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MECHANICS

On the Relationship between Control Theory and Nonholonomic Mechanics

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Abstract—This paper concerns transitions of a mechanical system, in a given time, from one state, in which the generalized coordinates and velocities are given, into another one, in which the system should have the required coordinates and velocities. It is assumed that such a transition may be effected using a single control force. It is shown that if one determines the force using the Pontryagin maximum principle (from the minimality condition of the time integral of the force squared during the time of motion), then a nonholonomic high-order constraint occurs in the defined process of motion of the system. As a result, this problem can be attacked by the theory of motion of nonholonomic systems with high-order constraints. According to this theory, in the set of different motions with a constraint of the same order, the optimal motion is the one for which the generalized Gauss principle is fulfilled. Thus, a control force that is chosen from the set of forces that provide the transition of a mechanical system from one state to another during a given time can be specified both on the basis of the Pontryagin maximum principle and on the basis of the generalized Gauss principle. Particular attention is paid to the comparison of the results that are obtained by these two principles. This is illustrated using the example of a horizontal motion of a pendula cart system to which a required force is applied. To obtain the control force without jumps at the beginning and the end of the motion, an extended boundary problem is formulated in which not only the coordinates and velocities are given at these times, but also the derivatives of the coordinates with respect to time up to the order $n \geq 2$. This extended boundary-value problem cannot be solved via the Pontryagin maximum principle, because in this case the number of arbitrary constants will be smaller than the total number of the boundary-value conditions. At the same time, the generalized Gauss principle is capable of solving the problem, because in this case it is only required to increase the order of the principle to a value that agrees with the number of given boundary-value conditions. The results of numerical calculations are presented.

Keywords: control theory, nonholonomic mechanics, high-order constraints, generalized Gauss principle, generalized principle of Hamilton–Ostrogradsky.

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1. INTRODUCTION

One of the most important problems in control theory is finding a control force that transfers a system from one phase state (with generalized coordinates and velocities of the system) of the system to a different given phase state, in which the system should have the required coordinates and velocities. As a rule such problems are solved using the Pontryagin maximum principle [1, 2].

In the present paper we use this principle to show that nonholonomic constraints of high order are satisfied during the motion of the system. Consequently, the theory of motion of nonholonomic systems with high-order constraints, as developed in the book [3], proves useful in attacking such problems. Of particular importance here is the fact that now the generalized Gauss principle [4] proves useful. The present paper is devoted to the comparison of the results that are obtained via the generalized Gauss principle and the Pontryagin maximum principle. For illustrative purposes we consider a controlled horizontal motion of a pendulum cart system subject to a sought-for force. To obtain a control without jumps at the start and at the end of the motion, we pose an extended boundary-value problem, in which at the terminal times not only the coordinates and the velocities are known, but also the derivatives of coordinates in time up to order $n \geq 2$. It is noted that this extended boundary-value problem may not be solved using the Pontryagin maximum principle, inasmuch as in this case the number of arbitrary constants will be smaller than the number of the boundary-value conditions. At the same time, the problem may be solved using the gener-

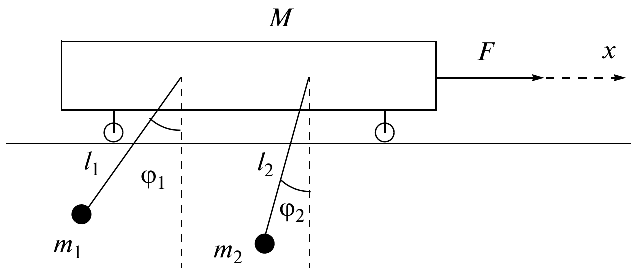

Fig. 1. A two-pendulum cart system.

alized Gauss principle, because in this case it is only required to increase the order of the principle to be used to a value agreeing with the number of given boundary-value conditions.

Some numerical results are given.

2. STATEMENT OF THE PROBLEM

To illustrate the theories under consideration we state the following specific problem.

