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Exact Asymptotics for the Distribution of the Time of Attaining the Maximum for a Trajectory of a Compound Poisson Process with Linear Drift

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Abstract—We consider the random process $at - \nu_+(pt) + \nu_-(-qt)$, $t \in (-\infty, \infty)$, where ν_- and ν_+ are independent standard Poisson processes if $t \geq 0$ and $\nu_-(t) = \nu_+(t) = 0$ if $t < 0$. Under certain conditions on the parameters a, p, and q, we study the distribution function $G = G(x)$ of the time of attaining the maximum for a trajectory of this process. In the present article, we find an exact asymptotics for the tails of G. We also find a connection between this problem and the statistical problem of estimation of an unknown discontinuity point of a density function.

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

For $t \geq 0$, let $\nu_- = \nu_-(t)$ and $\nu_+ = \nu_+(t)$ be independent standard Poisson processes. For $t < 0$, let $\nu_-(t) = \nu_+(t) = 0$. We put

$$Y(t) = at - \nu_+(pt) + \nu_-(-qt), \;\; t \in (-\infty, \infty). \tag{1}$$

The parameters a, p, and q are positive and satisfy the conditions

$$p > q, \quad a = \frac{p-q}{\ln(p/q)}. \tag{2}$$

By (2) and Lagrange's theorem, we obtain the inequalities

$$p > a = \frac{p-q}{\ln p - \ln q} > q. \tag{3}$$

They guarantee that the average drift of stochastic process (1) is negative; namely, we have

$$\mathbb{E}Y(t) = \begin{cases} (a-p)t < 0, & t > 0, \\ (a-q)t < 0, & t < 0. \end{cases}$$

Taking results of [15, Sec. 26] into account, we conclude that there exists a unique (with probability 1) proper random variable $t^*(Y) = \operatorname{argmax}\{Y(t)\}$ whose distribution function G is continuous. An explicit formula for G was found in [11] and later clarified in [12].

Put

$$\Lambda(z) = z - 1 - \ln z, \;\; z > 0. \tag{4}$$

Notice that $\Lambda(z)$ is positive if $z \neq 1$. By (2), we have

$$\Lambda(p/a) = \Lambda(q/a). \tag{5}$$

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We denote $\lambda = \Lambda(p/a) = \Lambda(q/a)$.

In [6], the exact asymptotic formulas

$$\begin{aligned} G(-x) &\sim \frac{c_-(p,q)}{x^{3/2}} e^{-\lambda a x}, \\ 1 - G(x) &\sim \frac{c_+(p,q)}{x^{3/2}} e^{-\lambda a x}, \end{aligned} \qquad x \to \infty, \tag{6}$$

are obtained for the distribution tails

$$\mathbb{P}\big(t^*(Y) < -x\big) = G(-x) \quad \text{and} \quad \mathbb{P}\big(t^*(Y) > x\big) = 1 - G(x),$$

where the constants $c_\pm$ depend on the parameters of stochastic process (1) only. The author of [6] mentions that he cannot find explicitly one of these constants but it is possible to find the other constant with the use of analytical methods from [3–5].

The aim of the present article is to find the constants $c_\pm$ in their explicit form and to obtain a series of results that clarify the analytical form of the distribution function G.

In the conclusion of this section, we mention that stochastic processes of the form (1) arise in the mathematical theory of insurance [14, p. 719] and in mathematical statistics [1, 2, 7–10] where we need to estimate an unknown discontinuity point of the density function for a given sample. Indeed, assume that the density $f = f(x,\theta)$, $\theta \in \Theta$, of an absolutely continuous distribution admits a unique jump at a point $x = \theta$, i.e., we have

$$0 < q(\theta) = f(\theta - 0, \theta) < f(\theta + 0, \theta) = p(\theta), \;\; \theta \in \Theta,$$

and $\{\widehat{\theta}_n\}$ is the series of maximal likelihood estimates (MLEs) of the parameter θ_0. The limit distribution (as $n \to \infty$) of the normed MLEs $n\big(\widehat{\theta}_n - \theta_0\big)$ coincides with the distribution of the time of attaining the maximum for a trajectory of the process

$$Z(t) = \big(p(\theta_0) - q(\theta_0)\big)t - \ln\big(p(\theta_0)/q(\theta_0)\big)\Big(\nu_+\big(p(\theta_0)t\big) - \nu_-\big(-q(\theta_0)t\big)\Big),$$

see [7, Ch. 5]. Put $p(\theta_0) = p$ and $q(\theta_0) = q$. We obtain $Z(t) = \ln(p/q) Y(t)$ and, consequently, $t^*(Z) = t^*(Y)$.

2. MAIN ASSUMPTIONS AND RESULTS

In stochastic process (1), we substitute $t := at$. This linear substitution allows us to simplify formulations and proofs. We obtain the following stochastic process:

$$Y^*(t) = t - \nu_+(pt) + \nu_-(-qt), \;\; t \in (-\infty, \infty), \tag{7}$$

where $p := p/a$ and $q := q/a$. Conditions (2)–(3) for stochastic process (7) assume the following form:

$$\begin{aligned} p > a = 1 > q > 0, \\ p - q = \ln(p/q). \end{aligned} \tag{8}$$

In the sequel, it will be more convenient to use the following form of the latter equality:

$$pe^{-p} = qe^{-q}. \tag{9}$$

Denote by $G^* = G^*(x)$ the distribution function of the time of attaining the maximum for a trajectory of stochastic process (7). By (6), the following asymptotic formulas are valid as $x \to \infty$:

$$\begin{aligned} G^*(-x) &\sim \frac{c^*_-(p,q)}{x^{3/2}} e^{-\lambda x}, \\ 1 - G^*(x) &\sim \frac{c^*_+(p,q)}{x^{3/2}} e^{-\lambda x}, \end{aligned} \tag{10}$$

where (see (4) and (5))

$$\lambda = \Lambda(p) = \Lambda(q). \tag{11}$$

We find the constants $c^*_\pm$ from (10). We consider the analytical representations of $G^*(-x)$ and $G^*(x)$ for $x > 0$ obtained in [11, Theorem 3] and [12, Theorem 1]. We have

$$G^*(-x) = (1-q)q\left(\int_x^\infty \sum_{k=[z]+1}^\infty \pi_k(q)\,dz - \int_x^\infty e^{bz} \sum_{k=[z]+1}^\infty \pi_k(q)e^{-bk}\,dz\right), \tag{12}$$

$$G^*(x) = \frac{(p-1)q}{(p-q)} + \beta\int_0^x e^{-pz}\sum_{k=0}^{[z]} \frac{p^k z^{k-1}}{k!}(z-k)\psi(z-k)\,dz, \tag{13}$$

where

$$\begin{aligned} \pi_k(q) &= (qk)^{k-1}e^{-qk}/k!, \quad k = 1,2,\dots, \\ b &= \beta(1-q/p), \end{aligned} \tag{14}$$

β is the unique positive solution to the equation

$$1 - e^{-\beta} = \beta/p, \tag{15}$$

and $[z]$ denotes the integer part of a number z. The integrand ψ in (13) is the distribution function for a certain random variable; namely, we have

$$\begin{aligned} \psi(x) &= \mathbb{P}\Big(\sup_{t<0} Y^*(t) \leqslant x\Big) \\ &= (1-q)\sum_{m=0}^{[x]} (-1)^m q^m \frac{(x-m)^m}{m!} e^{q(x-m)}, \quad x \ge 0. \end{aligned} \tag{16}$$

An analytical form of this function was found in [15]. Its properties were studied in [11, Lemmas 2 and 4].

