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Density Function of Weighted Sum of Chi-Square Variables with Doubly Degenerate Weights

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Abstract—We examine a weighted χ^2-distribution of a variable ξ that is a weighted sum of squares of independent standard normal random variables, where the weights that can be positive or negative. In the case of doubly degenerated weights, we obtain general expressions and discuss in detail some special cases. We show that if the weights are positive, then the values of ξ are distributed over the interval $\xi \in [0,\infty)$ and when $\xi \to 0$ the distribution density decreases as $\xi^{n/2-1}$ where n is the number of degrees of freedom. This case corresponds to the χ^2-distribution of the sums of squares of normally distributed quantities. We find the expressions complementing the Euler-Lagrange equalities.

Keywords: weighted chi-square distribution, density function, doubly degenerate weights, residue method, convolution method

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1. INTRODUCTION

According to the definition the χ^2-distribution is the distribution of the variable

$$\xi = \lambda\sum_{k=1}^{n} x_k^2, \tag{1}$$

where x_k are independent standard normal random variables with a zero mean and unit variance. For ease of comparison with the results obtained below, we introduce the coefficient λ.

The distribution of the variable ξ is well known:

$$P(\xi) = \frac{1}{2\lambda\Gamma(n/2)}\left(\frac{\xi}{2\lambda}\right)^{\frac{n}{2}-1} e^{-\frac{\xi}{2\lambda}}. \tag{2}$$

It has a single maximum at the point $\xi = \lambda(n-2)$ and its mean and variance are

$$\langle\xi\rangle = n\lambda,\ \ \langle\xi^2\rangle - \langle\xi\rangle^2 = 2n\lambda^2, \tag{3}$$

respectively. The number n is usually called *the number of degrees of freedom* of this distribution.

2. BASIC EXPRESSIONS

In the present paper, we discuss a more general distribution that is the distribution of the weighted sum of random variables

$$\xi = \sum_{k=1}^{n} \lambda_k x_k^2, \tag{4}$$

where the weights λ_k can be positive or negative. Such distribution appears in different applications (see, for example, [1–4]). Although many authors examined the distribution of the variable (4), nobody succeeded in obtaining a closed form of the distribution function. They proposed different approximations of this function using, for example, series in the Laguerre polynomials [5, 6], gamma series expansions [7, 8], and so on. The papers [9, 10] present a comprehensive review of the approximation methods.

It is easy to see that the presence of the weights in the sum (4) can be treated as a sum or difference of the standard normal random variables x_k^2 with the variance $\sigma_k^2 = |\lambda_k|^{-1}$ that is not equal to one: $\xi = \sum_1^n \mathrm{sgn}(\lambda_k)(x_k/\sigma_k)^2$.

When the number of degrees of freedom is not too large, to obtain the distribution of the variable (4) we can use the convolution of distributions with fewer degrees of freedom. In what follows, we will obtain a general expression for the case of a large number of degrees of freedom.

2.1. Convolution of Distributions

Suppose we have two random variables $\xi_1 \geq 0$ and $\xi_2 \geq 0$ whose distributions are $P_1(\xi_1)$ and $P_2(\xi_2)$, respectively. Then the expression

$$P(\xi) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 P_1(\xi_1) P_2(\xi_2) \delta[\xi - (\xi_1 \pm \xi_2)] \tag{5}$$

defines the distribution of the variable $\xi = \xi_1 \pm \xi_2$.

(a) Let us examine the distribution of the sum $\xi = \xi_1 + \xi_2$. In this case, from Eq. (5) we obtain

$$P(\xi) = \begin{cases} \int_0^\xi P_1(x) P_2(\xi - x) dx & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}, \tag{6}$$

We see that the distribution of the sum $\xi = \xi_1 + \xi_2$ is nonzero only inside the interval $\xi \geq 0$.