Assume that on a lift crane trolley of mass M, which rides on horizontal rails along the x-axis, there are fixed ropes of lengths, respectively, l_1 and l_2. The loads of the masses m_1 and m_2 are suspended at the ends of these ropes (see Fig. 1). It is required, for a fixed time $\tilde{T}$, by choosing and applying a horizontal force $F(t)$ to a trolley, to transfer the suspended loads to a given distance a, from a state of rest to a state of rest.

The motion equations of this system are written, for small oscillations, as follows (g is the acceleration of gravity):

$$\begin{aligned} (M+m_1+m_2)\ddot{x} - m_1 l_1 \ddot{\varphi}_1 + -m_2 l_2 \ddot{\varphi}_2 &= F, \\ \ddot{x} - l_1\ddot{\varphi}_1 &= g\varphi_1, \\ \ddot{x} - l_2\ddot{\varphi}_2 &= g\varphi_2. \end{aligned} \tag{2.1}$$

To stop oscillation when the system is stopped, considering that the system is at rest ab initio, the control force should be such that the following boundary-value conditions are satisfied:

$$\begin{aligned} \varphi_1(0) = \varphi_1(\tilde{T}) = 0, &\quad \dot{\varphi}_1(0) = \dot{\varphi}_1(\tilde{T}) = 0, \\ \varphi_2(0) = \varphi_2(\tilde{T}) = 0, &\quad \dot{\varphi}_2(0) = \dot{\varphi}_2(\tilde{T}) = 0, \\ x(0) = \dot{x}(0) = \dot{x}(\tilde{T}) = 0, &\quad x(\tilde{T}) = a. \end{aligned} \tag{2.2}$$

The mechanical system in question has two nonzero eigenfrequencies, Ω_1 and Ω_2. Using the natural forms of oscillations corresponding to these frequencies, we introduce the principal dimensionless coordinates x_1 and x_2, specifying them as a linear combination of angles φ_1 and φ_2 (this fairly complicated transition was described in detail in E.A. Shatrov's paper "Using the principal coordinates in the problem of damping of oscillations of a two-pendulum cart system," which is published in the same issue of this journal). Changing to the dimensionless time $\tau = \Omega_1 t$ and introducing the third dimensionless principal coordinate x_0, which is proportional to the travel of the center of gravity of the mechanical system in question, we eventually have

$$x_0'' = u, \quad x_\sigma'' + \omega_\sigma^2 x_\sigma = u, \quad \sigma = \overline{1, s}. \tag{2.3}$$

Here, u is a control proportional to the force F, the prime denotes differentiation with respect to the dimensionless time τ, $\omega_\sigma = \Omega_\sigma/\Omega_1$, and $s = 2$ in the case of two pendula. In what follows, for the sake of generality, we assume that the number s of pendula is any positive integer.

The solution to problem (2.1)–(2.2) varies linearly with a, and hence, without loss of generality, we may assume that

$$x_0(T) = 1, \quad \text{where} \quad T = \Omega_1 \tilde{T}.$$

Thus, the boundary conditions for system (2.3) are as follows:

$$x_0(0) = x_0'(0) = x_0'(T) = 0, \quad x_0(T) = 1,$$
$$x_\sigma(0) = x_\sigma'(0) = x_\sigma(T) = x_\sigma'(T) = 0, \quad \sigma = \overline{1, s}. \tag{2.4}$$

3. THE SOLUTION OF THE PROBLEM VIA THE PONTRYAGIN MAXIMUM PRINCIPLE

To attack the above problem (2.3)–(2.4) another condition should be added. This condition should express the principle underlying the choice of the force $F(t)$ from the entire set of forces for which the problem has a solution. In solving similar problems in the book [1], the choice of a control is subject to the minimality condition for the functional

$$J = \int_0^T u^2 dt. \tag{3.1}$$

In accordance with the Pontryagin maximum principle, the control u in (2.3) will provide for the minimality of functional (3.1) if it is as follows.

