ISSN 1063-4541, Vestnik St. Petersburg University. Mathematics, 2016, Vol. 49, No. 1, pp. 77–80. © Allerton Press, Inc., 2016.
Original Russian Text © S.M. Bauer, E.B. Voronkova, 2016, published in Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1. Matematika, Mekhanika, Astronomiya, 2016, No. 1, pp. 105–109.

MECHANICS

On Natural Frequencies of Transversely Isotropic Circular Plates[1]

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Received October 22, 2015

Abstract—The paper discusses the impact of the material properties of transversely isotropic circular plates on its natural frequencies. Two refined theories of plates have been used to analyze the free vibration behavior of homogeneous plates. Both theories take into account normal and rotary inertias. Fundamental frequencies for plates with radial inhomogeneity have been obtained with the help of finite element package Comsol Multiphysics 5.0. It has been shown that the inhomogeneity of the plate have a profound impact on the first (lowest) frequency of the plate, while the plate orthotropy has a greater influence on the second and higher vibration mode [2] (Fig. 1, Table 1).

Key words: circular plate, transversely isotropic plate, vibrations.

DOI: 10.3103/S1063454116010027

1. INTRODUCTION

In the present paper, the problem of determining the natural frequencies of a transversely isotropic circular plate is considered and the impact of the material properties of the plate on its natural frequencies is studied. The classical Kirchhoff–Love (KL) theory only takes into account the material properties of the midplane, which is why the fundamental frequencies for isotropic and transversely isotropic plates are equal with respect to the classical theory. The Ambartsumyan theory of anisotropic shells [1] takes into account the impact of the shear deformation in the thickness direction on the stress–strain state of the plate. In the general case, the theory of anisotropic plates and shells developed by Rodionova, Titaev, and Chernykh [2] permits one to take into account not only the transverse shears, but also the deformation of normals to the midplane. In the present paper, the problem of determining the fundamental frequencies is solved with respect to the theories [1] and [2], which improve the KL theory. To study the impact of the radial inhomogeneity on the plate fundamental frequencies, the calculations are carried out using the Comsol Multiphysics 5.0 finite element package.

2. STATEMENT OF THE PROBLEM

The problem of determining the natural frequencies of a transversely isotropic circular plate with radius R and thickness h is considered. The material of the plate obeys Hooke's law [2], assuming the plate midplane to be the plane of isotropy as follows:

$$\sigma_{ii} = E_{ii}e_{ii} + E_{ij}e_{jj} + E_{ik}e_{kk}, \quad i \neq j \neq k, \quad \sigma_{ij} = G_{ij}e_{ij}, \quad i \neq j, \tag{1}$$

where

$$E_{11} = \frac{E_1(E_1 - E_3\nu_{13}^2)}{E_{den}}, \quad E_{12} = \frac{E_1(E_1\nu_{12} + E_3\nu_{13}^2)}{E_{den}}, \quad E_{13} = \frac{E_1E_3\nu_{13}(1+\nu_{12})}{E_{den}},$$

$$E_{33} = \frac{E_1E_3(1-\nu_{12}^2)}{E_{den}}, \quad E_{den} = (1+\nu_{12})(E_1(1-\nu_{12}) - 2E_3\nu_{13}^2).$$

Here, E_i, $(i = r, \theta, z)$ is the Young moduli in the ith directions; (r, θ, z) is the introduced cylindrical coordinate system, G_{ij} is the shear modulus in $(i–j)$ plane, ν_{ij} is Poisson's ratio. For a transversely isotropic plate, $G_{12} = E_1/2(1 + \nu_{12})$ and $G_{13} = G_{23}$.

[1] The article was translated by the authors.

The system of equations of motion for a transversely isotropic plate of the Ambartsumyan refined theory [1] is written as follows:

$$D\Delta\Delta w + \rho h(1-\kappa\Delta)\frac{\partial^2 w}{\partial t^2} = 0, \quad \Delta F - \delta^2 F = 0, \tag{2}$$

where

$$\kappa = \frac{h^2}{10(1-\nu_{12}^2)}\left(2\frac{G_{12}}{G_{13}} - \nu_{13}\right), \quad \delta^2 = \frac{10}{h^2}\frac{G_{13}}{G_{12}},$$

Here, w is the normal displacement, F is the function related to the normal displacement and the shear strain resultants, Δ is the Laplace operator, ρ is the density of the plate material; and $D = E_1h^3/12(1-\nu_{12}^2)$ is the bending stiffness of the plate.

