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MATHEMATICS

Dividing a Convex Figure by a System of Rays and Inscribed Polygons

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Abstract—The paper proposes several theorems on dividing the area of a convex figure by a system of rays with a common initial point. These results include the previous results of this type. As a limiting case, we also obtained several theorems on inscribing a polygon of some type into a convex figure.

Keywords: convex figure, convex equipartitioning the area, inscribed polygons.

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In the literature are many assertions on equipartitioning the area of a convex figure by a system of rays with a common initial point and on the possibility of inscribing (or circumscribing) a polygon of some type into a convex figure (see the survey [1]). The proofs of these assertions depend upon similar topological arguments, which may be outlined by approaches that differ in their form (we will follow a more geometric approach).

Below we will prove more general assertions on dividing the area of a convex figure by a system of rays with a common initial point. Under this approach, the assertions on inscribed polygons are obtained as a limiting case of theorems of dividing the area of a convex figure.

In what follows, by a convex figure we mean a compact convex solid subset of a plane. Below, $S(K)$ and ∂K will mean, respectively, the area and the boundary of a convex figure, K.

1. THE CASE OF AN ARBITRARY CONVEX FIGURE

It is well known that on the boundary of any planar convex figure one may find the vertices of some square (see [4] for a historical account of this problem). In [2] (see [3]) this is proven for an arbitrary smooth Jordan curve. It is also well known that the area of a convex figure can be divided into equal parts by a pair of perpendicular lines. The following theorem combines these two assertions.

Theorem 1. *Let K be a planar convex figure with a smooth boundary and let $C_1, \ldots, C_4$ be cones with a common apex in the center of some square, whose direction sets are segments of equal length that lie on the sides of the square and are symmetric about its midpoints* (Fig. 1). *Then an isometry exists that maps the common apex of the cones to an interior point of the figure K such that the images of the cones $C_1, \ldots, C_4$ contain equal parts of the area of the figure K.*

Proof. We require the following auxiliary assertion.

Lemma. *For any planar convex figure, K, with a smooth boundary, $\varepsilon > 0$ exists such that through each interior point of the figure that lie at a distance smaller than ε from its boundary exactly one chord of the figure bisected by this point passes.*

Proof. At each boundary point of the figure K we consider an auxiliary Cartesian frame by placing the origin in this point, directing the x-axis along the tangent line to the figure, and directing the y-axis along the inward normal vector to the boundary. By the smoothness of the boundary of K, $\varepsilon > 0$ exists such that at each boundary point of the figure in the auxiliary frame, the boundary points that lie at the distance smaller than ε from the x-axis lie on the graph of a function that is nondecreasing for $x > 0$ and is nonincreasing for $x < 0$. Using compactness arguments, ε can be made to serve all the boundary points of the figure K. We assume that an interior point, a, of the figure lies at the distance $<\varepsilon/2$ from the boundary of the figure. Consider the above frame with an origin at the nearest point of the boundary K to the point a. Clearly, the boundary of the figure that is symmetric to K about the point a of the figure has points of inter-

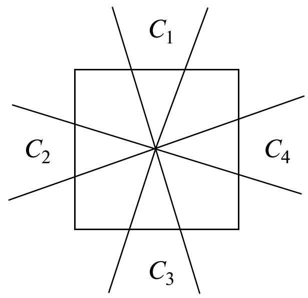

Fig. 1.

section with the boundary of the figure K only at a distance $<\varepsilon$ from the x-axis. There are precisely two of such points. This completes the proof of the lemma.

Let us prove Theorem 1. We assume that the sides of the square and of the cones are ordered anticlockwise; we denote by M the configuration space of squares that were obtained from the original one by an orientation that preserves the isometries of the plane for which the square center is an interior point of the figure K.

We define the mapping $f: M \times (0,\pi/2] \longrightarrow R^4$, where the second coordinate is the apex angle of the cone that is related to the square (which may vary in specified limits), by associating the set of numbers $(S(C_1 \cap K), ..., S(C_4 \cap K))$ with an ordered set of cones $(C_1, ..., C_4)$.

