Stability of a Laminar Flow at the Boundary of Uniform and Decelerating Flows

O. N. Melnikova[a], * and H. Yang[a]

[a] Faculty of Physics, Moscow State University, Moscow, 119991 Russia
e-mail: olamel@yandex.ru

Abstract—Conditions are obtained for the loss of stability of laminar motion at the boundary of uniform and decelerating flows, where the maximum drop in velocity occurs due to periodic deceleration of the fluid in the second section. At the end of the cycle, a local minimum with two inflection points appears on the flow velocity profile, and a vortex forms at the upper inflection point above the viscous layer. An estimate of the period of vortex formation is obtained.

INTRODUCTION

Prandtl in [1] was the first to assume the possibility of vortices appearing near the underlying surface in flows with a reverse pressure gradient that decelerates the liquid where viscosity is crucial. No mathematical description of such nonstationary boundary flows has been obtained so far. It was shown in [2, 3] that deceleration occurs periodically, and the deceleration cycle ends with the formation of a chain of cylindrical vortices with horizontal axis perpendicular to the flow. It was established experimentally in [4] that there is a phase of the periodic process of flow deceleration where a vertical velocity profile forms with a local minimum and two inflection points above the upper boundary of the viscous layer. It was found that a laminar flow with such a profile can lose its stability, resulting in the formation of vortices. The laminar flow is restored after the vortices depart. However, conditions of stability loss during such deformation of the vertical velocity profile were not studied. The aim of this work was to study the range of parameters of a flow with a reverse pressure gradient with stability of a laminar flow in a boundary layer of water with a free surface. The problem was solved experimentally. Experimental study of decelerated flows is complicated by the flow's parameters changing in the direction of its motion. To distinguish the narrow working area of a flow with predetermined local parameters, studies were performed were on the boundary of two regions of a straight channel. In the first region, flow velocity u and pressure p did not change in the direction of motion (uniform flow $\partial u/\partial x = 0$, $\partial p/\partial x = 0$). In the second, the velocity slowed in the motion direction. As the liquid decelerated along the boundary of the regions, the velocity gradient along the channel axis assumed its maximum value in the narrow area of

the longitudinal axis in a short time interval. Downstream of the boundary between the sections, the flow parameters determining the loss of stability did not have time to reach the critical value. As a result, there was no outflow of eddies downstream of the boundary, and no distortion of the velocity field.

EXPERIMENTAL

Our experiments were performed using a straight channel with transparent smooth walls 3.5 m long and 15 cm wide, and an adjustable bottom slope. The expenditure of water did not change over time, ensuring a stationary flow in the region with uniform motion of the liquid. In regions where velocity fell in the direction of motion, the flow was periodic. The longitudinal velocity profile was altered by the variable bottom slope of the channel and the change of water expenditure. The water layer $2.1 < h < 3.1$ cm thick. The maximum (in cross section) velocity of the flows in the series of experiments was $15 < U_{max} < 40$ cm s^{-1}, $u_x = \dfrac{\partial u}{\partial x} > -0.1$ s^{-1}. The velocity field was studied with a video recording of the displacement of particles of density $\rho = 1.05$ g cm^{-3} with an average diameter of 0.2 mm. Size of the particles allowed us to resolve the velocity of flow in a viscous layer around 2 mm thick. Recording was done through the side wall of the channel at a rate of 25 frames per second. The camera was focused manually on the channel axis in order to distinguish and use particles moving away from the vertical walls.

Processing the data with Adobe Photoshop allowed us to obtain the trajectories and velocities of the particles traveling along the working region at different horizons. To study the velocity field of the main flow,
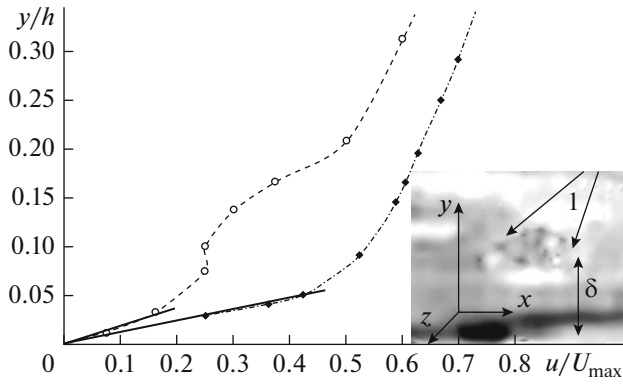
Fig. 1. Velocity profiles in the initial phase (black markers) and in the phase of maximum deceleration (white markers). Solid lines approximate the data in viscous layer. Video recording frame 1 shows the formation of the cylindrical vortex.