System (2.3) is written as

$$q_k' = f_k(q, u), \quad k = \overline{1, 2s+2},$$

where

$$q_1 = x_0, \quad q_2 = x_0', \quad q_{2\sigma+1} = x_\sigma, \quad q_{2\sigma+2} = x_\sigma',$$

$$f_1 = q_2, \quad f_2 = u, \quad f_{2\sigma+1} = q_{2\sigma+2}, \quad f_{2\sigma+2} = u - \omega_\sigma^2 q_{2\sigma+1}, \quad \sigma = \overline{1, s}.$$

Next, the associated functions $p_k(\tau)$, $k = \overline{1, 2s+2}$ and the Hamilton–Pontryagin function

$$H = -u^2 + \sum_{k=1}^{2s+2} p_k f_k(q, u)$$

are introduced. The variables $p_k(\tau)$ satisfy the equations

$$p_k' = -\frac{\partial H}{\partial q_k}, \quad k = \overline{1, 2s+2},$$

and the sought-for control $u(\tau)$ is determined from the condition

$$\frac{\partial H}{\partial u} = 0. \tag{3.2}$$

In the case in question, we have

$$H = -u^2 + p_1 q_2 + p_2 u + \sum_{\sigma=1}^{s} p_{2\sigma+1} q_{2\sigma+2} + \sum_{\sigma=1}^{s} p_{2\sigma+2}(u - \omega_\sigma^2 q_{2\sigma+1}), \tag{3.3}$$

$$p_1' = 0, \quad p_2' = -p_1, \quad p_{2\sigma+1}' = \omega_\sigma^2 p_{2\sigma+2}, \quad p_{2\sigma+2}' = -p_{2\sigma+1}, \quad \sigma = \overline{1, s}. \tag{3.4}$$

Using (3.2) and (3.3), this gives

$$u(\tau) = \frac{1}{2}\sum_{n=1}^{s+1} p_{2n}(\tau).$$

In accordance with system (3.4) the functions $p_{2n}(\tau)$, $n = \overline{1, s+1}$ satisfy the equations

$$p_2'' = 0, \quad p_{2\sigma+2}'' + \omega_\sigma^2 p_{2\sigma+2} = 0, \quad \sigma = \overline{1, s}.$$

Thus, the functional (3.1) is minimized for

$$u(\tau) = C_1 + C_2\tau + \sum_{\sigma=1}^{s} (C_{2\sigma+1}\cos\omega_\sigma\tau + C_{2\sigma+2}\sin\omega_\sigma\tau). \tag{3.5}$$

Here, C_k, $k = \overline{1, 2s+2}$, are arbitrary constants. Choosing these constants so as to satisfy the boundary-value conditions (2.4), we find the sought-for control $u(\tau)$ in a unique way.

4. THE RELATIONSHIP OF THE SOLUTION OBTAINED VIA THE PONTRYAGIN MAXIMUM PRINCIPLE WITH THE NONHOLONOMIC PROBLEM

In this form we can obtain a new remarkable point of view to the solution obtained using the Pontryagin maximum principle.

Let us consider in detail the control (3.5). It may be observed that the function $u(\tau)$ is a general solution of the equation

$$\frac{d^2}{d\tau^2}\left(\frac{d^2}{d\tau^2}+\omega_1^2\right)\left(\frac{d^2}{d\tau^2}+\omega_2^2\right)\ldots\left(\frac{d^2}{d\tau^2}+\omega_s^2\right)u = 0. \tag{4.1}$$

After converting to the dimensional variables and to the case $s = 2$, we get

$$\frac{d^2}{dt^2}\left(\frac{d^2}{dt^2}+\Omega_1^2\right)\left(\frac{d^2}{dt^2}+\Omega_2^2\right)F = 0.$$

Substituting here the expression for F from the first equation of the original system (2.1), we obtain an eighth-order differential equation in the generalized coordinates x, φ_1, and φ_2

$$a_{8,x}\frac{d^8x}{dt^8}+a_{8,\varphi_1}\frac{d^8\varphi_1}{dt^8}+a_{8,\varphi_2}\frac{d^8\varphi_2}{dt^8}+a_{6,x}\frac{d^6x}{dt^6}+a_{6,\varphi_1}\frac{d^6\varphi_1}{dt^6}+a_{6,\varphi_2}\frac{d^6\varphi_2}{dt^6}$$
$$+a_{4,x}\frac{d^4x}{dt^4}+a_{4,\varphi_1}\frac{d^4\varphi_1}{dt^4}+a_{4,\varphi_2}\frac{d^4\varphi_2}{dt^4} = 0. \tag{4.2}$$