(b) Let us examine the distribution of the difference $\xi = \xi_1 - \xi_2$. From Eq. (5) it follows that in this case

$$P(\xi) = \begin{cases} \int_0^\infty P_1(x+\xi) P_2(x) dx, & \xi \geq 0 \\ \int_0^\infty P_1(x) P_2(x + |\xi|) dx, & \xi < 0 \end{cases}. \tag{7}$$

We see that the distribution of the variable $\xi = \xi_1 - \xi_2$ is nonzero both for $\xi \geq 0$ and $\xi \leq 0$. Moreover, when $P_1(x) = P_2(x)$ the distribution $P(\xi)$ is symmetrical about the point $\xi = 0$.

2.2. General Case

In the general case, it is difficult to apply the convolution method when the number of degrees of freedom is large. The expression

$$P(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^\infty dx_1 \int_{-\infty}^\infty dx_2 \ldots \int_{-\infty}^\infty dx_n e^{-\frac{1}{2}\sum x_k^2} \delta\left(\xi - \sum_{k=1}^n \lambda_k x_k^2\right) \tag{8}$$

describes the most general form of the distribution of the variable (4). It is not difficult to calculate the moments of this distribution performing the integration in Eq. (8). As a result, we obtain

$$\langle \xi \rangle = \sum_{k=1}^n \lambda_k, \quad \langle \xi^2 \rangle - \langle \xi \rangle^2 = 2\sum_{k=1}^n \lambda_k^2. \tag{9}$$

We see that the expressions (9) are a generalization of the expressions (3).

When we replace the δ-function with its integral representation the expression (8) takes the form

$$P(\xi) = \frac{1}{(2\pi)^{n/2+1}} \int_{-\infty}^\infty d\omega e^{i\omega\xi} \prod_{k=1}^n \left(\int_{-\infty}^\infty dx_k e^{-\frac{1}{2}x_k^2(1+i2\omega\lambda_k)} \right). \tag{10}$$

Carrying out the integration over the variables x_k we obtain

$$P(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\xi}}{\prod_{k=1}^{n}\sqrt{1 + i2\omega\lambda_k}} d\omega. \tag{11}$$

The following analysis of Eq. (11) depends on the order of degeneracy of the weights in the sum (4). In particular, when $\lambda_1 = \lambda_2 = ...\lambda_n = \lambda$ and n is even, the integral (11) has one pole of the order $n/2$ and we obtain the standard χ^2-distribution (2) for the variable ξ/λ.

For definiteness of the calculations, let us suppose that n is even ($n = 2M$) and all the weights in Eq. (4) are doubly degenerated. In this case, for each pair $\lambda_k = \lambda_r$ we introduce $\Lambda_m = \lambda_k = \lambda_r$. In other words, we have $M = n/2$ different weights Λ_m and $m = 1, 2, \ldots, M$. Then we can rewrite the integral (11) as

$$P(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\xi}}{\prod_{m=1}^{n/2}(1 + i2\omega\Lambda_m)} d\omega. \tag{12}$$

We see that this integral has $n/2$ poles of the first order at the points $\omega = i/2\Lambda_m$.

When integrating over ω, we have to consider the cases $\xi > 0$ and $\xi < 0$ separately. When $\xi > 0$, only the poles in the upper half-plane ($\mathrm{Im}\,\omega > 0$) of the complex variable ω determined by the positive weights contribute to the integral (11). In the case $\xi < 0$, the integral (11) is the sum of the residues of the poles in the lower half-plane ($\mathrm{Im}\,\omega < 0$), which correspond to the negative weights Λ_m. With this in mind we obtain

$$P(\xi) = \frac{1}{2}\sum_{\Lambda_m > 0} \frac{e^{-\xi/2\Lambda_m}}{\Lambda_m \prod_{r \neq m}^{n/2}(1 - \Lambda_r/\Lambda_m)}, \text{ when } \xi > 0, \tag{13}$$

$$P(\xi) = \frac{1}{2}\sum_{\Lambda_m < 0} \frac{e^{-|\xi/2\Lambda_m|}}{|\Lambda_m| \prod_{r \neq m}^{n/2}(1 - \Lambda_r/\Lambda_m)}, \text{ when } \xi < 0. \tag{14}$$

Note that in Eq. (13) ($\xi > 0$) we sum up only over $\Lambda_m > 0$ and when $\xi < 0$ the summation in Eq. (14) is carried out over $\Lambda_m < 0$.

