partial t_n} \\ \frac{\partial g_{2k_2-1}}{\partial t_1} & \dots & \frac{\partial g_{2k_2-1}}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{2k_n-1}}{\partial t_1} + i\frac{\partial g_{2k_n}}{\partial t_1} & \dots & \frac{\partial g_{2k_n-1}}{\partial t_n} + i\frac{\partial g_{2k_n}}{\partial t_n} \end{pmatrix} + i\begin{pmatrix} \frac{\partial g_{2k_1}}{\partial t_1} & \dots & \frac{\partial g_{2k_1}}{\partial t_n} \\ \frac{\partial g_{2k_2-1}}{\partial t_1} + i\frac{\partial g_{2k_2}}{\partial t_1} & \dots & \frac{\partial g_{2k_2-1}}{\partial t_n} + i\frac{\partial g_{2k_2}}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{2k_n-1}}{\partial t_1} + i\frac{\partial g_{2k_n}}{\partial t_1} & \dots & \frac{\partial g_{2k_n-1}}{\partial t_n} + i\frac{\partial g_{2k_n}}{\partial t_n} \end{pmatrix}.$$

Performing this procedure for each row of the initial determinant, which is the sum of the row of partial derivatives of g_{2k_p-1} and the row of partial derivatives of g_{2k_p} multiplied by i, we obtain

$$\det f'_\beta = \sum_{j_1=2k_1-1}^{2k_1} \sum_{j_2=2k_2-1}^{2k_2} \dots \sum_{j_n=2k_n-1}^{2k_n} i^{q(j_1,\dots,j_n)} \det g'_{j_1,\dots,j_n} = \sum_{P(\alpha)=\beta} i^{q(\alpha)} \det g'_\alpha,$$

where $q(\alpha)$ is the number of even indices in the set $\alpha = \{j_1, \dots, j_n\}$; thus (see (4)),

$$\bar{\mu}_\beta(E) = \int\limits_{f^{-1}(E)} \det f'_\beta d\lambda_n.$$

(2.1) Suppose that $\bigcup_{k=1}^p E_k \subset E$ and $E \in \bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha)$, where the E_k belong to $\mathfrak{R}_M$ and are disjoint. It follows from item (1) that

$$\sum_{k=1}^p |\bar{\mu}_\beta(E_k)| = \sum_{k=1}^p \left| \int\limits_{f^{-1}(E_k)} \det f'_\beta d\lambda_n \right| \le \sum_{k=1}^p \int\limits_{f^{-1}(E_k)} |\det f'_\beta| d\lambda_n$$

$$= \int\limits_{\bigcup_{k=1}^p f^{-1}(E_k)} |\det f'_\beta| d\lambda_n \le \int\limits_{f^{-1}(E)} |\det f'_\beta| d\lambda_n.$$

By the definition of variation, we have

$$|\bar{\mu}_\beta|(E) = \sup\left\{ \sum_{k=1}^p |\bar{\mu}_\beta(E_k)| \right\} \le \int\limits_{f^{-1}(E)} |\det f'_\beta| d\lambda_n. \tag{5}$$

(2.2) Suppose that $E \in \mathfrak{R}_M$ and $E \subset K \subset U$, where K is compact. Then $f^{-1}(K)$ is compact as well (f is a homeomorphism of $\mathbb{R}^n$ onto U) and the continuous function $\det f'_\beta$ is uniformly continuous on $f^{-1}(K)$, i.e.,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall t, t' \in f^{-1}(K) \quad |t - t'| \le \delta \Rightarrow |\det f'_\beta(t) - \det f'_\beta(t')| \le \varepsilon. \tag{6}$$

Consider the cover of $f^{-1}(K)$ by open balls of diameter δ and choose its finite subcover: $f^{-1}(K) \subset \bigcup_{k=1}^{p} B_k$. It easily follows from $K \subset U = f(\mathbb{R}^n)$ that $K = f(f^{-1}(K)) \subset \bigcup_{k=1}^{p} f(B_k)$. The mapping f is a homeomorphism of $\mathbb{R}^n$ and U; therefore, the sets $f(B_k)$ are open in the open neighborhood $U \subset M$ and, hence, in M, i.e., they are α^+-measurable for any $\alpha = \{j_1, \ldots, j_n\}$ (see [2, Section 4, Lemma 8]). A similar argument proves the measurability of α^-- (see [2, Section 4, remark after Lemma 9]); thus,

$$f(B_k) \in \bigcap_{\alpha} (\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-) \Rightarrow E \cap f(B_k) \in \bigcap_{\alpha} (\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-),$$

$$\mu_\alpha^+(E \cap f(B_k)) \le \mu_\alpha^+(E) < \infty, \quad \mu_\alpha^-(E \cap f(B_k)) \le \mu_\alpha^-(E) < \infty,$$

i.e., $E \cap f(B_k) \in \mathfrak{R}_M$. We set

$$E_1 = E \cap f(B_1), \quad E_k = (E \cap f(B_k)) \setminus \bigcup_{s=1}^{k-1} (E \cap f(B_k)), \quad \text{for } k = 2, \ldots, p.$$