The characteristic equation for determining the natural frequencies of a clamped plate takes the form

$$\mu\frac{I_{n-1}(\mu)}{I_n(\mu)} - \lambda\frac{J_{n-1}(\lambda)}{J_n(\lambda)} + \frac{5}{2\delta^2R^2(1-\nu_{12})}\left(\lambda\mu\left[\lambda\frac{I_{n-1}(\mu)}{I_n(\mu)} + \mu\frac{J_{n-1}(\lambda)}{J_n(\lambda)}\right] - n(\lambda^2+\mu^2)\frac{I_{n-1}(\delta R)}{I'_n(\delta R)}\right) = 0. \tag{3}$$

and the natural frequencies are connected with the roots $\lambda_{n,m}$ of Eq. (3) by the relation

$$\omega_{n,m}^A = \frac{\lambda_{n,m}^2}{\sqrt{1+\kappa\lambda_{n,m}^2}}\sqrt{\frac{D}{\rho h R^4}}, \quad \mu_{n,m} = \frac{\lambda_{n,m}}{\sqrt{1+\kappa\lambda_{n,m}^2}}.$$

Here, J_n, I_n are Bessel functions and n, m are the number of nodal diameters and circles, respectively.

An equation of the classical Kirchhoff–Love (KL) theory, which describes the motion of the plate, and the corresponding characteristic equation can be obtained from (2) and (3) by assuming $\kappa = 0$ and passing to the limit $\delta \to \infty$.

3. VIBRATIONS WITH THE RODIONOVA–TIATEV–CHERNYKH THEORY

In general, the theory of anisotropic plates and shells developed by Rodionova, Titaev, and Chernykh (RTC) [2] permits one to take into account the deformation of normals to the midplane. When constructing the theory, it is assumed that the transverse tangential stresses are distributed along the thickness of the shell according to the quadratic law, and the normal stresses are distributed according to the cubic law of the z coordinate. In the present paper, we consider the problem of free vibrations based on theory described in [2] by taking into account only the transverse shear.

When there is no external surface forces, the vector-matrix equation of motion taking into account the normal inertia forces and the inertia of rotation has the form

$$A_{11}\frac{\partial^2\mathbf{V}}{\partial r^2} + A_{12}\frac{\partial^2\mathbf{V}}{r\partial r\partial\theta} + A_{22}\frac{\partial^2\mathbf{V}}{r^2\partial\theta^2} + A_{10}\frac{\partial\mathbf{V}}{\partial r} + A_{02}\frac{\partial\mathbf{V}}{r\partial\theta} + A_{00}\mathbf{V} + A_{tt}\frac{\partial^2\mathbf{V}}{\partial t^2} = 0, \tag{4}$$

where $\mathbf{V} = (w^*(r,\theta,t), \gamma_1^*(r,\theta,t), \gamma_2^*(r,\theta,t))$ is the sought vector-function and, in the square matrices $A_{\rm ij}$, the following elements are distinct from zero:

$$A_{11}^{(11)} = A_{22}^{(11)} = \frac{5}{6}G_{13}h, \quad A_{11}^{(22)} = A_{22}^{(55)} = d_{11}, \quad A_{11}^{(33)} = A_{22}^{(33)} = d_{66},$$

$$A_{12}^{(23)} = A_{12}^{(32)} = d_{12} + d_{66}, \quad A_{10}^{(11)} = \frac{5}{6r}G_{13}h, \quad A_{10}^{(12)} = \frac{5}{3}G_{13}, \quad A_{10}^{(21)} = -\frac{5}{6}G_{13}h,$$

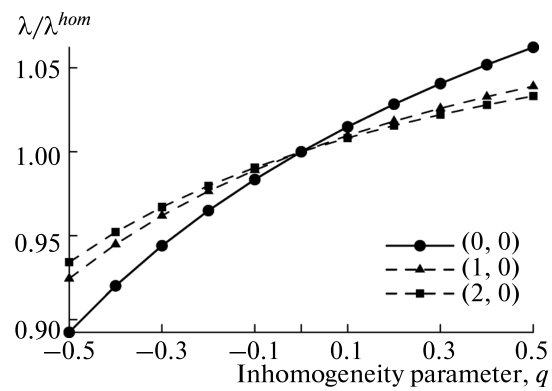
$$A_{10}^{(22)} = \frac{d_{11}}{r}, \quad A_{10}^{(33)} = \frac{d_{66}}{r}, \quad A_{02}^{(13)} = \frac{5}{3}G_{13}, \quad A_{02}^{(23)} = -A_{02}^{(32)} = \frac{(d_{12}+d_{66})}{r},$$

$$A_{02}^{(31)} = -\frac{5}{3}G_{13}h, \quad A_{00}^{(12)} = \frac{5}{3r}G_{13}, \quad A_{00}^{(22)} = \frac{-d_{11}}{r^2} - \frac{5}{3}G_{13}, \quad A_{00}^{(33)} = -d_{66}/r^2 - \frac{5}{3}G_{13},$$

$$A_{tt}^{(11)} = -\rho h, \quad A_{tt}^{(22)} = -\rho\frac{h^2}{6}, \quad A_{tt}^{(33)} = -\rho\frac{h^2}{6}, \quad d_{11} = \frac{E_1h^2}{6(1-\nu_{12}^2)},$$