We see that in the general case the distribution $P(\xi)$ is asymmetric with respect to the point $\xi = 0$. It becomes symmetric only in the special case of a symmetric spectrum of the weights Λ_m when for each positive weight Λ_k there is a negative weight $\Lambda_r = -\Lambda_k$.

Before proceeding to the analysis of special cases, let us discuss the general properties of the weighted distribution $P(\xi)$. For this purpose, we note that for any set of different values $\Lambda_1, \Lambda_2, \ldots, \Lambda_M$ the well-known Euler-Lagrange equalities are valid [11]:

$$\sum_{m=1}^{M} \frac{\Lambda_m^k}{\prod_{r \neq m}(\Lambda_m - \Lambda_r)} = \begin{cases} 1, & k = M - 1 \\ 0, & 0 \le k \le M - 2 \end{cases}. \tag{15}$$

These equalities follow from the properties of the Vandermonde determinant.

When $k = M - 1$, Eq. (15) corresponds to a normalization of the distribution $P(\xi)$. Indeed, performing the integration in (13), (14) we obtain an evident result:

$$\int_{-\infty}^{\infty} P(\xi) d\xi = \sum_{m=1}^{M} \frac{\Lambda_m^{M-1}}{\prod_{r \neq m}(\Lambda_m - \Lambda_r)} = 1. \tag{16}$$

In addition, from Eq. (15) we see that if $k = M - 1$ the function $P(\xi)$ is continuous at the point $\xi = 0$. Really, from Eqs. (13) and (14) it follows that

$$P(\xi \to 0_+) - P(\xi \to 0_-) = \sum_{m=1}^{M} \frac{\Lambda_m^{M-2}}{\prod_{r \neq m} (\Lambda_m - \Lambda_r)} = 0 \text{ and } P(\xi \to 0_+) = P(\xi \to 0_-). \quad (17)$$

The similar relations are also valid for the derivatives $d^k P/d\xi^k\big|_{\xi\to 0_+} = d^k P/d\xi^k\big|_{\xi\to 0_-}$. This means that derivatives of the order $k \in [1, M-2]$ are continuous at the point $\xi = 0$. Moreover, in the case of a symmetrical spectrum of the weights, from Eqs. (13)–(15) we have $dP/d\xi^k\big|_{\xi=0} = 0$ and consequently the distribution $P(\xi)$ is symmetric with respect to the point $\xi = 0$ and reaches its maximum at this point.

Let us discuss the properties of the distribution in a more specific case of the same sign of all the weights Λ_m. For example, let them be positive. Then from Eqs. (13) and (14) it follows that $P(\xi) = 0$ when $\xi < 0$ and on the interval $\xi \geq 0$ we have

$$P(\xi) = \frac{1}{2} \sum_{m=1}^{M} \frac{\Lambda_m^{M-2} e^{-\xi/2\Lambda_m}}{\prod_{r \neq m} (\Lambda_m - \Lambda_r)}, \quad \xi \geq 0, \quad M = n/2. \quad (18)$$

Setting $\xi = 0$ and comparing the obtained expression with Eq. (15) for $k = M-2$ we see that at this point the density $P(\xi)$ is equal to zero. Moreover, when $k < M-2$ the equality imposes conditions on the derivatives of the function $P(\xi)$ at zero point. Indeed, differentiating the expression (18) and comparing the result with Eq. (15), we obtain

$$\left.\frac{d^r P}{d\xi^r}\right|_{\xi=0} = \frac{(-1)^r}{2^r} \sum_{m=1}^{M} \frac{\Lambda_m^{M-2-r}}{\prod_{r \neq m} (\Lambda_m - \Lambda_r)} = 0 \text{ when } M-2 \geq r \geq 0. \quad (19)$$

From Eq. (19) it follows that the same as in the case of the standard the χ^2-distribution (2), the weighted sum with the positive weights decreases as $P(\xi) \sim \xi^{n/2-1}$ when $\xi \to 0$.