The following assertions are the main results of the present article.

Theorem 1. *The constant c^*_- in (10) has the form*

$$c^*_- = \frac{(1-q)qe^{1-q}}{\sqrt{2\pi}(1-qe^{1-q})}\left(\frac{1}{1-qe^{1-q}} - \frac{p\big(\exp\{(p-q)^2/p\}-1\big)}{(p-q)^2\big(\exp\{(p-q)^2/p\} - qe^{1-q}\big)}\right).$$

Theorem 2. *The constant c^*_+ in (10) has the form*

$$c^*_+ = \frac{p(1-q)}{\sqrt{2\pi}(p-1)^2}\left(\frac{1}{1-qe^{1-q}} + \frac{(p^2-pq+q)\exp\{q(1-p)/p\}-1}{\big(p\,\exp\{q(1-p)/p\}-1\big)^2}\right).$$

3. PROOFS

Lemma 1. *The number $\beta = p - q$ is a solution to equation (15). The constant b in (12) and (14) is equal to $(p-q)^2/p$.*

Proof. We have $p - q > 0$. Substitution $\beta = p - q$ in (15) leads to relation (9). □

Lemma 2. *For every natural n, we have*

$$G^*(-n) = (1-q)q\left(\sum_{k=1}^\infty k\pi_{k+n}(q) - \frac{1}{b}\sum_{k=1}^\infty \big(1-e^{-bk}\big)\pi_{k+n}(q)\right), \tag{17}$$

where the sequence $\{\pi_k(q)\}$ is defined in (14) and the constant b is defined in Lemma 1.

Proof. We introduce the following notation for the integrals in (12):

$$A_n = \int_n^\infty \sum_{k=[z]+1}^\infty \pi_k(q)\,dz, \qquad B_n = \int_n^\infty e^{bz}\sum_{k=[z]+1}^\infty \pi_k(q)e^{-bk}\,dz.$$

We have

$$G^*(-n) = (1-q)q(A_n - B_n), \tag{18}$$

$$A_n = \sum_{m=n}^{\infty} \int_m^{m+1} \sum_{k=[z]+1}^{\infty} \pi_k(q)\, dz = \sum_{m=n}^{\infty} \sum_{k=m+1}^{\infty} \pi_k(q)$$
$$= \sum_{k=n+1}^{\infty} \sum_{m=n}^{k-1} \pi_k(q) = \sum_{k=n+1}^{\infty} (k-n)\pi_k(q) = \sum_{k=1}^{\infty} k\pi_{k+n}(q).$$

In a similar way, we find that

$$B_n = \sum_{m=n}^{\infty} \int_m^{m+1} e^{bz} \sum_{k=[z]+1}^{\infty} \pi_k(q) e^{-bk}\, dz = \sum_{m=n}^{\infty} \sum_{k=m+1}^{\infty} \pi_k(q) e^{-bk} \int_m^{m+1} e^{bz}\, dz$$
$$= \sum_{k=n+1}^{\infty} \pi_k(q) e^{-bk} \sum_{m=n}^{k-1} \int_m^{m+1} e^{bz}\, dz = \sum_{k=n+1}^{\infty} \pi_k(q) e^{-bk} \int_n^k e^{bz}\, dz$$
$$= \sum_{k=n+1}^{\infty} \pi_k(q) e^{-bk} \frac{e^{bk} - e^{bn}}{b} = \frac{1}{b} \sum_{k=1}^{\infty} \pi_{k+n}(q)(1 - e^{-bk}).$$

It remains to substitute the ultimate expressions for A_n and B_n into (18). □

Lemma 3. *The following asymptotic relations are valid*:

$$\sum_{k=1}^{\infty} \pi_{k+n}(q) \sim \frac{c_-^{(1)}}{\sqrt{2\pi} q n^{3/2}} e^{-\lambda n},$$
$$\sum_{k=1}^{\infty} e^{-bk} \pi_{k+n}(q) \sim \frac{c_-^{(2)}}{\sqrt{2\pi} q n^{3/2}} e^{-\lambda n},$$
$$\sum_{k=1}^{\infty} k\pi_{k+n}(q) \sim \frac{c_-^{(3)}}{\sqrt{2\pi} q n^{3/2}} e^{-\lambda n}$$

as $n \to \infty$, *where* $c_-^{(1)} = e^{-\lambda}/(1 - e^{-\lambda})$, $c_-^{(2)} = e^{-\lambda}/(e^b - e^{-\lambda})$, $c_-^{(3)} = e^{-\lambda}/(1 - e^{-\lambda})^2$, *and the constant* λ *is defined in* (11).

Proof. We use definitions in (4) and (11) and transform sequence (14) as follows:

$$\pi_k(q) = \frac{k^{k-1} e^{-k}}{qk!} e^{-(q-1-\ln q)k} = \frac{k^{k-1} e^{-k}}{qk!} e^{-\lambda k}, \quad k = 1, 2, \dots .$$

We have

$$\pi_{k+n}(q) = \frac{e^{-\lambda n}}{\sqrt{2\pi} q n^{3/2}} C_k(n) e^{-\lambda k}, \tag{19}$$

where

$$C_k(n) = \frac{\sqrt{2\pi} n^{3/2} (k+n)^{k+n-1} e^{-(k+n)}}{(k+n)!}, \quad k = 1, 2, \dots . \tag{20}$$

We find that

$$\sum_{k=1}^{\infty} \pi_{k+n}(q) = \frac{e^{-\lambda n}}{\sqrt{2\pi} q n^{3/2}} \sum_{k=1}^{\infty} C_k(n) e^{-\lambda k}. \tag{21}$$