The set of sought configurations of the cones is closed by continuity arguments. From the above lemma it follows that this set is compact. In fact, by the equality of the areas, inside each pair of symmetric cones a chord of the figure bisected by this common vertex passes. Hence, by the above lemma the vertices of all sought configurations of cones are separated from the boundary of the figure K.

The cyclic group Z_4 acts freely on the space $M \times (0, \pi/2]$ by cyclic permutations of the cones $(C_1, ..., C_4)$. The set A of sought configurations of cones is the inverse image of the line in R^4 given by the equations $x_1 = ... = x_4$. By the construction, A is invariant relative to this action of Z_4. In the general position setting (that is, for the open C_1-dense set of figures K), A is a one-dimensional smooth submanifold $M \times (0, \pi/2]$, and $A_1 = A/Z_4$ is a smooth submanifold $M/Z_4 \times (0, \pi/2]$. For $a > 0$, on the manifold with the boundary $M/Z_4 \times [a, \pi/2]$, the part of A_1 that lies in it (which we denote by A_2) is a one-dimensional smooth manifold with boundaries in M/Z_4 and $M/Z_4 \times (\pi/2)$.

Note that in the general position setting, $A_2 \cap (M/Z_4) \times (\pi/2)$ consists of an odd number of points; that is, in this case an odd number of pairs of perpendicular lines exists that divide the area of the figure K into equal parts. This follows from the fact that a typical smooth function that assumes the values of different signs at the ends of the segment has an odd number of zeros (recall the standard proof of the existence of such a pair of lines; see, for example, [5]).

However, in this case $A_2 \cap (M/Z_4) \times a$ also consists of an odd number of points, which proves Theorem 1 in the general position setting. In the remaining cases, the theorem follows by passage to the limit.

Remark. If the apex angle of the cone is $\pi/2$, then we obtain the theorem on equipartition of the area of a convex figure by a pair of perpendicular lines. If the apex angle of the cone tends to 0, then in the limit we obtain the theorem on the inscribed square.

In [6] it was proven that the vertices of an affine image of a regular hexagon lie on the boundary of a convex figure, while in [7] it was proven that the area of this figure can be equipartitioned by three lies that pass through one point (see [5]).

Theorem 2. *Let K be a planar convex figure with a smooth boundary and let $C_1, ..., C_6$ be cones with a common apex in the center of some regular hexagon, whose direction sets are segments of equal length that lie on the sides of the regular hexagon and symmetrical about its midpoints* (Fig. 2). *Then, an affine transformation that maps the common apex of the cones to an interior point of the figure K exists such that the images of the cones $C_1, ..., C_6$ under study contain equal parts of the area of the figure K.*

Proof. We argue by analogy with the proof of Theorem 1. Let us assume that the sides of the hexagon and the cones are ordered anticlockwise and denote by M the configuration space of the hexagons, which are obtained from the original one by an orientation that preserves the affine transformations of the plane with a unit determinant, for which the center of the hexagon is an interior point of the figure K.

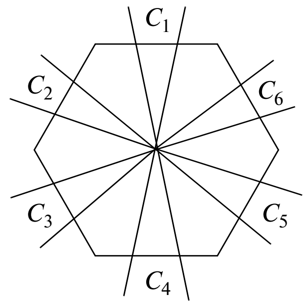

Fig. 2.

We consider the mapping $f\colon M \times (0, \pi/3] \longrightarrow R^6$, where the second coordinate is the apex angle of the cone that is related to the original hexagon, by assigning the set of numbers $(S(C_1 \cap K), ..., S(C_6 \cap K))$ to the affinely transformed set of cones $(C_1, ..., C_6)$.