$$y = 0.0029u^3 - 0.1068u^2 + 1.318u - 4.9153$$

Fig. 2. Dependence $y(u)$ in the layer containing the upper inflection point. Markers are experimental data; the dashed line, the approximating function; the solid line, the tangent at the inflection point.

particles were used that did not change their position along the vertical, which excluded allowing for perturbations introduced by vortices [2, 5].
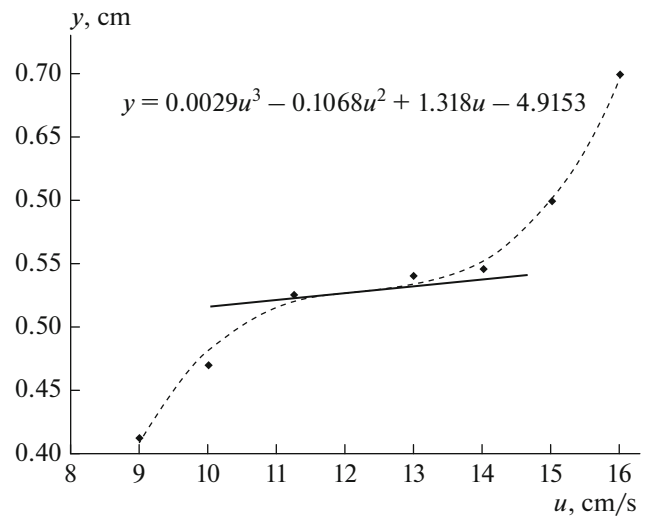
DEFORMATION OF THE VERTICAL VELOCITY PROFILE

It was shown in [4] that there was periodic deformation of the vertical profile of instantaneous velocity as the liquid in the boundary layer of a flow with a reverse pressure gradient is decelerated. A local minimum with two inflection points appears on the profile in the phase of maximum deceleration (Fig. 1). If we split the layer where the velocity minimum is observed into two sublayers containing one inflection point each, the lower $0.04 < \dfrac{y}{h} < 0.14$ and the upper

$0.14 < \dfrac{y}{h} < 0.23$, the stability of laminar flow can be estimated in each sublayer by the shape of the vertical profile of the flow velocity. According to the data presented in Fig. 1, the flow velocity grows monotonously along vertical axis y, and velocity derivative $\dfrac{\partial u}{\partial y} = u_y(y)$ has a minimum at the lower inflection point and a maximum at the upper inflection point. A number of theorems were obtained for such flows that allow us to estimating the stability of laminar motion from the shape of the velocity profile [6]. Conditions of Fjortoft's theorem are satisfied at the lower inflection point, where $u_y(y)$ has a minimum. This guarantees the resistance of laminar motion to low perturbations [6]. This is one of conditions of the Rosenbluth–Simon theorem, which ensures the necessary and sufficient condition of stability of laminar motion at the upper inflection points, where $u_y(y)$ has a maximum. This is true if $u_y(y)$ has a maximum at the

inflection point and the following expression has a negative sign [6]:

$$I = -\frac{dy}{du}\frac{1}{u-u_d}\bigg|_{u(y_1)}^{u(y_2)} + \int_{u(y_1)}^{u(y_2)} \frac{d^2y}{du^2}\frac{du}{u-u_d} < 0, \qquad (1)$$

where u_d is the flow velocity at the inflection point, and y_1, y_2 are coordinates of the boundaries of the layer with the inflection point. If I in (1) is greater than zero, then the flow is unstable. To check the stability of the flow in the phase of maximum deceleration in the layer containing the upper inflection point, we used a data obtained in a flow at a depth of $h = 3$ cm and a flow velocity on the free surface of $U_{max} = 30$ cm/s. Dependence $y(U)$ in the layer with the upper inflection point was approximated by the polynomial (Fig. 2):

$$y = au^3 + bu^2 + cu + d, \quad a = 0.0029, \\ b = -0.1068, \quad c = 1.318, \quad d = -4.9153. \qquad (2)$$