We apply Stirling's approximation $(k+n)! \sim \sqrt{2\pi}(k+n)^{k+n+1/2}e^{-(k+n)}$, $n \to \infty$, to sequence (20). We obtain

$$\lim_{n\to\infty} C_k(n) = 1, \quad k = 1, 2, \dots .$$

Since $(k+n)! > \sqrt{2\pi}(k+n)^{k+n+1/2}e^{-(k+n)}$, we have

$$C_k(n) < \frac{n^{3/2}}{(k+n)^{3/2}} < 1$$

for all k and n. We conclude that the following equalities are valid:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} C_k(n)e^{-\lambda k} = \sum_{k=1}^{\infty} \lim_{n\to\infty} C_k(n)e^{-\lambda k} = \sum_{k=1}^{\infty} e^{-\lambda k} = c_-^{(1)}.$$

Together with (21), they yield the first asymptotic formula of the lemma. The proofs of the second and third formulas are similar and are based on the same expression (19). For example, we deduce

$$\sum_{k=1}^{\infty} k\pi_{k+n}(q) = \frac{e^{-\lambda n}}{\sqrt{2\pi}qn^{3/2}} \sum_{k=1}^{\infty} C_k(n)ke^{-\lambda k},$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} C_k(n)ke^{-\lambda k} = \sum_{k=1}^{\infty} ke^{-\lambda k} = e^{-\lambda}/(1-e^{-\lambda})^2 = c_-^{(3)}. \qquad \square$$

Proof of Theorem 1. By (17) and Lemma 3, we have

$$G^*(-n) \sim \frac{1-q}{\sqrt{2\pi}}\left(c_-^{(3)} - \frac{c_-^{(1)} - c_-^{(2)}}{b}\right)\frac{e^{-\lambda n}}{n^{3/2}}, \quad n \to \infty.$$

We find the constant c_-^*. We deduce

$$\begin{aligned} c_-^* &= \frac{1-q}{\sqrt{2\pi}}\left(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} - \frac{1}{b}\left(\frac{e^{-\lambda}}{1-e^{-\lambda}} - \frac{e^{-\lambda}}{e^b - e^{-\lambda}}\right)\right) \\ &= \frac{(1-q)e^{-\lambda}}{\sqrt{2\pi}(1-e^{-\lambda})}\left(\frac{1}{1-e^{-\lambda}} - \frac{e^b-1}{b(e^b-e^{-\lambda})}\right). \end{aligned} \tag{22}$$

We replace $e^{-\lambda}$ by qe^{1-q} (which is also equal to pe^{1-p}) and b by $(p-q)^2/p$. We finally obtain

$$c_-^*(p,q) = \frac{(1-q)qe^{1-q}}{\sqrt{2\pi}(1-qe^{1-q})}\left(\frac{1}{1-qe^{1-q}} - \frac{p\big(\exp\{(p-q)^2/p\}-1\big)}{(p-q)^2\big(\exp\{(p-q)^2/p\}-qe^{1-q}\big)}\right). \qquad \square$$

Remark 1. It is not difficult to see that $c_-^* > 0$. Indeed, since $e^{-b} > 1-b$, we obtain the following lower estimate for the expression in the last parentheses in (22):

$$\begin{aligned} &\frac{1}{1-e^{-\lambda}} - \frac{e^b-1}{b(e^b-e^{-\lambda})} \\ &\quad = \frac{1}{1-e^{-\lambda}} - \frac{e^b-1}{be^b(1-e^{-\lambda-b})} > \frac{1}{1-e^{-\lambda}} - \frac{e^b-1}{be^b(1-e^{-\lambda})} \\ &\quad = \frac{1}{1-e^{-\lambda}}\left(1 - \frac{1-e^{-b}}{b}\right) > 0. \end{aligned}$$

For $z \geq 0$, we define

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{(qe^{-q})^k(z+k)^k}{k!}, \tag{23}$$

$$J_k(z) = \frac{(pe^{-p})^k (z+k)^k}{k!} e^{-pz}, \quad k = 0, 1, \dots,$$
$$J(z) = \sum_{k=0}^{\infty} J_k(z) = e^{-pz} \sum_{k=0}^{\infty} \frac{(pe^{-p})^k (z+k)^k}{k!}, \tag{24}$$

where we assume that $0^0 = 1$. Recall that p and q satisfy relations (8)–(9). Hence, we have

$$J(z) = e^{-pz} \varphi(z). \tag{25}$$

Lemma 4. *The following representation is valid*:

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{(qe^{-q})^k \varphi(k)}{k!} z^k.$$

Proof. Applying Newton's binomial theorem to (23), we find that

$$\begin{aligned}\varphi(z) &= \sum_{k=0}^{\infty} \frac{(qe^{-q})^k}{k!} \sum_{m=0}^{k} C_k^m z^m k^{k-m} \\ &= \sum_{m=0}^{\infty} \frac{(qe^{-q})^m}{m!} z^m \sum_{k=m}^{\infty} \frac{(qe^{-q})^{k-m} k^{k-m}}{(k-m)!} \\ &= \sum_{m=0}^{\infty} \frac{(qe^{-q})^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(qe^{-q})^k (m+k)^k}{k!}.\end{aligned}$$

It is obvious that the second sum in the rightmost expression is equal to $\varphi(m)$. □

The following assertion is of intrinsic interest too.

Lemma 5. *If p and q satisfy relations* (8) *then the functions φ and J from* (23) *and* (24) *can be represented as follows*:

$$\varphi(z) = e^{qz}/(1-q),$$
$$J(z) = e^{-(p-q)z}/(1-q). \tag{26}$$

Proof. We formally differentiate the series $\varphi(z)$ from (23). We find that

$$\varphi'(z) = \sum_{k=1}^{\infty} \frac{(qe^{-q})^k (z+k)^{k-1}}{(k-1)!} = qe^{-q} \sum_{k=0}^{\infty} \frac{(qe^{-q})^k (z+1+k)^k}{k!}. \tag{27}$$

We obtain the equality

$$\varphi'(z) = qe^{-q} \varphi(z+1) \tag{28}$$

because the following estimate is valid for the series in (27):

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(qe^{-q})^k (z+1+k)^k}{k!} &= \sum_{k=0}^{\infty} \frac{(qe^{-q})^k k^k (1+(z+1)/k)^k}{k!} \\ &< \sum_{k=0}^{\infty} \frac{(qe^{-q})^k k^k e^{z+1}}{\sqrt{2\pi} k^{k+1/2} e^{-k}} = \frac{e^{z+1}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{e^{-\lambda k}}{\sqrt{k}}.\end{aligned}$$

We recall that $\lambda = q - 1 - \ln q$, see (4) and (11).