The set A of the sought configurations of cones is the inverse image of the line in R^6, which is given by the equations $x_1 = ... = x_6$. By continuity, the set A is closed. From the above lemma it follows that the set of apexes of the cones from A is separated from the boundary of the figure K. In fact, by the equality of the areas inside each pair of symmetric cones, a chord of the figure that is bisected by its common vertex passes. Hence, by the above lemma the vertices of all sought configurations of the cones are separated from the boundary of the figure K. The intersection of A with each set $M \times [a, \pi/3]$, $a > 0$, is compact. Otherwise $M \times [0, \pi/3]$ would contain elements of A that are obtained from the original system of cones by compression in some direction with an arbitrary small coefficient. However, then (since the apexes of the cones are separated from the boundary of the figure), in choosing the limit direction, applying the inverse affine transformation, and passing to the limit, we would build an infinite strip between parallel lines that intercepts equal areas on the original cones, which may not be the case.

The cyclic group Z_6 acts freely on the space $M \times (0, \pi/3]$ by cyclic permutations of the cones $(C_1, ..., C_6)$. By the construction, A is invariant relative to this action of Z_6. In the general position setting (that is, for the open C_1-dense set of figures K), the set A is a one-dimensional smooth submanifold $M \times (0, \pi/3]$, and $A_1 = A/Z_6$ is a smooth submanifold $M/Z_6 \times (0, \pi/3]$. For $a > 0$, on the manifold with the boundary $M/Z_6 \times [a, \pi/3]$, the part of A_1 that is contained in it (which we denote by A_2) is a one-dimensional smooth manifold with a boundary in $M/Z_6 \times a$ and in $M/Z_6 \times (\pi/3)$.

We note that in the general position setting, $A_2 \cap M/Z_6 \times (\pi/3)$ consists of an odd number of points; that is, in this setting an odd number of triads of intersecting lines that divide the area of the figure K into equal parts exists. This follows from the fact that a typical smooth function that assumes values of different signs at the ends of the segment has an odd number of zeros (we recall the proof of the existence of such a triad of lines; see, for example, [5], [7]).

However, then $A_2 \cap M/Z_6 \times a$ is also composed of an odd number of points, which proves Theorem 2 in the general position setting. In the remaining cases, the theorem follows by passage to the limit.

Remark. If the apex angle of the cone is $\pi/3$, then we obtain the theorem on the possibility of equipartition of the area of a convex figure by a triad of intersection lines [7]. If the apex angle of the cone tends to 0, then in the limit we obtain the theorem on the inscribed affinely regular hexagon [6].

In [8], [9] it was proven that in any planar convex figure one may inscribe an affine image of a regular pentagon with a vertex at an a priori given boundary point of the figure K (see [10]). In [11], the same result was proven for a convex pentagon for which the sum of any neighboring angles is $>\pi$.

Theorem 3. *Let K be a planar convex figure with a smooth boundary and let $C_1, ..., C_5$ be cones with a common apex in the center of some pentagon, whose direction sets are segments of equal length that lie on the sides of the pentagon and are symmetric about its midpoints such that the apex angles of the cones are smaller than $\pi/5$. Then, an affine transformation exists that maps the common apex of the cones to an interior point of the figure K such that the images of the cones $C_1, ..., C_5$ contain equal parts of the area of the figure K, and a selected ray that bounds one of the cones intersects the boundary of the figure K at a given point.*

Proof. We assume that the sides of the pentagon and the cones are ordered anticlockwise, and denote by M the configuration space of system of cones, as obtained from the original one by orientation preserving affine transformations of a plane with a unit determinant, for which the common apex of the cones is an interior point of the figure K.

With a fixed apex angle $\varphi < \pi/5$ of the cone, we define the mapping $f: M \longrightarrow R^5$, by assigning to the ordered set of cones $(C_1, \ldots, C_5)$ the family of numbers $(S(C_1 \cap K), \ldots, S(C_5 \cap K))$.

We let A denote the set of such aforementioned affine transformations that $S(C_1 \cap K) = \ldots = (C_5 \cap K)$ for the areas of the images of the cones. We claim that in the general position setting (that is, for the open C^1-dense set of figures K the mapping f is transverse to the line $x_1 = \ldots = x_5$ in R^5) the set A is a one-dimensional smooth compact orientable submanifold of the five-dimensional noncompact manifold of the affine transformations under consideration, which realizes the generator of the fundamental group of M.