Boundaries of the layer $y_1 = 0.41$ cm, $y_2 = 0.65$ cm were selected so that the values of derivative $y'(u)$ on the boundaries would coincide. The first component in (1) was then $I_1 = -\dfrac{2y_b'}{u(y_2) - u(y_1)}$. In follows from (2) that for the integrand (1) we have

$$\frac{y''}{u - u_d} = \frac{6au + 2b}{u + \dfrac{b}{3a}} = 6a\frac{u + \dfrac{b}{3a}}{u + \dfrac{b}{3a}} = 6a. \qquad (3)$$
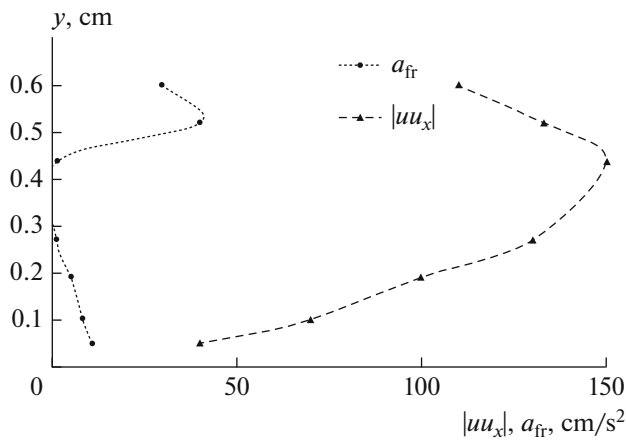
Fig. 3. Components of forces of reverse pressure gradient $|u \cdot u_x|$ and friction (a_{fr}) in the phase of the maximum deceleration of the liquid along the boundary of the regions of uniform and decelerating flows.

For the second component in (1) we therefore obtain $I_2 = 6a[u(y_2) - u(y_1)]$, and the condition of stability loss $I > 0$ can be written as

$$y'_b < 3a[u(y_2) - u(y_1)]^2. \qquad (4)$$

Substituting values $u(y_1) = 9$ cm/s, $u(y_2) = 15$ cm/s into (4), we obtain $0.103 < 0.379$ (i.e., the condition of instability (4) is satisfied in the layer with the upper inflection point). However, we should remember that the given stability criteria were obtained for profiles with one inflection point. This conclusion was confirmed the video in our experiment. In the video frame in Fig. 1, it can be seen that the vortex collapses in the region above the upper boundary of the viscous layer in the zone where a local minimum forms on the current velocity profile. A similar result was obtained in [7]: vortices periodically formed above the viscous layer of the flow.

ESTIMATING THE PERIOD OF VORTEX EMISSION

According to criterion (4), there is loss of stability upon critical deformation of the profile, determined by the velocity gradient in the zone of the local minimum. This gradient is determined by liquid's velocity of deceleration on the boundary of a uniform and decelerated flow. Formation of vortices was recorded at the critical value of the velocity gradient on the boundary of uniform and decelerated regions. In this experiment, the vortices collapsed when $\frac{\partial u}{\partial x} < -8$ s^{-1} was reached. Assuming the velocity gradient in (4) is approximately equal to the one on the boundary of the regions in the phase of maximum deceleration, we can

estimate the time needed to reach the phase of maximum deceleration or the period of vortex emission.

Let us consider motion of a unit volume of liquid with mass m along axis x in the zone of flow deceleration at a predetermined horizon. To obtain approximate estimates, we shall assume the volume maintains its shape and the flow is vortex free. The equation of motion and Cauchy–Lagrange integral can then be written as

$$m\frac{du}{dt} = -\frac{dp}{dx} + F_{\mathrm{fr}}, \quad p + m\left(\frac{u^2}{2} + \frac{\partial \varphi}{\partial t}\right) = \mathrm{const}, \qquad (5)$$

where φ is the potential of velocity. The friction force is determined by that of shear stress at the given horizon: $F_{\mathrm{fr}} = -\mu u_y S$, where μ is the dynamic viscosity of the liquid, and S is the area of the horizontal surface of the volume. We assume that the height of the volume is equal to the thickness of the viscous layer. Acceleration is then conditioned by shear stress $a_{\mathrm{fr}} = \frac{F_{\mathrm{fr}}}{m} = -\frac{\nu}{\delta}u_y$, where ν is the kinematic viscosity of liquid, and δ is the thickness of the viscous layer. The horizontal component of the pressure gradient is determined by differentiating second Eq. (5):

$$\frac{1}{m}\frac{dp}{dx} = -uu_x - u_t, \quad u_t = \frac{\partial u}{\partial t}. \qquad (6)$$

For the acceleration of a unit volume, we obtain the expression

$$\frac{du}{dt} = uu_x + u_t - a_{\mathrm{fr}}, \quad \text{where } a_{\mathrm{fr}} = \frac{\nu}{\delta}u_y. \qquad (7)$$