We use induction and prove the first equality in (26) for integer values of n, i.e., we prove that

$$\varphi(n) = e^{qn}/(1-q), \;\; n = 0, 1, \dots. \tag{29}$$

For the base case, we use the well-known series

$$\sum_{k=1}^{\infty} \frac{(qe^{-q})^k k^k}{k!} = \frac{q}{1-q}, \;\; 0 < q < 1,$$

see formula 4 at [13, p. 707]. Combining the above formula with (23), we obtain

$$\varphi(0) = 1 + \sum_{k=1}^{\infty} \frac{(qe^{-q})^k k^k}{k!} = 1 + \frac{q}{1-q} = \frac{1}{1-q}\,.$$

This corresponds to equality (29) for $n = 0$. We assume that equality (29) holds for some natural number n. Applying (28), we deduce

$$\varphi(n+1) = \frac{\varphi'(n)}{qe^{-q}} = \frac{qe^{qn}}{(1-q)qe^{-q}} = \frac{e^{q(n+1)}}{(1-q)}\,.$$

We substitute the expression on the right-hand side of (29) into the representation of $\varphi(z)$ from Lemma 4. We find that

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{(qe^{-q})^k e^{qk}/(1-q)}{k!} z^k = \frac{1}{1-q} \sum_{k=0}^{\infty} \frac{(qz)^k}{k!} = \frac{e^{qz}}{1-q}\,,$$

which finishes the proof of the first equality in (26). Combining this equality with (25), we obtain the second equality in (26). □

Lemma 6. *The sequence* $\{J_k\}$ *of functions in* (24) *satisfies the equalities*

$$z \sum_{k=0}^{\infty} \frac{J_k(z)}{z+k} = (1-q)J(z) = e^{-(p-q)z}, \ \ z \geq 0.$$

Proof. We verify the equality

$$z \sum_{k=0}^{\infty} \frac{J_k(z)}{z+k} = J(z) - pJ(z+1). \tag{30}$$

Indeed, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{z}{z+k} J_k(z) &= \sum_{k=0}^{\infty} J_k(z) - \sum_{k=0}^{\infty} \frac{k}{z+k} J_k(z) \\
&= J(z) - \sum_{k=1}^{\infty} \frac{(pe^{-p})^k (z+k)^{k-1}}{(k-1)!} e^{-pz} \\
&= J(z) - p \sum_{k=0}^{\infty} \frac{(pe^{-p})^k (z+1+k)^k}{k!} e^{-p(z+1)} \\
&= J(z) - pJ(z+1).
\end{aligned}$$

By the second equality in (26) and equality (9), we find that

$$pJ(z+1) = pe^{-(p-q)} J(z) = pe^{-p} e^q J(z) = qJ(z).$$

Combining this relation with (30) and (26), we obtain the required assertion. □

We prove an assertion concerning the integral in (13).

Lemma 7. *For every natural* n*, we have*

$$\begin{aligned}
1 - G^*(n) = (p-q)\Bigg(&\int_0^{\infty} \sum_{k=1}^{n} \frac{z+k}{z+n} \psi(z+k) J_{n-k}(z+k)\, dz \\
&+ \int_0^{\infty} z\psi(z) \sum_{k=n}^{\infty} \frac{J_k(z)}{z+k}\, dz \Bigg).
\end{aligned} \tag{31}$$

Proof. From (13) and Lemma 1 it follows that

$$1 - G^*(n) = (p-q)\int_n^\infty e^{-pz}\sum_{k=0}^{[z]}\frac{p^k z^{k-1}}{k!}(z-k)\psi(z-k)\,dz = (p-q)\,I_n\,, \tag{32}$$

where I_n denotes the integral in (32). We have

$$\begin{aligned} I_n &= \sum_{m=n}^\infty \int_m^{m+1} e^{-pz}\sum_{k=0}^{m}\frac{p^k z^{k-1}}{k!}(z-k)\psi(z-k)\,dz \\ &= \sum_{m=n}^\infty\sum_{k=0}^{m}\int_m^{m+1}(z-k)\psi(z-k)\frac{p^k z^{k-1}}{k!}e^{-pz}\,dz. \end{aligned}$$

We transform the latter expression as follows (for brevity, we omit the integrand):

$$\begin{aligned} I_n &= \sum_{m=n}^\infty\sum_{k=0}^{m}\int_m^{m+1} = \left(\sum_{m=0}^\infty - \sum_{m=0}^{n-1}\right)\sum_{k=0}^{m}\int_m^{m+1} \\ &= \sum_{m=0}^\infty\sum_{k=0}^{m}\int_m^{m+1} - \sum_{m=0}^{n-1}\sum_{k=0}^{m}\int_m^{m+1} \\ &= \sum_{k=0}^\infty\sum_{m=k}^{\infty}\int_m^{m+1} - \sum_{k=0}^{n-1}\sum_{m=k}^{n-1}\int_m^{m+1} \\ &= \sum_{k=0}^\infty\int_k^\infty - \sum_{k=0}^{n-1}\int_k^n = \sum_{k=0}^{n-1}\int_n^\infty + \sum_{k=n}^\infty\int_k^\infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_n &= \sum_{k=0}^{n-1}\int_n^\infty (z-k)\psi(z-k)\frac{p^k z^{k-1}}{k!}e^{-pz}\,dz \\ &\quad + \sum_{k=n}^\infty\int_k^\infty (z-k)\psi(z-k)\frac{p^k z^{k-1}}{k!}e^{-pz}\,dz. \end{aligned} \tag{33}$$

We replace z by $z-n$ in the integral in the first sum from (33). We obtain

$$\begin{aligned} &\sum_{k=0}^{n-1}\int_0^\infty \frac{z+n-k}{z+n}\psi(z+n-k)J_k(z+n-k)\,dz \\ &\qquad = \int_0^\infty\sum_{k=1}^{n}\frac{z+k}{z+n}\psi(z+k)J_{n-k}(z+k)\,dz. \end{aligned} \tag{34}$$

We replace z by $z-k$ in the series of integrals from (33). We obtain

$$\sum_{k=n}^\infty\int_0^\infty z\psi(z)\frac{J_k(z)}{z+k}\,dz = \int_0^\infty z\psi(z)\sum_{k=n}^\infty\frac{J_k(z)}{z+k}\,dz.$$

Combining this equality with (34), (33), and (32), we obtain (31). □

Remark 2. For $n = 0$, we can represent the integral I_n in (32) as follows:

$$I_0 = \int_0^\infty e^{-pz}\sum_{k=0}^{[z]}\frac{p^k z^{k-1}}{k!}(z-k)\psi(z-k)\,dz = \frac{(1-q)p}{(p-q)^2}\,. \tag{35}$$