Normalized frequency parameter $\bar{\lambda}_{m,n} = \lambda_{n,m}/\lambda^{KL}_{m,n}$ at $h/R = 1/10$

n	m	$\bar{\omega}^{A}_{n,m}$	$\bar{\omega}^{RTC}_{n,m}$	$\bar{\omega}^{FEM}_{n,m}$	$\bar{\omega}^{A}_{n,m}$	$\bar{\omega}^{RTC}_{n,m}$	$\bar{\omega}^{FEM}_{n,m}$
		$E_1/E_3 = G_{12}/G_{13} = 1$			$E_1/E_3 = G_{12}/G_{13} = 10$		
0	0	0.97	0.97	0.98	0.77	0.79	0.8
1	0	0.95	0.96	0.96	0.67	0.68	0.7
2	0	0.92	0.92	0.92	0.6	0.6	0.61
0	1	0.92	0.92	0.92	0.58	0.59	0.6
3	0	0.90	0.90	0.89	0.54	0.55	0.56
1	1	0.89	0.89	0.89	0.51	0.52	0.53



Impact of the inhomogeneity of the plate on its natural frequencies.

$$d_{12} = \nu_{12}d_{11}, \quad d_{66} = \frac{1-\nu_{12}}{2}d_{11}.$$

After the separation of the variables $w^* = W_n(r)\cos(n\theta)\cos(\omega t)$, $\gamma_1^* = g_{1n}(r)\cos(n\theta)\cos(\omega t)$, $\gamma_2^* = g_{2n}(r)\sin(n\theta)\sin(\omega t)$, system (4) can be traced to

$$Y' = A(\omega^2)Y, \tag{5}$$

where $Y = (W_n, g_{1n}, g_{2n}, W_n', g_{1n}', g_{2n}')$, $A(\omega^2)$ is the 6×6 matrix, $(\,)' = \frac{d}{dr}(\,)$.

The solution of system (5) can be represented as follows:

$$Y(r) = L\begin{bmatrix} e^{\lambda_1 r} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_6 r} \end{bmatrix}L^{-1}C. \tag{6}$$

Here, λ_i are the eigenvalues, L is the matrix composed of eigenvectors of the matrix $A(\omega^2)$, and C is a vector of arbitrary constants. After the substitution of (6) into the corresponding boundary conditions, one can derive the equation for determination of natural frequencies $\omega_{n,m}$ from the condition for a nontrivial solution of system (5).

4. RESULTS

The table presents the normalized fundamental frequencies parameters of the homogeneous plate $\bar{\lambda}_{n,m} = \lambda_{n,m}/\lambda^{KL}_{n,m}$, which is computed using two refined theories [1], [2] and with the help of the finite-element method (FEM) in Comsol Multiphysics 5.0. software. Indexes *KL*, *A*, *RTC*, and *FEM* correspond to the values obtained by the classical KL theory, the Ambarsumyan theory, the refined RTC theory, and by FEM, respectively. It can be seen that the transverse isotropy impacts more influences on higher frequencies.

Fundamental frequencies of the inhomogeneous plate are calculated in the software package Comsol Multiphysics 5.0. It has been supposed that the in-plane modulus of elasticity changes from the center to the edge of the plate as $E_1 = E_{10}f(r)$, where $f(r)$ is a fairly differentiable positive function. When performing the calculations, the parameter E_{10} is taken to keep the average value of the elastic modulus E_{av} constant

$$E_{av} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^{R} E_1(r) r dr d\theta = \text{const}.$$

In the figure, based on the example of the first three (lowest) frequencies, the impact of the plate material properties on its natural frequencies is reported. It is assumed that the in-plane Young modulus varies linearly as $E_1 = E_{10}(1 + qr/R)$, and its average value is ten times greater than the Young modulus in the direction of the thickness. For an inhomogeneous plate, frequency parameters $\lambda_{1,0}$ and $\lambda_{2,0}$ differ from the corresponding frequencies of the homogeneous plate ($q = 0$) by at most 7%, while for the first (lowest) frequency $\lambda_{0,0}$, the difference is greater than 10%.

5. CONCLUSIONS

The results have shown that the difference in the tangential and normal elastic moduli influences more the second and higher vibration modes. The inhomogeneity of the plate has a significant impact on the first (lowest) frequency of the plate.

ACKNOWLEDGMENTS

The study was partially supported by the Russian Foundation for Basic Research, project No. 15-01-06311 and was performed at the Observatory of Environmental Safety at St. Petersburg State University.

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