ISSN 1055-1344, Siberian Advances in Mathematics, 2017, Vol. 27, No. 3, pp. 187–195. © Allerton Press, Inc., 2017.
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On Integration of Systems of Stochastic Differential Equations

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Received March 2, 2016

Abstract—We suggest a method for solving systems of stochastic differential equations and describe the structure of solutions.

DOI: 10.3103/S1055134417030038

Keywords: *system of stochastic differential equations, structure of solutions*

Dedicated to A. A. Borovkov on his 85th birthday.

1. INTRODUCTION

The main result of the present article is a new method for solving systems of stochastic differential equations (SDEs) with the help of a finite chain of systems of ordinary differential equations (ODEs). This method was first found by the author for 1-dimensional SDEs and their pathwise versions, see [3, 4] (a more detailed exposition can be also found in [5]). The essence of this method is as follows. We consider a 1-dimensional SDE

$$d\eta(t) = b\big(t, \eta(t), W(t)\big)dt + \sigma\big(t, \eta(t)\big) * dW(t), \;\; \eta(0) = \eta_0,$$

with the Stratonovich integral with respect to a Wiener process $W(t)$. We find a solution of the form $\eta(t) = \phi\big(t, W(t) + C(t)\big)$, where ϕ is a deterministic function and C is a smooth predictable function. Moreover, we find the function ϕ from the ODE $\phi'_u = \sigma(t, \phi)$ and the function C from the ODE

$$C'(t) = \frac{b\Big(t, \phi\big(t, W(t) + C(t)\big), W(t)\Big) - \phi'_t(t, u)\Big|_{u=W(t)+C(t)}}{\sigma\Big(t, \phi\big(t, W(t) + C(t)\big)\Big)}$$

with a stochastic right-hand side. The initial condition has the form $\phi\big(0, W(0) + C(0)\big) = \eta_0$.

It turns out that a version of this method can be applied to parabolic and hyperbolic stochastic partial differential equations with a 1-dimensional Wiener process. Notice that, in [5], only empirical arguments (with no justification of the method) are presented on construction of solutions of systems of SDEs with the help of chains of ODEs.

It is important that our construction allows us to describe the structure of solutions of a system of SDEs. Namely, *solutions of systems of* SDEs *are deterministic functions of a multidimensional Wiener process and smooth adapted stochastic functions*. The latter functions are solutions of a normal systems of ODEs such that Wiener processes occur on the right-hand sides. This provides us with a new tool for studying certain problems of stochastic analysis.

No general method for solving systems of SDEs was previously known. Usually, the solutions were constructed by "guessing" with the help of the Itô formula. Sometimes, this was preceded by a stochastic change of the time variable or a drift removal transform, i.e., an application of the Girsanov theorem. On the other hand, there are examples of construction of solutions for SDEs by solving ODEs that arise during the construction, see [2, Ch. IV]. In [7], a method is found for constructing solutions for a certain class of systems of SDEs by solving related systems of total differential equations.

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2. MAIN RESULTS

1. We consider a d-dimensional Wiener process

$$\overline{W}(t) = \big(W_1(t), \dots, W_d(t)\big)$$

on a filtered probability space $\big(\Omega, F, (F_t)_{t\geq 0}\mathbb{P}\big)$. Consider a system of SDEs with the stochastic Stratonovich integrals of the form

$$\begin{cases} \eta_i(t) = \eta_i^0 + \displaystyle\int_0^t B^i\big(s, \overline{\eta}(s), \overline{W}(s)\big)\, ds \\ \qquad\qquad + \displaystyle\sum_{j=1}^{d} \int_0^t \sigma^{ij}\big(s, \overline{\eta}(s)\big) * dW_j(s), \;\; i = 1, 2, \dots, n, \end{cases} \tag{1}$$

where $\overline{\eta}(s) = \big(\eta_1(s), \dots, \eta_n(s)\big)$.

In the sequel, we assume that the coefficients $B^i(t, \overline{\eta}, \overline{x})$ and $\sigma^{ij}(t, \overline{\eta})$ are presented by deterministic functions unless explicitly stated otherwise. The fact that system (1) is considered together with the Stratonovich integrals is not essential. Indeed, a well-known transform allows us to rewrite such a system into a system of SDEs with the Itô integrals and vice versa.