Indeed, from (33) and Lemma 6 it follows that

$$\begin{aligned} I_0 &= \sum_{k=0}^{\infty} \int_k^{\infty} (z-k)\psi(z-k)\frac{p^k z^{k-1}}{k!}e^{-pz}\,dz \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} z\psi(z)\frac{J_k(z)}{z+k}\,dz = \int_0^{\infty} z\psi(z)\sum_{k=0}^{\infty}\frac{J_k(z)}{z+k}\,dz \\ &= \int_0^{\infty} \psi(z)e^{-(p-q)z}\,dz = \frac{(1-q)p}{(p-q)^2}\,. \end{aligned}$$

The latter equality is a consequence of the equality

$$\int_0^{\infty} \psi(z)e^{-\beta z}\,dz = \frac{(1-q)p}{(p-q)\beta}$$

from [11, Lemma 4] and the equality $\beta = p - q$, see Lemma 1 (and Lemma 9 below). By (35) and (13), we obtain

$$G^*(+\infty) = \frac{(p-1)q}{p-q} + (p-q)\frac{(1-q)p}{(p-q)^2} = 1.$$

The following two assertions describe exact asymptotics for the integrals in (31) as $n \to \infty$.

Lemma 8. *The following representation is valid*:

$$\int_0^{\infty} \sum_{k=1}^{n} \frac{z+k}{z+n}\psi(z+k)J_{n-k}(z+k)\,dz \sim \frac{c_+^{(1)}}{n^{3/2}}e^{-\lambda n} \tag{36}$$

as $n \to \infty$, *where*

$$c_+^{(1)} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \sum_{k=1}^{\infty} \frac{(z+k)\psi(z+k)}{p^k}e^{-(p-1)z}\,dz. \tag{37}$$

The following upper estimate is valid for the constant $c_+^{(1)}$:

$$c_+^{(1)} < \frac{p+1}{\sqrt{2\pi}(p-1)^3}\,. \tag{38}$$

Proof. Since

$$\sum_{k=1}^{\infty} p^{-k} = 1/(p-1), \qquad \sum_{k=1}^{\infty} kp^{-k} = p/(p-1)^2, \tag{39}$$

estimate (38) is immediate from the inequalities $0 < \psi(\cdot) < 1$, where ψ is the distribution function in (16). Indeed, we have

$$\begin{aligned} c_+^{(1)} &< \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\sum_{k=1}^{\infty}\frac{z+k}{p^k}e^{-(p-1)z}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}p^{-k}\int_0^{\infty} ze^{-(p-1)z}\,dz + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}kp^{-k}\int_0^{\infty} e^{-(p-1)z}\,dz \\ &= \frac{1}{\sqrt{2\pi}(p-1)^2}\sum_{k=1}^{\infty}p^{-k} + \frac{1}{\sqrt{2\pi}(p-1)}\sum_{k=1}^{\infty}kp^{-k} \\ &= \frac{p+1}{\sqrt{2\pi}(p-1)^3}\,. \end{aligned}$$

We denote by f_n the integrand in (36). We use the definition of the functions J_k in (24) and transform:

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{n} \frac{z+k}{z+n}\psi(z+k)J_{n-k}(z+k) \\ &= \sum_{k=1}^{\infty} \frac{(z+k)\psi(z+k)(pe^{-p})^{n-k}(z+n)^{n-k-1}}{(n-k)!}e^{-p(z+k)} \\ &= \frac{(pe^{-p})^n n^{n-1}}{n!}e^{-pz}\left(1+\frac{z}{n}\right)^{n-1}\sum_{k=1}^{n}\frac{(z+k)\psi(z+k)n!}{p^k(n-k)!(z+n)^k}. \end{aligned} \tag{40}$$

We denote

$$\begin{aligned} a_n &= \frac{(pe^{-p})^n n^{n-1}}{n!}, \\ B_n^k(z) &= \begin{cases} \dfrac{n!}{(n-k)!(z+n)^k}, & k \le n, \\ 0, & k > n, \end{cases} \end{aligned} \tag{41}$$

where $k, n = 1, 2, \dots$ and $z \ge 0$. By (40) and (41), we have

$$\frac{f_n(z)}{a_n} = e^{-pz}\left(1+\frac{z}{n}\right)^{n-1}\sum_{k=1}^{\infty}\frac{(z+k)\psi(z+k)}{p^k}B_n^k(z). \tag{42}$$

The asymptotics of the expressions in (41) has the following form as $n \to \infty$:

$$\begin{aligned} a_n &\sim \frac{(pe^{-p})^n n^{n-1}}{\sqrt{2\pi}n^{n+1/2}e^{-n}} = \frac{e^{-\lambda n}}{\sqrt{2\pi}n^{3/2}}, \\ \lim_{n\to\infty} B_n^k(z) &= \lim_{n\to\infty}\frac{(1-1/n)\cdots(1-(k-1)/n)}{(1+z/n)^k} = 1; \end{aligned} \tag{43}$$

moreover, we have $B_n^k(z) \le 1$ for all $k, n = 1, 2, \dots$ and $z \ge 0$. Using (43) and (39), we find an integrable majorant (that is independent of n) and the limit of the sequence of functions in (42). Indeed, we have

$$\begin{aligned} \frac{f_n(z)}{a_n} &\le e^{-(p-1)z}\sum_{k=1}^{\infty}\frac{(z+k)}{p^k} = e^{-(p-1)z}\left(\frac{z}{p-1}+\frac{p}{(p-1)^2}\right), \\ \lim_{n\to\infty}\frac{f_n(z)}{a_n} &= e^{-(p-1)z}\sum_{k=1}^{\infty}\frac{(z+k)\psi(z+k)}{p^k}. \end{aligned} \tag{44}$$

From (44) it follows that

$$\begin{aligned} \lim_{n\to\infty}\frac{\int_0^\infty f_n(z)\,dz}{a_n} &= \int_0^\infty \lim_{n\to\infty}\left(\frac{f_n(z)}{a_n}\right)dz \\ &= \int_0^\infty e^{-(p-1)z}\sum_{k=1}^{\infty}\frac{(z+k)\psi(z+k)}{p^k}\,dz. \end{aligned}$$

Combining the last formula with the first relation in (43), we obtain relations (36)–(37). □

We turn to the asymptotics of the second integral in (31).