It is easy to verify that the sets E_k are disjointed, $E = \bigcup_{k=1}^{p} E_k$ and, based on the properties of a ring of sets, all values of E_k belong to $\mathfrak{R}_M$. The mapping f is injective; therefore, $f^{-1}(f(B_k)) = B_k$, from which we have

$$f^{-1}(E_k) \subset f^{-1}(E \cap f(B_k)) \subset f^{-1}(E) \cap B_k \subset B_k \Rightarrow \operatorname{diam}(f^{-1}(E_k))$$
$$\le \delta \Rightarrow (\text{see } (6)) \quad \forall t, t_k \in E_k \quad \left|\det f'_\beta(t) - \det f'_\beta(t_k)\right| \le \varepsilon$$
$$\Rightarrow \left|\det f'_\beta(t)\right| \le \left|\det f'_\beta(t_k)\right| + \left|\det f'_\beta(t) - \det f'_\beta(t_k)\right| \le \varepsilon + \left|\det f'_\beta(t_k)\right|.$$

Moreover, the sets $f^{-1}(E_k)$ are Lebesgue measurable (see [3, Theorem 2]). Thus, for fixed $t_k \in E_k$, we obtain

$$\int_{f^{-1}(E_k)} \left|\det f'_\beta\right|(t) d\lambda_n \le \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\det f'_\beta(t_k)\right| \cdot \lambda_n(f^{-1}(E_k))$$

$$= \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\det f'_\beta(t_k) \cdot \lambda_n(f^{-1}(E_k))\right| = \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\int_{f^{-1}(E_k)} \det f'_\beta(t_k) d\lambda_n\right|$$

$$= \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\int_{f^{-1}(E_k)} \det f'_\beta(t) d\lambda_n + \int_{f^{-1}(E_k)} (\det f'_\beta(t_k) - \det f'_\beta(t)) d\lambda_n\right|$$

$$\le \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\int_{f^{-1}(E_k)} \det f'_\beta(t) d\lambda_n + \int_{f^{-1}(E_k)} \left|\det f'_\beta(t_k) - \det f'_\beta(t)\right| d\lambda_n\right|$$

$$\le \varepsilon \cdot \lambda_n(f^{-1}(E_k)) + \left|\int_{f^{-1}(E_k)} \det f'_\beta(t) d\lambda_n\right| + \varepsilon \cdot \lambda_n(f^{-1}(E_k))$$

$$= 2\varepsilon \cdot \lambda_n(f^{-1}(E_k)) + |\bar{\mu}_\beta(E_k)| \quad (\text{see item } (1)).$$

Since the sets E_k are disjointed and $E = \bigcup_{k=1}^{p} E_k$, it follows that the sets $f^{-1}(E_k)$ are disjoint as well and $f^{-1}(E) = \bigcup_{k=1}^{p} f^{-1}(E_k)$. As a result, we obtain

$$\int_{f^{-1}(E)} \left|\det f'_\beta\right|(t) d\lambda_n = \sum_{k=1}^{p} \int_{f^{-1}(E_k)} \left|\det f'_\beta\right|(t) d\lambda_n \le 2\varepsilon \sum_{k=1}^{p} \lambda_n(f^{-1}(E_k)) + \sum_{k=1}^{p} |\bar{\mu}_\beta(E_k)|$$

$$\le 2\varepsilon \cdot \lambda_n(f^{-1}(E)) + |\bar{\mu}_\beta|(E) \quad (\text{by the definition of the variation of a measure}),$$

$$\text{i.e.,} \quad \int\limits_{f^{-1}(E)} |\det f'_\beta|(t)d\lambda_n \le 2\varepsilon \cdot \lambda_n(f^{-1}(E)) + |\bar{\mu}_\beta|(E). \tag{7}$$

We have $f^{-1}(E) \subset f^{-1}(K)$, and the set $f^{-1}(K)$ is compact; therefore, $\lambda_n(f^{-1}(E)) < \infty$. Passing to the limit as $\varepsilon \longrightarrow 0$ in (7), we obtain

$$\int\limits_{f^{-1}(E)} |\det f'_\beta|(t)d\lambda_n \le |\bar{\mu}_\beta(E)| \overset{\text{see (5)}}{\Rightarrow} \int\limits_{f^{-1}(E)} |\det f'_\beta|(t)d\lambda_n = |\bar{\mu}_\beta|(E).$$