We denote

$$\overline{W}_{[k]}(t, v_k) = \big(W_1(t), \dots, W_{k-1}(t), v_k, W_{k+1}(t), \dots, W_d(t)\big),$$

where the subscript $[k]$ means that the kth coordinate $W_k(t)$ of the vector $\overline{W}(t)$ is replaced by the variable v_k. Let $U_{\overline{W}(t)}$ be a neighborhood of the point $\overline{v}^* = \overline{W}(t)$. Our method for solving systems of SDEs is based on the following assertion.

Theorem 1. *Let*

$$\sigma^{ik}(t, \overline{\eta}), \;\; k = 1, 2, \dots, d, \;\; i = 1, 2, \dots, n,$$
$$B^i(t, \overline{\eta}, \overline{v}), \;\; i = 1, 2, \dots, n,$$

be continuously differentiable functions. Assume that, for $t \in [0, T]$ and $\overline{v} \in U_{\overline{W}(t)}$, the components of the vector-valued function $\overline{\varphi}(t, \overline{v}) = \big(\varphi_1(t, \overline{v}), \dots, \varphi_n(t, \overline{v})\big)$ are continuously differentiable with respect to all arguments and satisfy the following finite chain of relations:

$$\Big\{ (\varphi_i)'_{v_1}\big(t, \overline{W}_{[1]}(t, v_1)\big) = \sigma^{i1}\Big(t, \overline{\varphi}\big(t, \overline{W}_{[1]}(t, v_1)\big)\Big), \quad i = 1, 2, \dots, n, \tag{2}$$

$$\dots$$

$$\Big\{ (\varphi_i)'_{v_k}\big(t, \overline{W}_{[k]}(t, v_k)\big) = \sigma^{ik}\Big(t, \overline{\varphi}\big(t, \overline{W}_{[k]}(t, v_k)\big)\Big), \quad i = 1, 2, \dots, n, \tag{3}$$

$$\dots$$

$$\Big\{ (\varphi_i)'_{v_d}\big(t, \overline{W}_{[d]}(t, v_d)\big) = \sigma^{id}\Big(t, \overline{\varphi}\big(t, \overline{W}_{[d]}(t, v_d)\big)\Big), \quad i = 1, 2, \dots, n, \tag{4}$$

$$\begin{cases} (\varphi_i)'_t(t, \overline{v})\Big|_{\{v_j = W_j(t), j = 1, 2, \dots, d\}} = B^i\Big(t, \overline{\varphi}\big(t, \overline{W}(t)\big), \overline{W}(t)\Big), \\ \qquad\qquad \varphi_i\big(0, \overline{W}(0)\big) = \eta_i^0, \;\; i = 1, 2, \dots, n. \end{cases} \tag{5}$$

Then the function

$$\overline{\eta}(t) = \big(\eta_1(t), \dots, \eta_n(t)\big) = \overline{\varphi}\big(t, \overline{W}(t)\big), \;\; t \in [0, T], \;\; \overline{W}(t) \in \mathbb{R}^d,$$

is a solution of the Cauchy problem (1).

Proof. Let $\overline{\varphi}$ satisfy the relations (2)–(5). For each component of $\overline{\varphi}\big(t,\overline{W}(t)\big)$, we apply the formula for the Itô differential with the Stratonovich integral. We obtain

$$d\varphi_i\big(t,\overline{W}(t)\big)=\sum_k(\varphi_i)'_{v_k}\big(t,\overline{W}(t)\big)*dW_k(t)+(\varphi_i)'_t\big(t,\overline{W}(t)\big)\,dt,\quad i=1,2,\dots,n.$$

Taking into account the relations (2)–(5), we conclude that $\overline{\varphi}\big(t,\overline{W}(t)\big)$ is a solution of the Cauchy problem (1). □

Remark. Below, we show that the relations (2)–(5) are reduced to a chain of systems of ODEs in certain "good cases." In the sequel, we call these relations *chain* (2)–(5) *of systems of* ODEs.

The first questions on a system of SDEs are the questions on the existence and uniqueness of a solution. Therefore, it is natural to assume that the following conditions hold.