Concluding this Section, let us note that the comparison of the moments of the distribution $P(\xi)$ following from the general definitions (9) and the series representations (13) and (14) allows us to obtain expressions that complementing the Euler-Lagrange equalities (see Appendix).

3. EXAMPLES OF SPECIFIC DISTRIBUTIONS

To understand more clearly the basic properties of the weighted χ^2-distribution, let us discuss some simple examples.

Example I. The simplest case of the distribution of the weighted variable corresponds to the number of degrees of freedom $n = 2$. Then $\xi = \xi_1 \pm \xi_2$, where $\xi_1 = \lambda_1 x_1^2$, $\xi_2 = \lambda_2 x_2^2$, $\lambda_{1,2} > 0$. The equation (2) describes the distributions of the variables ξ_1 and ξ_2 with the numbers of degrees of freedom $n_1 = n_2 = 1$. Consequently, $P(\xi_k) = \exp(\xi_k/2\lambda_k)/\sqrt{2\pi\lambda_k\xi_k}$, $k = 1, 2$.

(Ia) $\xi = \xi_1 + \xi_2$. The variable $\xi = \xi_1 + \xi_2$ is positive definite. Therefore $P(\xi) = 0$ when $\xi < 0$ and in the case $\xi \geq 0$ we obtain from Eq. (6):

$$P(\xi) = \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} e^{-\frac{\xi}{2\lambda_2}} \int_0^{\xi} e^{-\frac{1}{2}x\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)} \frac{dx}{\sqrt{x(\xi - x)}}, \quad \xi \geq 0. \quad (20)$$

When substituting $x = \xi \cos^2(\varphi/2)$, after some transformations of the integral we obtain

$$P(\xi) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} e^{-\frac{\xi}{4}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)} \mathrm{I}_0(\xi_-), \quad \xi_- = \left|\frac{\xi}{4}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right|, \quad (21)$$

where $\mathrm{I}_0(\xi_-)$ is the modified Bessel function [12].

Since the asymptotics of the modified Bessel function are $\mathrm{I}_0(z) \to 1$ when $z \to 0$ and $\mathrm{I}_0(z) \to e^z/\sqrt{2\pi z}$ when $z \to \infty$ we have

$$P(\xi) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} \begin{cases} e^{-\frac{\xi}{4}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)}, & \xi \ll 1 \\ e^{-\frac{\xi}{2\lambda_{\min}}}, & \xi \gg 1 \end{cases} \text{ where } \lambda_{\min} = \min(\lambda_1, \lambda_2). \quad (22)$$

As we expected when $\lambda_1 = \lambda_2 = \lambda$ the expression (21) turns into the expression (2).

(Ib) $\xi = \xi_1 - \xi_2$. From Eq. (7) it follows that the distribution of the variable $\xi = \xi_1 - \xi_2$ for the interval $\xi \geq 0$ is

$$P(\xi) = \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} e^{-\frac{\xi}{2\lambda_1}} \int_0^{\infty} e^{-\frac{1}{2}x\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)} \frac{dx}{\sqrt{x(\xi+x)}}. \quad (23)$$