(2.3) Let $C_k = \prod_{j=1}^{n}[-k; k]$; then $\mathbb{R}^n = \bigcup_{k=1}^{\infty} C_k$ and $U = f(\mathbb{R}^n) = \bigcup_{k=1}^{\infty} f(C_k)$. For any $E \in \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-)$, $E \subset U$, we set

$$E_1 = E \cap f(C_1), \quad E_k = (E \cap f(C_k)) \backslash \bigcup_{s=1}^{k-1}(E \cap f(C_k)), \quad \text{for } k = 2, 3, \ldots.$$

It is easy to verify that the E_k are disjoint and $E = \bigcup_{k=1}^{\infty} E_k$. The mapping f is a homeomorphism; therefore, the sets $f(C_k)$ are compact and, by Theorem 6 of [2, Section 4], $f(C_k) \in \mathfrak{R}_M \subset \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-)$. Properties of a σ-algebra imply

$$f(C_k) \in \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-) \Rightarrow E_k \in \bigcap_\alpha(\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-),$$

$$\mu_\alpha^+(E_k) \le \mu_\alpha^+(f(C_k)) < \infty, \quad \mu_\alpha^-(E_k) \le \mu_\alpha^-(f(C_k)) < \infty,$$

i.e., all E_k belong to $\mathfrak{R}_M$. Since $\bar{\mu}_\beta$ is countably additive, it follows that the variation $|\bar{\mu}_\beta|$ is as well (see [4, Chapter 1, Section 3]). From the inclusions $E_k \in f(C_k)$ and item (2.2), we conclude that

$$|\bar{\mu}_\beta|(E) = \sum_{k=1}^{\infty}|\bar{\mu}_\beta|(E_k) = \sum_{k=1}^{\infty}\int\limits_{f^{-1}(E_k)} |\det f'_\beta|(t)d\lambda_n = \int\limits_{f^{-1}(E)} |\det f'_\beta|(t)d\lambda_n. \quad \bullet$$

Definition 2. Let $m \ge n$, and let M be an oriented n-manifold in $\mathbb{C}^m$ with locally finite variations. A function $\omega : M \times (\mathbb{C}^m)^n \longrightarrow \mathbb{C}$ is called a step differential n-form on M if there exist finite families of sets $\{E_k\}_{k=1}^p$, $E_k \in \mathfrak{R}_M$, and exterior n-forms $\{\varphi_k\}_{k=1}^p$ over the field $\mathbb{C}$ such that

$$\omega = \sum_{k=1}^{p}\varphi_k\chi_{E_k} \quad (\chi_{E_k} \text{ denotes the characteristic function of the set } E_k).$$

Since $\mathfrak{R}_M$ is a δ-ring, we can assume that the sets E_k are disjoint [4, Chapter 2, Section 6, Subsection 1].

As is known [5, Chapter 3, Section 5, note], the exterior n-form φ generates a $\mathbb{C}$-linear functional $\tilde{\varphi}$ on the space of n vectors, which is defined at the basis n vectors by

$$\tilde{\varphi}(\tilde{e}_{i_1} \wedge \tilde{e}_{i_2} \wedge \ldots \wedge \tilde{e}_{i_n}) = \varphi(\tilde{e}_{i_1}, \tilde{e}_{i_2}, \ldots, \tilde{e}_{i_n}). \tag{8}$$

Therefore, we can define the integral of a step differential form with respect to the vector measure $\tilde{\mu}_M$ as follows [4, Chapter 2, Section 7, Subsection 1]:

$$\text{if } \omega = \sum_{k=1}^{p}\varphi_k\chi_{E_k}, \quad \text{then } \int\omega d\tilde{\mu}_M = \sum_{k=1}^{p}\tilde{\varphi}_k(\tilde{\mu}_M(E_k)). \tag{9}$$

Defining the integral of not necessarily step differential forms requires that the vector measure have additional properties. In the spaces of exterior n-forms and n vectors, the dual norms of any exterior n-form φ and n vector v satisfy the inequality $\|\tilde{\varphi}(v)\| \le \|\varphi\| \cdot \|v\|$ ($\tilde{\varphi}$ is the linear functional corresponding to the form φ). Moreover, $\mathfrak{R}_M$ is a δ-ring, and the measure $\tilde{\mu}_M$ has finite variation. Thus, we can use the theory of integration of vector functions with respect to vector measures (see [4, Chapter 2, Section 8]).

The following definition is a reformulation of a general definition given in [4] for differential forms and the measure $\tilde{\mu}_M$.

Definition 3. Let M be an oriented real n-dimensional manifold with locally finite variations embedded in $\mathbb{C}^m$ ($m \geq n$). The differential form ω is as follows: $M \times (\mathbb{C}^m)^n \longrightarrow \mathbb{C}$ is said to be $\tilde{\mu}_M$-integrable if there is a sequence $\{\omega_k\}_{k=1}^{\infty}$ of step differential forms such that

(1) $\{\omega_k\}_{k=1}^{\infty}$ is a Cauchy sequence, i.e., $\lim\limits_{k, l \to \infty} \int \|\omega_k - \omega_l\| d|\tilde{\mu}_M| = 0$.