Here, the constant coefficients are determined from the parameters of the mechanical system as follows:

$$a_{8,x} = M+m_1+m_2, \quad a_{8,\varphi_1} = -m_1l_1, \quad a_{8,\varphi_2} = -m_2l_2,$$
$$a_{6,x} = (\Omega_1^2+\Omega_2^2)(M+m_1+m_2), \quad a_{6,\varphi_1} = -(\Omega_1^2+\Omega_2^2)m_1l_1, \quad a_{6,\varphi_2} = -(\Omega_1^2+\Omega_2^2)m_2l_2,$$
$$a_{4,x} = \Omega_1^2\Omega_2^2(M+m_1+m_2), \quad a_{4,\varphi_1} = -\Omega_1^2\Omega_2^2m_1l_1, \quad a_{4,\varphi_2} = -\Omega_1^2\Omega_2^2m_2l_2.$$

Note that having specified a different number s, we obtain, instead of equation (4.2), an equation of order $2s + 4$.

Thus, to the solution of the above problem, as obtained via the Pontryagin maximum principle, there corresponds a solution of some nonholonomic problem subject to the eighth-order constraint (4.2). In other words, if a mechanical system in question moves under the action of the control (obtained from the Pontryagin maximum principle), then during this motion the eighth-order nonholonomic constraint (4.2) is satisfied continuously (a constraint of order $2s + 4$ in the general setting).

This being so, solving the boundary-value problem (2.3), (2.4) with minimization of functional (3.1) via the Pontryagin maximum principle turns out to be equivalent to solving the problem of the motion of a mechanical system under some nonholonomic constraints of order $2s + 4$. Accordingly, it seems expedient to try to solve this mechanical problem based on the theory of motion of nonholonomic systems with high order constraints, as developed in the book [3]. In accordance with this theory, given a constraint of order $2s + 4$ one may put together an equation of order $2s + 2$ with respect to this constraint force. Thus, considering constraint (4.2) as some motion program to be followed by a mechanical system, this constraint force proves to be the required control force that provides the fulfillment of the given program. Hence, the differential equation of order $2s + 2$ with respect to the control may be interpreted as a differential equation with respect to the constraint force. However, having already started to employ the theory of motion of nonholonomic systems with high-order constraints, it is therefore natural instead of minimizing functional (3.1) via the Pontryagin maximum principle to use the variational principle peculiar to this theory. The principle we speak of is the generalized Gauss principle [4].

5. SOLVING THE PROBLEM VIA THE GENERALIZED GAUSS PRINCIPLE

As noted in [5], the system of equations (2.3) describes the controlled motion of a mechanical system that has a zero eigenfrequency and s different nonzero eigenfrequencies. It is only necessary that all natural modes of oscillations be excited by this control force. This is, of course, a fairly broad class of mechanical systems, which includes, as a classical example, a pendulum cart system. System (2.3) is written in the dimensionless form, and hence has a simple form. Thanks to this simplicity, it was shown above using simple computation that the minimality of functional (3.1) in accordance with the Pontryagin maximum principle is achieved in searching for the desired control in form (3.5). The generalized Gauss principle,