An analogous expression for the interval $\xi \leq 0$ we obtain making the change in the integral (23): $\lambda_1 \leftrightarrow \lambda_2$ and $\xi \to |\xi|$. Then after some transformations we have for $\xi \in (-\infty, \infty)$:

$$P(\xi) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} e^{-\frac{\xi}{4}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)} \mathrm{K}_0(\xi_+), \quad \xi_+ = \left|\frac{\xi}{4}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\right|, \quad (23)$$

and $K_0(z)$ is the modified Bessel function [12] whose asymptotics are $K_0(z) \sim |\ln z|$ when $z \to 0$ and $K_0(z) \sim e^{-z}\sqrt{\pi/2z}$ when $z \to \infty$. Consequently,

$$P(\xi) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} e^{\frac{\xi(\lambda_1-\lambda_2)}{4\lambda_1\lambda_2}} \ln\left(\frac{4\lambda_1\lambda_2}{|\xi|(\lambda_1+\lambda_2)}\right) \text{ when } |\xi| \ll 1 \quad (24)$$

and

$$P(\xi) = \frac{1}{\sqrt{2\pi|\xi|(\lambda_1+\lambda_2)}} \begin{cases} e^{-\frac{\xi}{2\lambda_1}}, & \xi \to +\infty \\ e^{-\frac{|\xi|}{2\lambda_2}}, & \xi \to -\infty \end{cases}. \quad (25)$$

Example II. Let us examine the distribution of the variable $\xi = \xi_1 \pm \xi_2$ with the number of degrees of freedom $n = 4$, where $\xi_1 = \lambda_1 \sum_{k=1}^{2} x_{1k}^2$ and $\xi_2 = \lambda_2 \sum_{k=1}^{2} x_{2k}^2$, $\lambda_{1,2} > 0$. The distributions of the variables $\xi_{1,2}$ are defined by Eq. (2) with the numbers of degrees of freedom $n_{1,2} = 2$.

(IIa) $\xi = \xi_1 + \xi_2$. The distribution of the variable in question is nonzero only in the interval $\xi \geq 0$. In this case, from Eq. (6) we obtain:

$$P(\xi) = \frac{e^{-\xi/2\lambda_1} - e^{-\xi/2\lambda_2}}{2(\lambda_1 - \lambda_2)}, \quad \xi \geq 0, \quad \lambda_{1,2} > 0 \quad (26)$$

and the moments of this distribution are

$$\langle\xi\rangle = 2(\lambda_1 + \lambda_2), \quad \langle\xi^2\rangle - \langle\xi\rangle^2 = 4\left(\lambda_1^2 + \lambda_2^2\right). \quad (27)$$

As we could expect when $\lambda_1 = \lambda_2 = \lambda$ the distribution (26) turns into the standard χ^2-distribution (2) with the number of degrees of freedom $n = n_1 + n_2 = 4$; the expressions for the moments of the distribution coincide with Eq. (3).

(IIb) $\xi = \xi_1 - \xi_2$. In this case, we obtain the distribution of the difference $\xi = \xi_1 - \xi_2$ from Eq. (7):

$$P(\xi) = \frac{1}{2(\lambda_1 + \lambda_2)} \begin{cases} e^{-\frac{\xi}{2\lambda_1}}, & \xi \geq 0 \\ e^{-\frac{|\xi|}{2\lambda_2}}, & \xi \leq 0 \end{cases}, \quad \lambda_{1,2} > 0. \quad (28)$$

The moments of this distribution have the form

$$\langle\xi\rangle = 2(\lambda_1 - \lambda_2), \quad \langle\xi^2\rangle - \langle\xi\rangle^2 = 4\left(\lambda_1^2 + \lambda_2^2\right). \quad (29)$$

The expressions (29) coincide with Eq. (27) if we make a change $\lambda_2 \to -\lambda_2$. However, this is where the coincidences end. We see that the distribution (28) is asymmetric with respect to the origin of coordinates $\xi = 0$. Moreover, the derivatives of $P(\xi)$ are discontinuous at the point $\xi = 0$. This example shows that the weighted distribution can be very different from the standard form (2).