Lemma 9. *We have*

$$\int_0^\infty z\psi(z)\sum_{k=n}^{\infty}\frac{J_k(z)}{z+k}\,dz \sim \frac{c_+^{(2)}}{n^{3/2}}e^{-\lambda n} \tag{45}$$

as $n \to \infty$, where

$$c_+^{(2)} = \frac{1}{\sqrt{2\pi}(1 - pe^{-p})} \int_0^\infty z\psi(z)e^{-(p-1)z}\,dz. \tag{46}$$

The following upper estimate is valid for the constant $c_+^{(2)}$:

$$c_+^{(2)} < \frac{1}{\sqrt{2\pi}(1 - pe^{1-p})(p-1)^2}\,.$$

Proof. We denote

$$g_n(z) = \sum_{k=n}^\infty \frac{J_k(z)}{z+k}\,, \quad n = 1, 2, \dots\,. \tag{47}$$

We use definition (24) and transform sequence (47) as follows:

$$\begin{aligned} g_n(z) &= \sum_{k=n}^\infty \frac{(pe^{-p})^k (z+k)^{k-1}}{k!} e^{-pz} \\ &= \sum_{k=0}^\infty \frac{(pe^{-p})^{k+n}(z+k+n)^{k+n-1}}{(k+n)!} e^{-pz} \\ &= \frac{(pe^{-p})^n e^n}{n^{3/2}} \sum_{k=0}^\infty \frac{(pe^{-p})^k n^{3/2}(k+n)^{k+n-1}\big(1 + z/(k+n)\big)^{k+n-1}}{(k+n)!e^n} e^{-pz}. \end{aligned}$$

We obtain the following representation:

$$g_n(z) = \frac{e^{-\lambda n}}{n^{3/2}} \sum_{k=0}^\infty C_n^k(z), \tag{48}$$

where

$$C_n^k(z) = \frac{(pe^{-p})^k n^{3/2}(k+n)^{k+n-1}\big(1 + z/(k+n)\big)^{k+n-1}}{(k+n)!e^n} e^{-pz},$$

$$k = 0, 1, \dots, \;\; n = 1, 2, \dots\,.$$

By (47) and (48), the integral in (45) assumes the form

$$\int_0^\infty z\psi(z) \sum_{k=n}^\infty \frac{J_k(z)}{z+k}\,dz = \frac{e^{-\lambda n}}{n^{3/2}} \int_0^\infty z\psi(z) \sum_{k=0}^\infty C_n^k(z)\,dz. \tag{49}$$

Notice that

$$\begin{aligned} C_n^k(z) &\le \frac{(pe^{-p})^k n^{3/2}(k+n)^{k+n-1}e^z}{\sqrt{2\pi}(k+n)^{k+n+1/2}e^{-k-n}e^n} e^{-pz} \\ &= \frac{e^{-\lambda k}}{\sqrt{2\pi}} \left(\frac{n}{k+n}\right)^{3/2} e^{-(p-1)z} \le \frac{e^{-\lambda k}}{\sqrt{2\pi}} e^{-(p-1)z}, \quad k = 0, 1, \dots\,; \end{aligned}$$

moreover, we have

$$\lim_{n\to\infty} C_n^k(z) = \frac{e^{-\lambda k}}{\sqrt{2\pi}} e^{-(p-1)z}, \quad k = 0, 1, \dots\,.$$

This allows us to make passage to the limit:

$$\lim_{n\to\infty} \int_0^\infty z\psi(z) \sum_{k=0}^\infty C_n^k(z)\,dz$$

$$= \int_0^\infty z\psi(z) \sum_{k=0}^\infty \lim_{n\to\infty} C_n^k(z)\, dz$$
$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty e^{-\lambda k} \int_0^\infty z\psi(z) e^{-(p-1)z}\, dz$$
$$= \frac{1}{\sqrt{2\pi}(1-e^{-\lambda})} \int_0^\infty z\psi(z) e^{-(p-1)z}\, dz = c_+^{(2)},$$

where the last equality is a consequence of definitions (4) and (11), i.e., we have

$$e^{-\lambda} = pe^{1-p} (= qe^{1-q}).$$

Taking (49) into account, we obtain (45).

The upper estimate for the constant $c_+^{(2)}$ is immediate from (46) and the relations

$$c_+^{(2)} < \frac{1}{\sqrt{2\pi}(1-pe^{-p})} \int_0^\infty ze^{-(p-1)z}\, dz = \frac{1}{\sqrt{2\pi}(1-pe^{1-p})(p-1)^2}\,. \qquad \square$$

Lemma 10. *For every $s > 0$, we have*

$$\int_0^\infty \psi(z) e^{-sz}\, dz = \frac{1-q}{s-q+qe^{-s}}\,, \tag{50}$$
$$\int_0^\infty z\psi(z) e^{-sz}\, dz = \frac{(1-q)(1-qe^{-s})}{(s-q+qe^{-s})^2}\,, \tag{51}$$

where the function ψ is defined in (16).

Proof. We use the following properties of the function ψ, see [11, Lemma 2]:

$$\begin{aligned} &\psi'(z) = q\big(\psi(z) - \psi(z-1)\big), \qquad z \ge 1, \\ &\psi(0) = 1-q, \quad \psi(z) = 0, \qquad z < 0. \end{aligned} \tag{52}$$

We integrate by parts. We have

$$\begin{aligned} I(s) &= \int_0^\infty \psi(z) e^{-sz}\, dz = -\frac{1}{s}\int_0^\infty \psi(z)\, de^{-sz} \\ &= \frac{1-q}{s} + \frac{q}{s}\int_0^\infty \big(\psi(z)-\psi(z-1)\big) e^{-sz}\, dz \\ &= \frac{1-q}{s} + \frac{q}{s} I(s) - \frac{q}{s}\int_1^\infty \psi(z-1) e^{-sz}\, dz \\ &= \frac{1-q}{s} + \frac{q}{s} I(s) - \frac{qe^{-s}}{s} I(s). \end{aligned}$$

We solve the equation with respect to $I(s)$ and obtain (50). The proof of equality (51) is similar. Indeed, we denote by $I(s)$ the integral in (51) and deduce:

$$\begin{aligned} I(s) &= \int_0^\infty z\psi(z) e^{-sz}\, dz = -\frac{1}{s}\int_0^\infty z\psi(z)\, de^{-sz} \\ &= \frac{1}{s}\int_0^\infty e^{-sz}\, dz\psi(z) \\ &= \frac{1}{s}\int_0^\infty \psi(z) e^{-sz}\, dz + \frac{q}{s}\int_0^\infty z\big(\psi(z)-\psi(z-1)\big) e^{-sz}\, dz \\ &= \frac{1}{s}\int_0^\infty \psi(z) e^{-sz}\, dz + \frac{q}{s} I(s) - \frac{q}{s}\int_1^\infty z\psi(z-1) e^{-sz}\, dz. \end{aligned}$$

The last integral can be represented as follows:

$$\int_0^\infty (z+1)\psi(z)e^{-s(z+1)}\,dz = e^{-s}I(s) + e^{-s}\int_0^\infty \psi(z)e^{-sz}\,dz.$$

Taking into account the previous chain of equalities, we obtain the equation

$$I(s) = \frac{1}{s}\int_0^\infty \psi(z)e^{-sz}\,dz + \frac{q}{s}I(s) - \frac{qe^{-s}}{s}I(s) - \frac{qe^{-s}}{s}\int_0^\infty \psi(z)e^{-sz}\,dz.$$

We use the integral in (50) and find that

$$I(s) = \frac{1-qe^{-s}}{s-q+qe^{-s}}\int_0^\infty \psi(z)e^{-sz}\,dz. \qquad \square$$

The following assertion is immediate from (51) and (46).