Let us verify that the disk K is a figure in a general position. For the disk K we need to be concerned only with regular configurations of the cones with a common apex in the disk center. We assume that the above regular system of cones has a common apex in the center of the disk K and that an affine transformation a with unit determinant maps it into a system of cones with the same property. Then, the system of cones $a(C_1), \ldots, a(C_5)$ intercepts sectors of equal area on the figures K and $a(K)$ (the latter are bounded by an ellipse). Hence, inside each of the cones $a(C_1), \ldots, a(C_5)$ a ray passes that emanates from their common apex on which the figures K and $a(K)$ intercept equal segments; that is, their boundaries intersect at five points (that lie on a circle); therefore they coincide by Bezout's theorem. Thus, in the case in question the set A consists of an orientation that preserves rotation about the disk center. By compressing the system of cones with a common apex in the center of the disk K towards the bisector of the angle of one of the cones with subsequent translation along the bisector of their common apex inside this cone, we will ensure that the sectors that are intercepted by other cones will have equal areas and the selected cone will reduce its area at a nonzero rate in comparison with them. This proves that the disk K is a figure in a general position.

For a figure K with a smooth boundary, we define the continuous mapping $g: A \longrightarrow \partial K$ by associating a point of intersection of the image of the selected ray that bounds one of the cones with the boundary of the figure with the affine transform. Any two planar smooth convex figures are smoothly deformable into one another in the class of such figures (for example, their convex linear combinations). As well, the set of required configurations of cones for figures from the one-parameter family under consideration is compact. Otherwise, as above, we apply affine transformations to assure that the thus-obtained systems of cones are regular systems and consider the disks with centers at the common apexes of the cones, from which the cones intercept sectors of the same area as the images of the figures under the same transformations. The boundaries of the figures and disks intersect inside the cones at the points $A_1, \ldots, A_5$. By the above restriction on the apex angles of the cones, for any four subsequent vertices of the pentagons $A_1, \ldots, A_5$, for example $A_1, \ldots, A_4$, the following situation holds. The extensions of the sides A_1A_2, A_4A_3, and A_2A_3 bound the triangle A_2A_3 B with the angle $\angle B$ exceeding $\pi/5 - \varphi$. In fact, the angular measure of the arc A_2A_3 is smaller than $2\pi/5 + \varphi$, and for the arc A_1A_4 that has no intersection with this arc, is greater than $4\pi/5 - \varphi$. The angle B is measured by the semi-differences of the angular measures of theses arc; thus, it is greater than $\pi/5 - \varphi$.

Thus, the affine image of the figure contains the pentagon $A_1, \ldots, A_5$ inscribed in a circle and is contained in a pentagram bounded by the extensions of the sides of this pentagon, with acute angles greater than $\pi/5 - \varphi$. Consequently, the ratio of the area of the affine image of the figure to the area of the pentagon $A_1, \ldots, A_5$ is majorized by the ratio star area to the pentagon area, which, in turn, is bounded by a constant that depends on φ. This proves the compactness of the set of the sought configurations of the cones for figures from the one-parameter families under study here.

As a result, an appeal to standard topological consideration shows that for a figure in a general position the class of singular bordism of the mapping g is thus well defined; thus, so is its degree. The above example of the disk K shows that this degree is 1. The proof of Theorem 3 is complete.

2. THE CASE OF A CENTRALLY SYMMETRIC CONVEX FIGURE

Theorem 4. *Let K be a planar centrally symmetric convex figure and let $C_1, \ldots, C_6$ be cones with a common apex in the center of some regular hexagon whose sides are equal and symmetric about the middles of these sides as a direction sets. Then an affine transformation exists that maps the common apex cones to the symme-*

try center of the figure K such that the images of the cones C_1, ..., C_6 contain equal parts of the area of the figure K, and a selected ray that bounds one of the cones intersects the boundary of the figure K at a given point.