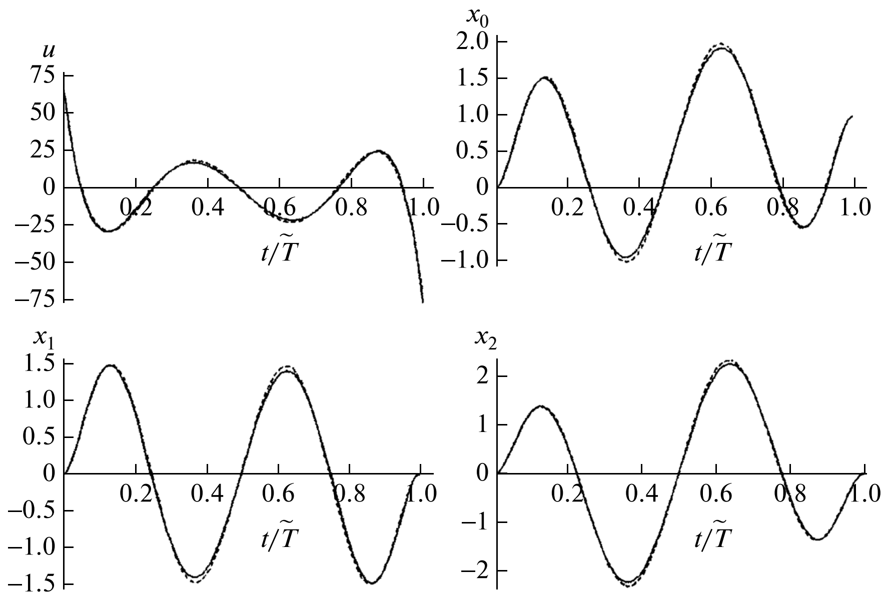

Fig. 2. Short-time motion of the mechanical system, $T = T_2$, $T_2 = 0.5T_1$.

as well as the Pontryagin maximum principle, is by no means related to whether the motion equations are written in the dimensional or in the dimensionless form and which coordinates (principal or standard) are used. Taking this into account, we formulate, for simplicity, the generalized Gauss principle in the context the system of equations (2.3), assuming that in (2.3) the derivatives in time are ordinary.

In the tangent space to system (2.3) one vector equation [3] corresponds

$$\mathbf{W} = \mathbf{Y} + \mathbf{R}, \tag{5.1}$$

where

$$\mathbf{W} = \sum_{\alpha=0}^{s} \ddot{x}_\alpha \mathbf{i}_\alpha, \quad \mathbf{Y} = -\sum_{\sigma=1}^{s} \omega_\sigma^2 x_\sigma \mathbf{i}_\alpha, \quad \mathbf{R} = \sum_{\alpha=0}^{s} u \mathbf{i}_\alpha.$$

Here, $\mathbf{i}_\alpha$, $\alpha = \overline{0, s}$, are the unit vectors that form the basis for the tangent space.

It was noted above that the control u that satisfies equation (4.1) may be looked upon as a force of a linear nonholonomic constraint of order $2s + 4$. Hence, in the vector equation (5.1), the vector corresponding to the presence of control u is denoted by the letter $\mathbf{R}$, which is conventionally used to denote the vector of the constraint force. According to the generalized Gauss principle of order $2s + 2$, a linear nonholonomic constraint of order $2s + 4$ is ideal if the quantity

$$\left(\overset{(2s+2)}{\mathbf{W}} - \overset{(2s+2)}{\mathbf{Y}}\right)^2 = \left(\overset{(2s+2)}{u} \sum_{\alpha=0}^{s} \mathbf{i}_\alpha\right)^2 \tag{5.2}$$

is minimal [4]. Here, the superscript $(2s + 2)$ denotes the order of the derivative in time.

From all possible linear nonholonomic constraints of order $2s + 4$ we single out a subset such that (5.2) equals its sharp lower bound (which is zero) for all elements from this set. To all these elements there corresponds a control u that satisfies the equation

$$\overset{(2s+2)}{u} = 0.$$

The general solution of this equation is given by

$$u(t) = \sum_{k=1}^{2s+2} C_k t^{k-1}. \tag{5.3}$$

As distinct from the control $u(t)$ given by (3.5), the control sought in the form of polynomial (5.3) will not have oscillations that agree with the corresponding eigenfrequencies of the system. The thus-obtained function will be sufficiently smooth, which is an unquestioned advantage of this function.