Lemma 11. *The constant* $c_+^{(2)}$ *in* (45)–(46) *has the form*

$$c_+^{(2)} = \frac{(1-q)(1-qe^{1-p})}{\sqrt{2\pi}(1-pe^{1-p})(p-q-1+qe^{1-p})^2}\,.$$

The following assertions will be needed for finding the constant $c_+^{(1)}$ from (37).

Lemma 12. *We put*

$$Q(z) = \sum_{k=1}^\infty \frac{\psi(z+k)}{p^k}\,, \qquad Q(+\infty) = \frac{1}{p-1}\,, \tag{53}$$

$$H(z) = \sum_{k=1}^\infty \frac{(z+k)\psi(z+k)}{p^k}\,, \quad z \ge 0. \tag{54}$$

Then the functions Q and H satisfy the differential equations

$$Q'(z) = \frac{q(p-1)}{p}Q(z) - \frac{q}{p}\psi(z), \tag{55}$$

$$H'(z) = \frac{q(p-1)}{p}H(z) + \frac{p-q}{p}Q(z) - \frac{q}{p}z\psi(z) - \frac{q}{p}\psi(z); \tag{56}$$

moreover, we have

$$Q(0) = \sum_{k=1}^\infty \frac{\psi(k)}{p^k} = \frac{1-q}{p\,\exp\{q(1-p)/p\}-1}\,, \tag{57}$$

$$H(0) = \sum_{k=1}^\infty \frac{k\psi(k)}{p^k} = \frac{(1-q)(p-q)\exp\{q(1-p)/p\}}{\Big(p\,\exp\{q(1-p)/p\}-1\Big)^2}\,. \tag{58}$$

Proof. By (52), the distribution function ψ and its derivative are bounded. Therefore, we may differentiate series (53)–(54) term-by-term. We differentiate (53). We use equality (52) and obtain (55):

$$\begin{aligned} Q'(z) &= q\sum_{k=1}^\infty \frac{\psi(z+k)}{p^k} - q\sum_{k=1}^\infty \frac{\psi(z+k-1)}{p^k} \\ &= qQ(z) - \frac{q}{p}\sum_{k=0}^\infty \frac{\psi(z+k)}{p^k} = qQ(z) - \frac{q}{p}Q(z) - \frac{q}{p}\psi(z) \\ &= \frac{q(p-1)}{p}Q(z) - \frac{q}{p}\psi(z). \end{aligned}$$

We substitute

$$Q(z) = \frac{q}{p}\int_0^\infty \psi(z+t)\exp\left\{-\frac{q(p-1)}{p}t\right\}dt \tag{59}$$

into equation (55) and find that Q is a solution to (55). Indeed, we have

$$\begin{aligned} Q'(z) &= \frac{q}{p}\int_0^\infty \exp\left\{-\frac{q(p-1)}{p}t\right\}d\psi(z+t) \\ &= \frac{q}{p}\psi(z+t)\exp\left\{-\frac{q(p-1)}{p}t\right\}\Big|_0^\infty \\ &\quad + \frac{q(p-1)}{p}\cdot\frac{q}{p}\int_0^\infty \psi(z+t)\exp\left\{-\frac{q(p-1)}{p}t\right\}dt \\ &= -\frac{q}{p}\psi(0) + \frac{q(p-1)}{p}Q(z). \end{aligned}$$

Since

$$Q(+\infty) = \frac{q}{p}\int_0^\infty \exp\left\{-\frac{q(p-1)}{p}t\right\}dt = \frac{1}{p-1}\,,$$

this function satisfies the boundary condition in (53). From (59) and (50) we obtain (57):

$$Q(0) = \frac{q}{p}\int_0^\infty \psi(t)\exp\left\{-\frac{q(p-1)}{p}t\right\}dt = \frac{1-q}{p\,\exp\{q(1-p)/p\}-1}\,.$$

Differentiating series (57) with respect to p, we obtain (58):

$$\begin{aligned} H(0) &= \sum_{k=1}^\infty \frac{k\psi(k)}{p^k} = -p\left(\sum_{k=1}^\infty \frac{\psi(k)}{p^k}\right)'_p \\ &= -p\left(\frac{1-q}{p\,\exp\{q(1-p)/p\}-1}\right)'_p \\ &= \frac{(1-q)(p-q)\exp\{q(1-p)/p\}}{(p\,\exp\{q(1-p)/p\}-1)^2}\,. \end{aligned}$$

It remains to verify equality (56). By (54) and (52), we have

$$\begin{aligned} H'(z) &= \sum_{k=1}^\infty \frac{\psi(z+k)}{p^k} + q\sum_{k=1}^\infty \frac{(z+k)\big(\psi(z+k)-\psi(z+k-1)\big)}{p^k} \\ &= Q(z) + qH(z) - \frac{q}{p}\sum_{k=1}^\infty \frac{(z+k)\psi(z+k-1)}{p^{k-1}} \\ &= Q(z) + qH(z) - \frac{q}{p}\sum_{k=0}^\infty \frac{(z+k+1)\psi(z+k)}{p^k} \\ &= Q(z) + qH(z) - \frac{q}{p}\sum_{k=1}^\infty \frac{(z+k+1)\psi(z+k)}{p^k} - \frac{q}{p}(z+1)\psi(z) \\ &= Q(z) + qH(z) - \frac{q}{p}H(z) - \frac{q}{p}Q(z) - \frac{q}{p}z\psi(z) - \frac{q}{p}\psi(z) \\ &= \frac{q(p-1)}{p}H(z) + \frac{p-q}{p}Q(z) - \frac{q}{p}z\psi(z) - \frac{q}{p}\psi(z). \end{aligned}$$

□

Lemma 13. *The function Q in* (53) *satisfies the formula*

$$A := \int_0^\infty Q(z) e^{-(p-1)z}\, dz = \frac{1-q}{(p-1)(p-q)} \left(\frac{p}{p\,\exp\{q(1-p)/p\} - 1} - \frac{q}{p-q-1+qe^{1-p}} \right). \tag{60}$$