The proof is similar to that of Theorem 3. We assume that the sides of the hexagon and the cones are ordered anticlockwise, and denote by M the configuration space of the system of cones, as obtained from the original one by an orientation that preserves the affine transformations of the plane with a unit determinant, for which the common apex of the cones is the center of the figure K. We let A denote the set of such affine transformations such that the images of the cones satisfy the equality $S(C_1 \cap \mathrm{K}) = ... = \mathrm{S}(C_6 \cap K)$.

Having fixed some apex angle of the cone, we define the mapping $f: M \longrightarrow R^3$ by assigning the set of numbers $(S(C_1 \cap K), S(C_3 \cap K_0, S(C_5 \cap K))$ with the ordered family of cones $(C_1, ..., C_6)$. We claim that in a general position setting (that is, for the open C^1-dense set of figures K the mapping f is transverse to the line $x_1 = x_3 = x_5$ in R^3) the set A is a one-dimensional smooth compact orientable submanifold of the three-dimensional noncompact manifold of the affine transformations under consideration that is the generator of the fundamental group of M.

Let us verify that the disk K is a figure in a general position. For the disk K, only the regular configurations of cones with a common apex in the disk center are of interest. We assume that the above regular system of cones has a common apex in the center of the disk K and that an affine transformation a with unit determinant maps it in the system of cones with the same property. Hence, the system of cones $a(C_1)$, ..., $a(C_6)$ intercepts sectors of equal area on the figures K and $a(K)$. Hence, inside each of the cones $a(C_1)$, ..., $a(C_6)$ a ray passes that emanates from their common apex ray on which the figures K and $a(K)$ intercept equal segments; that is, their boundaries intersect at six points. As a result, they coincide by Bezout's theorem. Thus, in the case in question, the set A consists of orientation-preserving rotations about the disk center. Compressing the regular system of cones with a common apex in the center of the disk K to the bisector of the angle between the neighboring cones, we will assure that the sectors that are bounded by the cones nearest to the bisector cones will decrease their the area (while remaining equal), and the remaining two symmetric cones will increase its area with a nonzero rate. This proves that the disk K is a figure in a general position.

Let is define the continuous mapping $g: A \longrightarrow \partial K$ by associating the point of intersection of the image of a selected ray that bounds one of the cones with the boundary of the figure with an affine transformation. Any two planar centrally symmetric smooth convex figures with a common center are smoothly deformable into one another in the class of such figures (for example, by their convex linear combinations). As well, the set of the sought configurations of cones for figures from the above one-parameter family is compact. In fact, assuming the contrary, we will convert the thus-obtained system of cones by affine transformations into regular systems and then make a passage to the limit (as above). Thus, we will construct an infinite strip bounded by parallel lines from which the cones intercept equal areas; this may not be the case. Thus, a standard topological argument shows that for a figure in a general position the class of singular bordism of the mapping g is well defined; thus, so is its degree. The above example of the disk K shows that this degree is 1. The theorem is proven in the general position setting. In the remaining cases the required assertion follows by passage to the limit.

Remark. If the apex angle of the cone tends to 0, then in the limit we obtain the well-known theorem on the inscribed affinely regular hexagon with a vertex at an a priori given boundary point of the figure K.

It is known that in any centrally symmetric convex figure one may inscribe an affine image of the regular octagon [9],[10].

Theorem 5. *Let K be a planar centrally symmetric convex figure and let C_1, ..., C_8 be cones with a common apex in the center of some octagon, whose direction sets are segments of equal length that lie on the sides of the octagon and are symmetric about its midpoints. Then an affine transformation exists that maps the common apex of the cones to the symmetry center of the figure K such that the images of the cones C_1, ..., C_8 contain equal parts of the area of the figure K.*

Proof. We claim that in the general position setting (that is, for the open C^1-dense set of figures K with a smooth boundary) the sought configurations of the images of the cones form an uncountable set. Let us show that the system of cones with an angle $<\pi/4$ in the vertex of the square (smoothed in small neighborhoods of the vertices) is a figure in a general position, and that for it exactly the one sought system of cones exists.