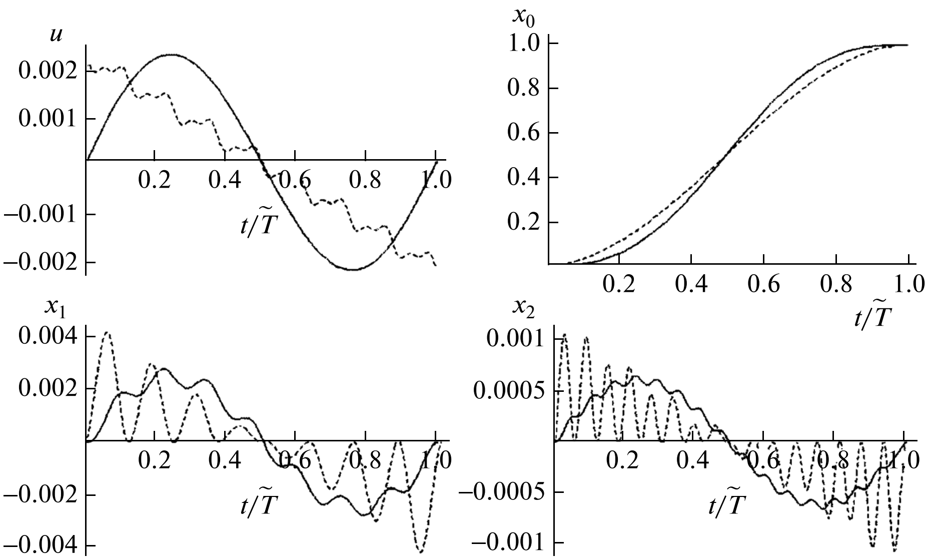

Fig. 3. Long-time motion of the mechanical system, $T = 16T_2$, $T_2 = 0.5T_1$.

6. NUMERICAL ANALYSIS

Figures 2 and 3 show the calculation results that were obtained via the two above methods. The graphs in Fig. 2 correspond to a short-time motion, when $T = T_2$, and in Fig. 3, to a long-time motion, when $T = 16T_2$. As well, it was adopted that $T_2 = 0.5T_1$ and assumed that $\omega_1 = 1$. The solutions that are obtained via the Pontryagin maximum principle are depicted in dashed lines and the solutions produced via the generalized Gauss principle are shown by solid lines.

From comparison of these two cases it is seen that in the case of a short-time motion the solutions that are obtained by these two methods are practically identical, while for a long-time motion they are substantially different. This difference may be explained by the fact that the control obtained through the Pontryagin maximum principle contains harmonics with eigenfrequencies of the system, which puts the system into resonance. At the same time, the control obtained via the generalized Gauss principle is given by a polynomial, which provides smoother motion of the system.

Another important point is worth pointing out: the application of the Pontryagin maximum principle always produces jumps in the control at the terminal times of motion. On the other side, if the generalized Gauss principle is employed, then such jumps pass away after a long motion. This suggests the following question: is it possible to eliminatejumps during short-time motion of the system? This question is addressed in the next section.

7. STATEMENT AND SOLUTION OF THE EXTENDED (GENERALIZED) BOUNDARY-VALUE PROBLEM. SINGULAR POINTS

It turns out that the generalized Gauss principle is capable of producing a control without jumps at terminal times for a short-time motion of a pendulum cart system. To this end one needs to augment the boundary-value conditions (2.4) with the following requirements:

$$x_0''(0) = x_0''(T) = 0. \tag{7.1}$$

It is worth noting that the extended (generalized) boundary-value problem (2.3), (2.4), (7.1) in this setting may not be solved by minimizing functional (3.1) via the Pontryagin maximum principle, because in this case the number of arbitrary constants in the solution is insufficient. Contrariwise, a solution of a similar extended boundary-value problem using the generalized Gauss principle may be constructed; to this end it suffices to increase its order by two. The numerical results for the generalized boundary-value problem with $T = T_2$, $T_2 = 0.5T_1$ is represented in Fig. 4. The graph of the dimensionless control shows that it proved possible to eliminate the jumps in the control force at the terminal times of motion of the system.