(2) The ω_k converge to ω almost everywhere with respect to the measure $|\tilde{\mu}_M|$.

In this case, we set

$$\int \omega d\tilde{\mu}_M = \lim_{k \to \infty} \int \omega_k d\tilde{\mu}_M.$$

If $A \in \bigcap_\alpha (\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-)$ and the differential form $\omega \cdot \chi_A$ is integrable, then the form ω is said to be integrable on the set A, and its integral is defined as $\int_A \omega d\tilde{\mu}_M = \int \omega \chi_A d\tilde{\mu}_M$.

Theorem 2. *Suppose that M is an oriented n-manifold in* $\mathbb{C}^m$ *with locally finite variations, a set* $K \subset M$ *is compact, and the restriction* $\omega|_K$ *of a differential form ω is continuous on K. Then, ω is integrable on K.*

The proof of this theorem is a word-for-word repetition of that of Theorem 7 in [2] (with $|\mu_M|$ replaced by $|\tilde{\mu}_M|$).

Theorem 3. *Let M be an oriented manifold of real dimension n smoothly embedded in* $\mathbb{C}^m$ ($m \geq n$). *Then,*

(1) *M is a manifold with locally finite variations*;

(2) *If* $E \subset U = f(\mathbb{R}^n)$, *where f is a positive parameterization* (*a* C^1 *diffeomorphism*) *of a neighborhood* $U \subset M$, *and E is small and* α^+*- and* α^-*-measurable for all* $\alpha = \{i_1, \ldots, i_n\}$, *then* $f^{-1}(E)$ *is Lebesgue measurable*;

(3) *If a differential form* $\omega(z) = \sum_\beta a_\beta(z) dz_\beta$ *is integrable on E with respect to the measure* $\tilde{\mu}_M$, *then*

$$\int_E \omega d\tilde{\mu}_M = \int_{f^{-1}(E)} \sum_\beta (a_\beta \circ f) \cdot \det f'_\beta d\lambda_n. \quad (*)$$

Proof. Assertions (1) and (2) were proved in [3, Theorem 2, items 1 and 2]. Let us prove (*).

(1) Suppose that $E \subset U = f(\mathbb{R}^n)$, $E \in \mathfrak{R}_M$, and $\omega = dz_{k_1} \wedge \ldots \wedge dz_{k_n}$ (a constant differential form). Let us show that

$$\int_E \omega d\tilde{\mu}_M = \int_{f^{-1}(E)} \det f'_\beta d\lambda_n, \quad \beta = \{k_1, \ldots, k_n\}.$$

By the definition of the integral of a step differential form, we have

$$\int_E \omega d\tilde{\mu}_M = \int \omega \chi_E d\tilde{\mu}_M = \tilde{\varphi}(\tilde{\mu}_M(E)).$$

Since the functional $\tilde{\varphi}$ is $\mathbb{C}$-linear, it follows that (see (2))

$$\tilde{\varphi}(\tilde{\mu}_M(E)) = \sum_\beta \bar{\mu}_\beta(E) \tilde{\varphi}(e_\beta), \quad \beta = \{j_1, \ldots, i_n\}, \quad 1 \leq i_1 < \ldots < i_n \leq m. \quad (10)$$

According to (2), we have

$$\tilde{\varphi}(\tilde{e}_{j_1} \wedge \tilde{e}_{j_2} \wedge \ldots \wedge \tilde{e}_{j_n}) = \varphi(\tilde{e}_{j_1}, \tilde{e}_{j_2}, \ldots, \tilde{e}_{j_n}) = (dz_{k_1} \wedge \ldots \wedge dz_{k_n})(\tilde{e}_{j_1}, \tilde{e}_{j_2}, \ldots, \tilde{e}_{j_n}).$$

Based on a lemma similar to Lemma 2 in [1, Vol. 3, Chapter 2, Section 2] (for an exterior form over $\mathbb{C}$), it follows that

$$(dz_{k_1} \wedge \ldots \wedge dz_{k_n})(\tilde{e}_{j_1}, \tilde{e}_{j_2}, \ldots, \tilde{e}_{j_n}) = \begin{cases} 1 & \text{if } \{j_1, \ldots, j_n\} = \{k_1, \ldots, k_n\}, \\ 0 & \text{if } \{j_1, \ldots, j_n\} \neq \{k_1, \ldots, k_n\} \end{cases}$$

(here, $k_1 < k_2 < \ldots < k_n$ and $j_1 < j_2 < \ldots < j_n$). Therefore, only one summand in (10) is nonzero. We obtain

$$\int_E \omega d\tilde{\mu}_M = \tilde{\varphi}(\tilde{\mu}_M(E)) = \bar{\mu}_{\{k_1, \ldots, k_n\}}(E) = \int_{f^{-1}(E)} \det f'_\beta d\lambda_n \quad \text{(by Theorem 1)}.$$

(2) Suppose that $E \subset U = f(\mathbb{R}^n)$, $E \in \bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha)$, and ω is a step differential form. For this form, let us prove (*).