Example III. Suppose we have two variables ξ_1 and ξ_2 that are subject to the χ^2-distribution with the numbers of the degrees of freedom n_1 and n_2, respectively. The distribution of the variable $\xi = \xi_1 + \xi_2$ is evident: it is defined by Eq. (2) with the number of degrees of freedom $n = n_1 + n_2$.

Let us examine the distribution of the variable $\xi = \xi_1 - \xi_2$. From Eq. (7) we obtain

$$P(\xi) = \text{const}\int_0^\infty \varepsilon^{m_1}(\varepsilon + |\xi|)^{m_2} e^{-\varepsilon - \frac{1}{2}|\xi|} d\varepsilon, \quad m_{1,2} = \frac{n_{1,2}}{2} - 1. \tag{30}$$

For the sake of simplicity we set $n_1 = n_2$ and $m_{1,2} = m$. In this case, the distribution is symmetric with respect to the point $\xi = 0$. To analyze the asymptotics of this distribution we use the saddle-point method. We suppose that $m \gg 1$ and rewrite Eq. (30) as

$$P(\xi) = P_0\int_0^\infty e^{-S(\varepsilon)} d\varepsilon, \tag{31}$$

where P_0 is a normalization constant whose form is not significant. In Eq. (31)

$$S(\varepsilon) = x - m\ln\left(x^2 - \frac{1}{4}\xi^2\right), \quad x = \varepsilon + \frac{1}{2}|\xi|, \tag{32}$$

$$\frac{dS}{dx} = 1 - \frac{2mx}{x^2 - \frac{1}{4}\xi^2}, \quad \frac{d^2S}{dx^2} = \frac{x-m}{mx}.$$

Setting $dS/dx = 0$ we define the saddle-point $x_0 = m\left(1 + \sqrt{1 + \xi^2/4m^2}\right)$ and the form of the distribution

$$P(\xi) = \frac{P_0}{\sqrt{4\pi(x_0 - m)}}(2mx_0)^{m - \frac{1}{2}} e^{-x_0}. \tag{33}$$

From Eq. (33) we obtain the form of the function $P(\xi)$ near the center of the distribution ($|\xi| \ll m$) and at the tails ($|\xi| \gg m$):

$$P(\xi) \sim \begin{cases} e^{-\frac{1}{16m}\xi^2}, & |\xi| \ll m \\ |\xi|^{m-1} e^{-\frac{1}{2}|\xi|}, & |\xi| \gg m \end{cases}, \quad m = \frac{n}{2}. \tag{34}$$

We see that the behavior of $P(\xi)$ resembles the χ^2-distribution only at the far ends of the tails.

Example IV. Let us examine the distribution of the variable (4) when the doubly degenerated weights ($n = 2M$) have the form

$$\frac{1}{\Lambda_k} = \frac{1}{\Lambda_0} + (k-1)\Delta, \quad \lambda_0 > 0, \quad \Delta \ge 0. \tag{35}$$

In this case, Eq. (13) takes the form

$$P(\xi) = \frac{1}{2} e^{-\frac{1}{2\Lambda_0}\xi} \sum_{k=1}^{M} \frac{1}{\Lambda_k \prod_{r \ne k}^{M} \Lambda_r\left(\Lambda_r^{-1} - \Lambda_k^{-1}\right)} e^{-\frac{1}{2}(k-1)\xi\Delta}. \tag{36}$$

When we take into account the relation

$$\frac{1}{\Lambda_k \prod_{r\neq k}^{M}\left[\Lambda_r\left(\Lambda_r^{-1}-\Lambda_k^{-1}\right)\right]} = \frac{(-1)^{k-1}}{(k-1)!(M-k)!\Delta^{M-1}\prod_{m=1}^{M}\Lambda_r}$$

from Eq. (36) we obtain:

$$P(\xi) = P_0 e^{-\frac{\xi}{2\Lambda_0}}\left(\frac{1-e^{-\frac{1}{2}\xi\Delta}}{\Delta}\right)^{M-1}, \quad P_0 = \frac{1}{2(M-1)!\Lambda_0^M}\prod_{k=1}^{M}(1+k\Lambda_0\Delta). \tag{37}$$

As we could expect, when $\Delta \to 0$ this expression passes to the non-degenerate case (2).