We point out that the application of the generalized Gauss principle to dampen the oscillations of the mechanical system in question is preferable. However, this method is rather restricted in scope. Numerical

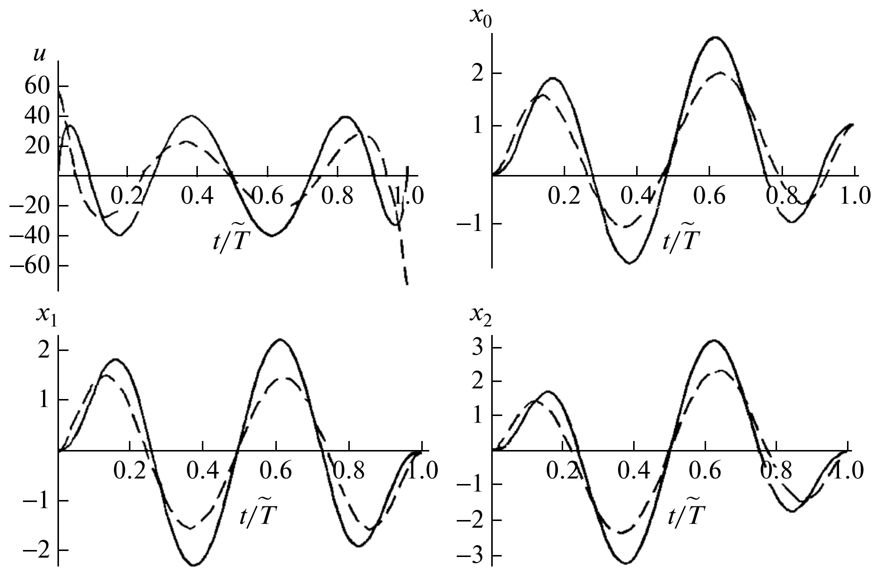

Fig. 4. Short-time motion without jumps in the control force, $T = T_2$, $T_2 = 0.5T_1$.

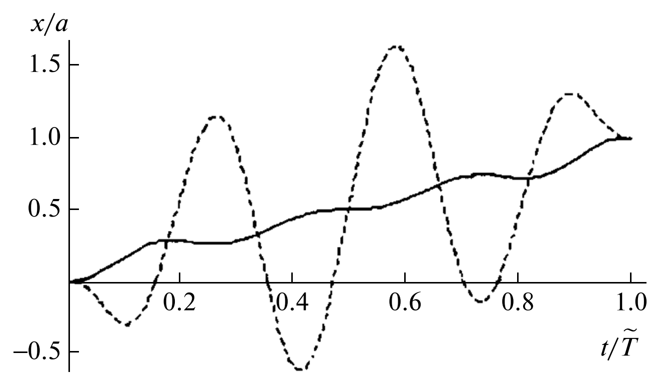

Fig. 5. The motion of a cart near a singular point.

analysis has shown that the solutions depend quite essentially on the parameter $\lambda = T/T_1$. In [5] it was shown that application of the generalized Gauss principle in solving boundary-value problems produces an infinite countable set of singular points λ near which the control force unboundedly increases. The solid line in Fig. 5 represents a movement of the cart, as expressed in fractions of a, with $\lambda = 1.5$. This corresponds to the solution of the boundary-value problem (2.1), (2.2) via the generalized Gauss principle with

$$\frac{l_2}{l_1} = \frac{1}{4}, \quad \frac{m_1}{M + m_1 + m_2} = \frac{4}{5}, \quad \frac{m_2}{M + m_1 + m_2} = \frac{1}{10},$$

when $\Omega_2/\Omega_1 = 2.242$.

At the same time, if with given parameters of the system one extends the boundary-value conditions by assuming that, for all points of the system, the accelerations are zero at terminal times, then it will turn out that the chosen parameter $\lambda = 1.5$ would lie near the first critical number, which equals 1.522. The evolution of intensive oscillations of the cart in this case is shown in Fig. 5 by the dashed line. It is seen that in this case the cart passes to the left of its initial position three times and passes to the right of its final position three times. However, even for such intensive oscillations the extended problem is solved and the resulting control force does not have any jumps near the terminal times.

A control without singular points may be produced by applying, for example, the generalized Hamilton—Ostrogradskii principle [6]. In this case the control is built using basis functions. Note that this solution is also polynomial in time, although the order of the polynomial is $4s + 3$.

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