We have $\omega = \sum_{k=1}^p \varphi_k \chi_{E_k}$, where the φ_k are constant differential forms and all E_k belong to the ring $\mathfrak{R}_M$. Hence,

$$\omega_{\chi_E} = \sum_{k=1}^p \varphi_k \chi_{E_k} \cdot \chi_E = \sum_{k=1}^p \varphi_k \chi_{E_k \cap E}, \quad (E_k \cap E) \in \bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha),$$

$$\text{and} \ \ \mu^+_\alpha(E_k \cap E) \le \mu^+_\alpha(E_k) < \infty, \quad \mu^-_\alpha(E_k \cap E) \le \mu^-_\alpha(E_k) < \infty, \quad \text{i.e., } E_k \cap E \in \mathfrak{R}_M,$$

and the form ω_{χ_E} is step, too. Therefore, without a loss of generality, we can assume that $\omega = \omega_{\chi_E}$.

Thus, we have $\omega = \sum_{k=1}^p \varphi_k \chi_{E_k}$, where the sets E_k can be assumed to be disjointed (see [4, Chapter 2, Section 6, Subsection 1]) and $E_k \subset E$. Let $\varphi_k = \sum_\beta a_{\beta k} dz_{i_1} \wedge \ldots \wedge dz_{i_n}$; then, for $z \in E_k$, we have $\omega(z) = \sum_\beta a_{\beta k} dz_{i_1} \wedge \ldots \wedge dz_{i_n}$, and for $z \in E \backslash \left(\bigcup_{k=1}^p E_k\right)$, we have $\omega(z) = 0$. Therefore,

$$\text{if} \ \ \omega(z) = \sum_\beta a_{\beta k} dz_{i_1} \wedge \ldots \wedge dz_{i_n}, \quad \text{then} \quad a_\beta(z) = \begin{cases} a_{\beta k} & \text{if } z \in E_k, \\ 0 & \text{if } z \in E \backslash \left(\bigcup_{k=1}^p E_k\right). \end{cases}$$

The linearity of integral and the result obtained for the form $dz_{i_1} \wedge \ldots \wedge dz_{i_n}$ imply

$$\int \varphi_k \chi_{E_k} d\tilde{\mu}_M = \int_{E_k} \left(\sum_\beta a_{\beta k} dz_{i_1} \wedge \ldots \wedge dz_{i_n} \right) d\tilde{\mu}_M = \sum_\beta a_{\beta k} \int_{E_k} (dz_{i_1} \wedge \ldots \wedge dz_{i_n}) d\tilde{\mu}_M$$

$$= \sum_\beta a_{\beta k} \int_{f^{-1}(E_k)} \det f'_\beta d\lambda_n = \int_{f^{-1}(E_k)} \left(\sum_\beta a_{\beta k} \det f'_\beta \right) d\lambda_n;$$

the integrand is continuous because $f \in C^{(1)}(\mathbb{R}^n)$.

If $t \in f^{-1}(E_k)$, then $f(t) \in E_k$, i.e., $a_\beta(f(t)) = a_{\beta k}$. In this case,

$$\int_E \omega d\tilde{\mu}_M = \int \left(\sum_{k=1}^p \varphi_k \chi_{E_k} \right) d\tilde{\mu}_M = \sum_{k=1}^p \int \varphi_k \chi_{E_k} d\tilde{\mu}_M = \sum_{k=1}^p \int_{f^{-1}(E_k)} \left(\sum_\beta a_\beta(f(t)) \cdot \det f'_\beta \right) d\lambda_n.$$

If $t \in f^{-1}(E)\Big\backslash\Big(\bigcup_{k=1}^{p} f^{-1}(E_k)\Big) = f^{-1}\Big(E\Big\backslash\bigcup_{k=1}^{p} E_k\Big)$, then $a_\beta(f(t)) = 0$ for all β. Using the disjointedness of the sets $f^{-1}(E_k)$, we obtain

$$\sum_{k=1}^{p}\int_{f^{-1}(E_k)}\Big(\sum_\beta a_\beta(f(t))\det f'_\beta\Big)d\lambda_n. = \int_{\bigcup_{k=1}^{p} f^{-1}(E_k)}\Big(\sum_\beta a_\beta(f(t))\det f'_\beta\Big)d\lambda_n$$

$$+ \int_{f^{-1}(E)\backslash\bigcup_{k=1}^{p} f^{-1}(E_k)}\Big(\sum_\beta a_\beta(f(t))\det f'_\beta\Big)d\lambda_n = \int_{f^{-1}(E)}\Big(\sum_\beta (a_\beta\circ f)\det f'_\beta\Big)d\lambda_n,$$

which proves (*) for step differential forms. Note that the integrand function is continuous on the Lebesgue measurable sets $f^{-1}(E_k)$ and $f^{-1}(E)\backslash\bigcap_{k=1}^{p} f^{-1}(E_k)$; i.e., it is Lebesgue measurable.