Proof. We use formula (55) and integrate by parts the expression in the definition of A above. We then obtain

$$A = -\frac{1}{p-1}\int_0^\infty Q(z)\, de^{-(p-1)z} = \frac{Q(0)}{p-1} + \frac{1}{p-1}\int_0^\infty Q'(z)e^{-(p-1)z}\, dz = \frac{Q(0)}{p-1} + \frac{q}{p}A - \frac{q}{p(p-1)}\int_0^\infty \psi(z)e^{-(p-1)z}\, dz.$$

We conclude that

$$A = \frac{pQ(0)}{(p-1)(p-q)} - \frac{q}{(p-1)(p-q)}\int_0^\infty \psi(z)e^{-(p-1)z}\, dz.$$

It remains to use formulas (50) and (57). □

Lemma 14. *The constant $c_+^{(1)}$ in* (36)–(37) *admits the representation*

$$c_+^{(1)} = \frac{1-q}{\sqrt{2\pi}(p-1)^2(p-q)} \left(\frac{p\big((p^2-pq+q)\exp\{q(1-p)/p\} - 1\big)}{\big(p\,\exp\{q(1-p)/p\} - 1\big)^2} - \frac{q\big(p^2-pq-1+qe^{1-p}\big)}{\big(p-q-1+qe^{1-p}\big)^2} \right). \tag{61}$$

Proof. By (37) and (54), we have

$$c_+^{(1)} = \frac{1}{\sqrt{2\pi}}\int_0^\infty H(z)e^{-(p-1)z}\, dz. \tag{62}$$

We use equality (56) and find the integral from (62). We deduce

$$\begin{aligned} B &:= \int_0^\infty H(z)e^{-(p-1)z}\, dz \\ &= -\frac{1}{p-1}\int_0^\infty H(z)\, de^{-(p-1)z} \\ &= \frac{H(0)}{p-1} + \frac{1}{p-1}\int_0^\infty H'(z)e^{-(p-1)z}\, dz \\ &= \frac{H(0)}{p-1} + \frac{q}{p}B + \frac{p-q}{p(p-1)}\int_0^\infty Q(z)e^{-(p-1)z}\, dz \\ &\quad - \frac{q}{p(p-1)}\int_0^\infty z\psi(z)e^{-(p-1)z}\, dz - \frac{q}{p(p-1)}\int_0^\infty \psi(z)e^{-(p-1)z}\, dz. \end{aligned}$$

Hence, one has

$$\begin{aligned} B &= \int_0^\infty H(z)e^{-(p-1)z}\, dz \\ &= \frac{pH(0)}{(p-1)(p-q)} + \frac{1}{p-1}\int_0^\infty Q(z)e^{-(p-1)z}\, dz \\ &\quad - \frac{q}{(p-1)(p-q)}\int_0^\infty z\psi(z)e^{-(p-1)z}\, dz - \frac{q}{(p-1)(p-q)}\int_0^\infty \psi(z)e^{-(p-1)z}\, dz. \end{aligned}$$

We take into account the representations for the integrals in Lemmas 10 and 13 and relation (58). We find that

$$\begin{aligned}\int_0^\infty H(z)e^{-(p-1)z}\,dz = &\frac{p(1-q)\exp\{q(1-p)/p\}}{(p-1)\big(p\,\exp\{q(1-p)/p\}-1\big)^2}\\ &+\frac{p(1-q)}{(p-1)^2(p-q)\big(p\,\exp\{q(1-p)/p\}-1\big)}\\ &-\frac{q(1-q)}{(p-1)^2(p-q)\big(p-q-1+qe^{1-p}\big)}\\ &-\frac{q(1-q)\big(1-qe^{1-p}\big)}{(p-1)(p-q)\big(p-q-1+qe^{1-p}\big)^2}\\ &-\frac{q(1-q)}{(p-1)(p-q)\big(p-q-1+qe^{1-p}\big)}.\end{aligned}$$

Converting positive (and then negative) fractions into fractions with the same denominator, we obtain

$$\begin{aligned}\int_0^\infty H(z)e^{-(p-1)z}\,dz = &\frac{p(1-q)\big((p^2-pq+q)\exp\{q(1-p)/p\}-1\big)}{(p-1)^2(p-q)\big(p\,\exp\{q(1-p)/p\}-1\big)^2}\\ &-\frac{q(1-q)\big(p^2-pq-1+qe^{1-p}\big)}{(p-1)^2(p-q)\big(p-q-1+qe^{1-p}\big)^2}.\end{aligned}$$

Taking (62) into account, we find the constant $c_+^{(1)}$. □

Proof of Theorem 2. From representation (31) and asymptotic formulas (36) and (45) we obtain the asymptotic relation

$$1-G^*(n)\sim(p-q)\frac{c_+^{(1)}+c_+^{(2)}}{n^{3/2}}e^{-\lambda n},\quad n\to\infty.$$

We substitute the representations from Lemmas 11 and 14 for the constants $c_+^{(1)}$ and $c_+^{(2)}$. We find that

$$\begin{aligned}c_+^* = \frac{(p-q)}{\sqrt{2\pi}}\Bigg(&\frac{p(1-q)\big((p^2-pq+q)\exp\{q(1-p)/p\}-1\big)}{(p-1)^2(p-q)\big(p\,\exp\{q(1-p)/p\}-1\big)^2}\\ &-\frac{q(1-q)\big(p^2-pq-1+qe^{1-p}\big)}{(p-1)^2(p-q)\big(p-q-1+qe^{1-p}\big)^2}\\ &+\frac{(1-q)\big(1-qe^{1-p}\big)}{\big(1-pe^{1-p}\big)\big(p-q-1+qe^{1-p}\big)^2}\Bigg).\end{aligned}$$

The sum of the last two fractions assumes the form

$$\frac{p(1-q)\big(p-q-1+qe^{1-p}\big)^2}{(p-1)^2(p-q)\big(1-pe^{1-p}\big)\big(p-q-1+qe^{1-p}\big)^2}=\frac{p(1-q)}{(p-1)^2(p-q)\big(1-pe^{1-p}\big)}.$$

We conclude that

$$c_+^* = \frac{p(1-q)}{\sqrt{2\pi}(p-1)^2}\left(\frac{(p^2-pq+q)\exp\{q(1-p)/p\}-1}{\big(p\,\exp\{q(1-p)/p\}-1\big)^2}+\frac{1}{1-pe^{1-p}}\right). \tag{63}$$

It remains to notice that we may replace pe^{1-p} by qe^{1-q} in (63). □

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