4. CONCLUSIONS

Our analysis shows that in the general case the distribution of the weighted sum bears little resemblance with the standard (not weighted) χ^2-distribution. Although the function $P(\xi)$ is continuous, its derivatives are generally discontinuous at the point $\xi = 0$ except for a few special cases. In the general case, our attempt to determine the maximum of the function $P(\xi)$ failed. The expressions (13) and (14) allow one to calculate the distribution $P(\xi)$ in the most general case.

APPENDIX

Using the found expressions we can obtain many different equalities for different sets of the weights $\Lambda_1, \Lambda_2, \ldots, \Lambda_M$.

(i) From Eq. (8) it is not difficult to obtain directly the expressions for the moments of the distribution $P(\xi)$. The same expressions follow from Eqs. (13) and (14). Indeed, integrating the series in these equations we can easily calculate the first two moments, which are

$$\langle\xi\rangle = \sum_{m=1}^{M}\frac{\Lambda_m^M}{\prod_{r\neq m}(\Lambda_m-\Lambda_r)}, \tag{A1}$$

$$\langle\xi^2\rangle = 2\sum_{m=1}^{M}\frac{\Lambda_m^{M+1}}{\prod_{r\neq m}(\Lambda_m-\Lambda_r)}. \tag{A2}$$

Comparing Eqs. (A.1) and (A.2) with the general expressions (9) we obtain a generalization of Eq. (15) to the case $k \geq M$:

$$\sum_{m=1}^{M}\frac{\Lambda_m^M}{\prod_{r\neq m}(\Lambda_m-\Lambda_r)} = 2\sum_{m=1}^{M}\Lambda_m, \tag{A3}$$

$$\sum_{m=1}^{M}\frac{\Lambda_m^{M+1}}{\prod_{r\neq m}(\Lambda_m-\Lambda_r)} = 4\sum_{m=1}^{M}\Lambda_m^2 + \left(2\sum_{m=1}^{M}\Lambda_m\right)^2. \tag{A4}$$

We can obtain the analogous equalities comparing the higher moments calculated using Eqs. (8), (13) and (14).

(ii) The approach described below allows us to derive some useful equalities. Suppose we have a series

$$\sum_{m=1}^{M}\frac{\Lambda_m^{M-1}}{(Z-\Lambda_m)\prod_{r\neq m}(\Lambda_m-\Lambda_r)}.$$

We (conventionally) add a new variable $\Lambda_{M+1} = Z$ to the variables Λ_m and rewrite this series in the following forms

$$\sum_{m=1}^{M}\frac{\Lambda_m^{M-1}}{(Z-\Lambda_m)\prod_{r\neq m}^{M}(\Lambda_m-\Lambda_r)}=-\sum_{m=1}^{M}\frac{\Lambda_m^{M-1}}{\prod_{r\neq m}^{M+1}(\Lambda_m-\Lambda_r)}=\frac{\Lambda_{M+1}^{M-1}}{\prod_{r=1}^{M}(\Lambda_{M+1}-\Lambda_r)}-\sum_{m=1}^{M+1}\frac{\Lambda_m^{(M+1)-2}}{\prod_{r\neq m}^{M+1}(\Lambda_m-\Lambda_r)}.$$

Since the last sum in this chain of equalities is strictly equal to zero we obtain

$$\sum_{m=1}^{M}\frac{\Lambda_m^{M-1}}{(Z-\Lambda_m)\prod_{r\neq m}^{M}(\Lambda_m-\Lambda_r)}=\frac{Z^{M-1}}{\prod_{r=1}^{M}(Z-\Lambda_r)}. \tag{A6}$$

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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