(3) Let ω be a differential form integrable on $E \subset U = f(\mathbb{R}^n)$. Then, by definition, there is a sequence $\{\omega_k\}_{k=1}^{\infty}$ of step forms such that $\omega_k(z) \longrightarrow \omega(z)$ almost everywhere with respect to the measure $|\tilde{\mu}_M|$. Since the space of exterior forms is finite-dimensional, the convergence $\omega_k(z) \longrightarrow \omega(z)$ is equivalent to coordinatewise convergence in any basis. This means that, if $\omega_k(z) = \sum_\beta a_{\beta k}(z)dz_\beta$ and $\omega(z) = \sum_\beta a_\beta(z)dz_\beta$, then, for any β, we have $a_{\beta k}(z) \longrightarrow a_\beta(z)$ $|\tilde{\mu}_M|$ almost everywhere. Let us prove that the functions $a_{\beta k}(f(t))\det f'_\beta(t)$ converge to $a_\beta(f(t))\det f'_\beta(t)$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^n$.

Let $E_\beta = \{z \in E | a_{\beta k}(z) \not\longrightarrow a_\beta(z)\}$; then, $E_\beta \in \mathfrak{R}_M$ and $|\tilde{\mu}_M|(E_\beta) = 0$.

Let $P_\beta = \{t \in \mathbb{R}^n | a_{\beta k}(f(t))\det f'_\beta(t) \not\longrightarrow a_\beta(f(t))\det f'_\beta(t)\}$. Then,

$$t \in P_\beta \Leftrightarrow a_{\beta k}(f(t)) \not\longrightarrow a_\beta(f(t)) \quad \text{and} \quad \det f'_\beta(t) \neq 0$$

$$\Leftrightarrow f(t) \in E_\beta \quad \text{and} \quad \det f'_\beta(t) \neq 0 \Leftrightarrow t \in f^{-1}(E_\beta) \cap \{t \in \mathbb{R}^n | \det f'_\beta(t) \neq 0\}.$$

Since $E_\beta \in \mathfrak{R}_M$, it follows that the set $f^{-1}(E_\beta)$ is Lebesgue measurable (by Theorem 1). The function $\det f'_\beta(t)$ is continuous and, therefore, Lebesgue measurable; hence, the set $\{t \in \mathbb{R}^n | \det f'_\beta(t) \neq 0\}$ is measurable as well. We have proved that

$$P_\beta = f^{-1}(E_\beta) \cap \{t \in \mathbb{R}^n | \det f'_\beta(t) \neq 0\} \quad \text{is measurable.}$$

Using the definition of the variations in the measure, we have

$$|\tilde{\mu}_M|(E_\beta) = \sup\left\{\sum_{k=1}^{p}\|\tilde{\mu}_M(E_k)\| \,\middle|\, \bigcup_{k=1}^{p} E_k \subset E_\beta, \text{the } E_k \text{ belong to } \mathfrak{R}_M \text{ and are disjoint}\right\} = 0.$$

It follows that, for any $E' \subset E_\beta$, $E' \in \mathfrak{R}_M$, we must have $\tilde{\mu}_M(E') = \mathbf{0}$; thus, $\bar{\mu}_\beta(E') = 0$ for any β, from which we have $|\bar{\mu}_\beta|(E_\beta) = 0$. Based on Theorem 1,

$$|\bar{\mu}_\beta|(E_\beta) = \int_{f^{-1}(E_\beta)}|\det f'_\beta|d\lambda_n = \int_{P_\beta}|\det f'_\beta|d\lambda_n + \int_{f^{-1}(E_\beta)\backslash P_\beta}|\det f'_\beta|d\lambda_n.$$

Since $\det f'_\beta = 0$ outside the set P_β, as required, we have

$$|\bar{\mu}_\beta|(E_\beta) = \int_{P_\beta}|\det f'_\beta|d\lambda_n = 0, \quad \text{on } P_\beta, \quad |\det f'_\beta| > 0 \Rightarrow \lambda_n(P_\beta) = 0.$$

As noted at the end of the proof of item (2), all functions $a_{\beta k}(f(t))\det f'_\beta(t)$ are measurable. Since $a_{\beta k}(f(t))\det f'_\beta(t) \longrightarrow a_\beta(f(t))\det f'_\beta(t)$ almost everywhere, it follows that the functions $(a_\beta \circ f)\det f'_\beta$ and $|(a_{\beta k} \circ f)\det f'_\beta - (a_\beta \circ f)\det f'_\beta|$ are Lebesgue measurable on $f^{-1}(E)$.