In fact, the line that intercepts triangles of equal areas from two neighboring cones should be perpendicular to the diagonal of the octagon that separates them. Hence, for the square, the sought system of cones is one when the diagonals of the octagon are perpendicular to the square sides. Exchanging a pair of parallel sides of the square, thus making it into a parallelogram, one easily obtains that the areas of the parts of the figure in three pairs of symmetric cones will not change, and in the fourth pair they will change with a nonzero rate. By applying affine transformations to convert the thus-obtained parallelograms into the original square, we see that the smoothed square is a figure in a general position.

Any two planar centrally symmetric smooth convex figures with a common center are smoothly deformable into one another in the class of such figures (for example, by their convex linear combinations). As well, the set of the sought configurations of cones for figures from the one-parameter family under consideration is compact. In fact, assuming the contrary, we will convert the thus-obtained system of cones by affine transformations into a regular system, making a passage to the limit (as above), and build an infinite stripe that is bounded by parallel lines from which the cones intercept equal areas; however, this may not be the case. Consequently, an appeal to standard topological considerations shows that the residue modulo 2 of the number of sought configurations of cones for a figure in a general position is well defined. The above example of a smoothed square shows that this residue is 1. This proves the theorem in the general position setting. In the remaining cases the required assertion follows by passage to the limit.

Remark. If the apex angle of the cone tends to 0, then in the limit we obtain the theorem on the inscribed affinely regular octagon.

In [12] it was proven that in any planar centrally symmetric convex figure K one may either inscribe an affine image of a regular decagon or there are two affine images of a regular decagon for which four pairs of opposite vertices lie on the boundary of K; the two remaining vertices of the first polygon lie outside K and the two remaining vertices of the second polygon lie inside K.

Theorem 6. *Let K be a planar centrally symmetric convex figure. We consider ten cones with a common apex in the center of some regular decagon, whose direction sets are segments of equal length that lie on the sides of the decagon and are symmetric about its midpoints. Then, either an affine transformation exists that maps the common apex of the cones to the symmetry center of the figure K such that the images of the cones contain equal parts of area of the figure K or two affine images of the system of cones exist, with a common apex in the symmetry center of the figure K, such that the images of four pairs of symmetric cones contain equal parts of the area of the figure K, which in one case are larger than the parts that are contained in the cones of the remaining pair and in the other cases are smaller than the corresponding parts.*

Proof. Consider a right cylinder C with base K, whose height is much greater than the base diameter and let O be the symmetry center of C. We denote by $C_1, \ldots, C_5$ the pairs of symmetrical cones with respect to their common apex. We let M denote the manifold of the above system of cones with a common apex at the center O of the cylinder C, assuming that pairs of cones are ordered anticlockwise in the ambient oriented planes from the Grassman manifold of oriented two-dimensional planes in R^3. It is clear that M is homeomorphic to the Stiefel manifold $V_2(R^3)$ of 2-frames in R^3.

We define the continuous mapping $f: M \longrightarrow R^5$ by assigning the set of numbers $(S(C_1 \cap C), \ldots, (C_5 \cap$ C)) to an ordered set of cones. On the manifold M and R^5 the cyclic group Z_5 acts; it on the first manifold it acts by cyclic permutations of pairs of symmetric cones, while on the other ones it acts by cyclic permutations of coordinates. By this construction, the mapping f preserves the above group action.

According to [13], in this case the image $f(M)$ either contains a point with equal coordinates or contains two points with four equal coordinates such that for one point the fifth coordinate is smaller than the other ones, and for the second point the fifth coordinate is greater than the other ones. Clearly, by the central symmetry of C and since its height is much greater than the base diameter, it follows that the systems of cones that are thus obtained may not intersect the bases of the cylinder C. The required result then follows by projecting the above system of cones to the base K of the cylinder C.

Remark. The above assertion from [12] now follows in the limit if we make the apex angle of the cones 0.

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