According to experimental data, the maximum intensity of deceleration at the beginning of the process falls on friction force in the viscous layer. In the upper sublayer, contributions from the friction force and reverse pressure gradient are equal and an order of magnitude less than the friction force in the viscous layer. The contribution from the reverse pressure gradient grows rapidly during deceleration, since the velocity gradient grows along with derivative u_x on the boundary of the regions with uniform and decelerated flow. Figure 3 shows that the force of the reverse pressure gradient in the phase of the maximum deceleration of the liquid is an order of magnitude greater than the friction force in the layer containing the upper inflection point. This allows us to make an approximate estimate of the time needed to reach the critical velocity gradient. Assuming the acceleration of the volume is mainly determined by the equation $\frac{du}{dt} = u_x u$, we find that the volume velocity on the boundary between uniform and decelerated flow falls during short time Δt to $u_1 = u_0 \exp(u_{x0}\Delta t)$. Here u_{x0}, the derivative of the velocity according to the longitudinal coordinate at the start of deceleration, is

assumed as predetermined. The change in the volume velocity at n equal and short intervals Δt is

$$u_n = u_0 \exp\left(\Delta t \sum_{i=0}^{n} u_{xi}\right), \qquad (8)$$

where for short interval Δt we may assume that $u_{xi} \approx 2\dfrac{u_i - u_0}{\Delta t(u_0 + u_i)}$. When velocity gradient $u_0 - u_n$ on the boundary of the regions of uniform and a decelerated flow reaches the value determined by criterion (4), we can expect loss of stability of the laminar flow. The period of deceleration is $T = n\Delta t$. Assuming that the flow velocity and u_x on the upper boundary of the viscous layer are half the values on the free surface [4], we calculate the period of deceleration for the initial data obtained in the experiment: $u_0 = 15$ cm s^{-1}, $u_{x0} = 0.05$ s^{-1}. We assume that time interval Δt is 0.04 s, which corresponds to the one between sequential frames. In our experiment, the velocity gradient in the upper sublayer was 6.6 cm/s in the phase of maximum deceleration (the end of the cycle). This value corresponds to $9 < n < 10$ and period $T = 0.38$ s. The average value of the period of vortice emission in the experiment was 0.45 s, and the confidence interval for the probability of 0.67 was 0.1 s. The calculated value does not extend beyond the confidence interval, which agrees with the proposed physical model of deceleration of the liquid on the boundary of uniform and decelerated flows.

CONCLUSIONS

The proposed way of studying the stability of laminar flow on the boundary of uniform and decelerated flows allowed us to determine the characteristics of a flow with a reverse pressure gradient that lead to loss of stability and the formation of vortices. It was shown that the vertical velocity profile in which a local minimum in the boundary layer appears rapidly deforms as the liquid decelerates on the boundary of the flows.

Maximum deformation is observed in the layer where the force of reverse pressure gradient grows in proportion to the velocity gradient on the boundary of the flows. There is a critical value of the velocity gradient at which laminar flow loses its stability and a vortex forms. This loss of stability occurs in the zone of local minimum on the vertical velocity profile, which forms in the phase of maximum deceleration. The profile contains two inflection points in the same zone. At the lower inflection point, the flow is stable; at the upper, it is not, and a vortex forms instead of a laminar flow. After the vortices depart from the layer containing the upper inflection point, the parameters of the velocity field return to the initial state. An estimate was obtained for the time interval from the onset of laminar flow deceleration to the departure of vortices. The result was confirmed by our experimental data.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Prandtl, L., *Proc. Third Int. Math. Congr.*, Heidelberg, 1904, p. 484.
2. Mel'nikova, O.N., *Izv. Atmos. Ocean. Phys.*, 2005, vol. 41, no. 5, p. 620.
3. Lebon, B., Nguyen, M.Q., Peixinho, J., et al., *Phys. Fluids*, 2018, vol. 30, 031701.
4. Melnikova, O.N., Pokazeev, K.V., and Yang, H., *Bull. Russ. Acad. Sci.: Phys.*, 2021, vol. 85, no. 1, p. 98.
5. Ahmadi, F., Sanders, S., and Ghaemi, S., *Phys. Rev. Fluids*, 2020, vol. 5, no. 1, 014302.
6. Dikii, L.A., *Gidrodinamicheskaya ustoichivost' i dinamika atmosfery* (Hydrodynamic Stability and Dynamics of the Atmosphere), Moscow: Gidrometeoizdat, 1976.
7. He, L., Yi, Sh.-H., Chen, Zh., and Zhu, Y.-Zh., *Proc. 14th Eur. Turbulence Conf.*, Lyon, 2013.

Translated by K. Gumerov