(4) Let us prove that $\int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta| = \int_{f^{-1}(E)} |(a_{\beta k} \circ f) - (a_\beta \circ f)| \cdot |\det f'_\beta| d\lambda_n$.

The function $a_{\beta k}$ is step and measurable with respect to the σ-algebra $\bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha)$, and $a_{\beta k}(z) \longrightarrow a_\beta(z)$ almost everywhere with respect to the measure $|\tilde{\mu}_M|$ (defined on the same σ-algebra); therefore, the a_β and $|a_{\beta k} - a_\beta|$ are measurable as well. The measure $|\bar{\mu}_\beta|$ is defined on $\bigcap_{P(\alpha) = \beta} (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha) \supset \bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha)$; hence, the integral $\int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta|$ makes sense.

If h is a step function and $h = \sum_{i=1}^p c_i \chi_{E_i}$, where all $c_i \ge 0$ are nonnegative, all E_i are disjoint and belong to the σ-algebra $\bigcap_\alpha (\mathfrak{A}^+_\alpha \cap \mathfrak{A}^-_\alpha)$, and $E = \bigcup_{i=1}^p E_i$, then, according to Theorem 1, we have

$$\int_E h\, d|\bar{\mu}_\beta| = \sum_{i=1}^p c_i |\bar{\mu}_\beta|(E_i) = \sum_{i=1}^p \left(c_i \int_{f^{-1}(E_i)} |\det f'_\beta| d\lambda_n \right)$$

$$= \sum_{i=1}^p \int_{f^{-1}(E_i)} h(f(t)) |\det f'_\beta(t)| d\lambda_n = \int_{f^{-1}(E_i)} h(f(t)) |\det f'_\beta(t)| d\lambda_n$$

(because $f(t) \in E_i \Rightarrow h(f(t)) = c_i$ for $t \in f^{-1}(E_i)$).

The nonnegative function $|a_{\beta k} - a_\beta|$ on E, which is measurable with respect to all of the measures μ^+_α and μ^-_α, is the limit of an increasing sequence of nonnegative step functions, i.e.,

$$\forall z \in E \quad 0 \le h_j(z) \le h_{j+1}(z), \quad h_j(z) \longrightarrow |a_{\beta k}(z) - a_\beta(z)| \quad (j \longrightarrow \infty)$$

(we can set $h_j(z) = \dfrac{k}{2^j}$ if $\dfrac{k}{2^j} \le |a_{\beta k}(z) - a_\beta(z)| < \dfrac{k+1}{2^j}$, where k is an integer, $k < j \cdot 2^j$, and $h_j(z) = j$ if $|a_{\beta k}(z) - a_\beta(z)| \ge j$). All properties of the sequence $\{h_j\}_{j=1}^\infty$ are easy to verify. Obviously, we have

$$\forall t \in f^{-1}(E) \quad 0 \le h_1(f(t))|\det f'_\beta(t)| \le h_2(f(t))|\det f'_\beta(t)| \le \ldots,$$

$$h_j(f(t))|\det f'_\beta(t)| \longrightarrow |a_{\beta k}(f(t)) - a_\beta(f(t))| \cdot |\det f'_\beta(t)|;$$

passing to the limit in the relation $\int_E h_j d|\bar{\mu}_\beta| = \int_{f^{-1}(E)} (h_j \circ f) |\det f'_\beta| d\lambda_n$ and applying Levi's monotone convergence theorem, we obtain

$$\int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta| = \int_{f^{-1}(E)} |(a_{\beta k} \circ f) - (a_\beta \circ f)| \cdot |\det f'_\beta| d\lambda_n.$$

(5) Let $g_k(t) = \sum_\beta a_{\beta k}(f(t))\det f'_\beta(t)$, and let $g(t) = \sum_\beta a_\beta(f(t))\det f'_\beta(t)$. The functions $(a_\beta \circ f)\det f'_\beta$ are Lebesgue measurable (by item (3)); therefore, so is g. By the remark to item (2), the functions g_k are

measurable; hence, $|g_k - g|$ is measurable and nonnegative, i.e., the integral $\int_{f^{-1}(E)} |g_k - g| d\lambda_n$ makes sense. Let us prove that $\int_{f^{-1}(E)} |g_k - g| d\lambda_n \longrightarrow 0$ as $k \longrightarrow \infty$. We have

$$\int_{f^{-1}(E)} |g_k - g| d\lambda_n = \int_{f^{-1}(E)} \left| \sum_\beta (a_{\beta k} \circ f) \det f'_\beta - \sum_\beta (a_\beta \circ f) \det f'_\beta \right| d\lambda_n$$

$$\leq \sum_\beta \int_{f^{-1}(E)} |(a_{\beta k} \circ f) - (a_\beta \circ f)| \cdot |\det f'_\beta| d\lambda_n.$$

According to item (4),

$$\sum_\beta \int_{f^{-1}(E)} |(a_{\beta k} \circ f) - (a_\beta \circ f)| \cdot |\det f'_\beta| d\lambda_n = \sum_\beta \int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta|.$$

By the definition of the measure $\tilde{\mu}_M$, for any $A \in \mathfrak{R}_M$, $\bar{\mu}_\beta(A)$ is a coordinate of the n vector $\tilde{\mu}_M(A)$ in the basis $\{\tilde{e}_\beta\}$. Since the mapping taking each n vector to its coordinate is linear and continuous in any norm on the space of n vectors (which is finite-dimensional), it follows that there is a constant $C_1 > 0$ such that $|\bar{\mu}_\beta| \leq C_1 \|\tilde{\mu}_M(A)\|$ for any β and any $A \in \mathfrak{R}_M$. Using the definition of the variation of a measure, we can easily check that $|\bar{\mu}_\beta|(A) \leq C_1 |\tilde{\mu}_M|(A)$ for any $A \in \bigcap_\alpha (\mathfrak{A}_\alpha^+ \cap \mathfrak{A}_\alpha^-)$, from which we have

$$\sum_\beta \int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta| \leq C_1 \int_E \sum_\beta |a_{\beta k} - a_\beta| d|\tilde{\mu}_M|.$$

Note that $a_{\beta k}(z) - a_\beta(z)$ is a coefficient of the exterior form $\omega_k(z) - \omega(z)$. The mapping taking each exterior form to the set of its coefficients is linear and continuous (the spaces of exterior forms and of coefficients are finite-dimensional). The sum of the absolute values of coordinates of a vector is a norm on a finite-dimensional space; therefore, for any norm in the space of exterior forms, there is a constant $C_2 > 0$ such that, for any exterior form $\varphi = \sum_\beta a_\beta dz_\beta$, we have $\sum_\beta |a_\beta| \leq C_2 \|\varphi\|$; in particular,

$$\sum_\beta |a_{\beta k}(z) - a_\beta(z)| \leq C_2 \|\omega_k(z) - \omega(z)\|.$$

The space of integrable differential forms is linear [4, Chapter 2, Section 8, Subsection 1, Proposition 1]; therefore, $\omega_k - \omega$ is integrable on E with respect to the measure $\tilde{\mu}_M$, and $\|\omega_k - \omega\|$ is integrable with respect to $|\tilde{\mu}_M|$ [4, Chapter 2, Section 8, Subsection 1, Proposition 4]. Thus,

$$\int_E \sum_\beta |a_{\beta k} - a_\beta| d|\tilde{\mu}_M| \leq C_2 \int_E \|\omega_k - \omega\| d|\tilde{\mu}_M|$$

Using the preceding inequalities in this item, we obtain

$$\int_{f^{-1}(E)} |g_k - g| d\lambda_n \leq \sum_\beta \int_{f^{-1}(E)} |(a_{\beta k} \circ f) - (a_\beta \circ f)| \cdot |\det f'_\beta| d\lambda_n$$

$$= \sum_\beta \int_E |a_{\beta k} - a_\beta| d|\bar{\mu}_\beta| \leq C_1 \int_E \sum_\beta |a_{\beta k} - a_\beta| d|\tilde{\mu}_M| \leq C_1 C_2 \int_E \|\omega_k - \omega\| d|\tilde{\mu}_M|.$$

This inequality implies that the function g is Lebesgue integrable on $f^{-1}(E)$. Since $\{\omega_k\}_{k=1}^{\infty}$ is a Cauchy sequence and $\omega_k(z) \longrightarrow \omega(z)\,|\tilde{\mu}_M|$ almost everywhere, it follows that [4, Chapter 2, Section 8, item 2, Proposition 12]

$$\int_E \|\omega_k - \omega\| d|\tilde{\mu}_M| \longrightarrow 0, \qquad \int_{f^{-1}(E)} |g_k - g| d\lambda_n \longrightarrow 0.$$

Therefore,

$$\int_{f^{-1}(E)} g d\lambda_n = \int_{f^{-1}(E)} (a_\beta \circ f) \det f'_\beta d\lambda_n = \lim_{k \to \infty} \int_{f^{-1}(E)} g_k d\lambda_n = \lim_{k \to \infty} \int_E \omega_k d\tilde{\mu}_M = \int_E \omega d\tilde{\mu}_M$$

(according to item (2) for step forms ω_k and the definition of the integral of the form ω). ●

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