(A) The functions $B^i(t,\overline{\eta},\overline{x})$ and $\sigma^{ij}(t,\overline{\eta})$, where $t\in[0,T]$, $\overline{\eta}\in\mathbb{R}^n$, $\overline{x}\in\mathbb{R}^d$, $i=1,2,\dots,n$, and $j=1,2,\dots,d$, are jointly continuous with respect to all variables and satisfy the Lipschitz conditions and the following linear growth condition with respect to $\overline{\eta}$: There exists $N>0$ such that, for every t and all values $\overline{\eta}_1$ and $\overline{\eta}_2$ of the variable $\overline{\eta}$, the inequalities

$$\big|B^i(t,\overline{\eta}_1,\overline{x})-B^i(t,\overline{\eta}_2,\overline{x})\big|\le N\,|\overline{\eta}_1-\overline{\eta}_2|\,,$$
$$\big|\sigma^{ij}(t,\overline{\eta}_1)-\sigma^{ij}(t,\overline{\eta}_2)\big|\le N\,|\overline{\eta}_1-\overline{\eta}_2|\,,$$
$$\big|B^i(t,\overline{\eta},\overline{x})\big|^2\le N\big(1+|\overline{\eta}|^2\big),$$
$$\big|\sigma^{ij}(t,\overline{\eta})\big|^2\le N\big(1+|\overline{\eta}|^2\big)$$

hold for all $i=1,2,\dots,n$ and $j=1,2,\dots,d$.

(B) Each column of the "diffusion matrix"

$$\big\{\sigma^{ij}(t,\overline{\eta})\big\},\ \ i=1,2,\dots,n,\ \ j=1,2,\dots,d,$$

contains an element that is separated from zero.

(C) The functions σ^{ij} and B^i are twice continuously differentiable with respect to $\overline{\eta}$.

2. The problem naturally arises on finding solutions of system (1) and describing their structure. Together with chain (2)–(5) of ODEs, we consider finite chains of systems of the form

$$\Big\{(\varphi_i)'_{v_1}(t,\overline{v})=\sigma^{i1}\big(t,\overline{\varphi}(t,\overline{v})\big),\quad i=1,2,\dots,n, \tag{6}$$
$$\dots$$
$$\Big\{(\varphi_i)'_{v_k}(t,\overline{v})=\sigma^{ik}\big(t,\overline{\varphi}(t,\overline{v})\big),\quad i=1,2,\dots,n, \tag{7}$$
$$\dots$$
$$\Big\{(\varphi_i)'_{v_d}(t,\overline{v})=\sigma^{id}\big(t,\overline{\varphi}(t,\overline{v})\big),\quad i=1,2,\dots,n, \tag{8}$$

$$\begin{cases}(\varphi_i)'_t(t,\overline{v})\Big|_{\{v_j=W_j(t),j=1,2,\dots,d\}}=B^i\Big(t,\overline{\varphi}\big(t,\overline{W}(t)\big),\overline{W}(t)\Big),\\ \varphi_i\big(0,\overline{W}(0)\big)=\eta_i^0,\ \ i=1,2,\dots,n,\end{cases} \tag{9}$$

where each φ_i is a sufficiently smooth function. Moreover, we assume that, for every t, each of these relations (except for the last one) holds in some neighborhood of the point $\overline{W}(t)$.

We first present two remarks on ODEs (see [6, pp. 267–268; 1, pp. 160–161]) that will be needed in the sequel. Then we turn to solving systems of SDEs.

(a) Consider an autonomous system of ODEs

$$\Big\{x'_k(v)=H_k(x_1,\dots,x_m),\quad k=1,2,\dots,m. \tag{10}$$

We assume that each function H_k on the right-hand side is continuous and satisfies the Lipschitz conditions with respect to the variables $x_1, \dots, x_m$. Hence, there exists a unique solution of system (10). Assume that one of these functions (say, H_1) is a nonzero function. We reduce the system to the form

$$\frac{dx_k}{dx_1} = \frac{H_k(x_1, \dots, x_m)}{H_1(x_1, \dots, x_m)}, \ \ k = 2, \dots, m, \quad \frac{dx_1}{H_1(x_1, \dots, x_m)} = dv. \tag{11}$$

Let $x_k = x_k^*(x_1, C_2, \dots, C_m)$, $k = 2, \dots, m$, be the general solution of the subsystem of (11) which consists of the former $m-1$ equations. We substitute each x_k into the latter equation and obtain

$$x_1 = x_1^*(v + C_1, C_2, \dots, C_m);$$

hence, we have

$$\begin{aligned} x_k &= x_k^*\big(x_1^*(v + C_1, C_2, \dots, C_m), C_2, \dots, C_m\big) \\ &= x_k(v + C_1, C_2, \dots, C_m), \ \ k = 2, \dots, m. \end{aligned}$$

(b) We again consider system (10) of ODEs. Assume that each function on the right-hand side is continuous and the partial derivatives $\frac{\partial H_k}{\partial x_j}$ are continuous in some domain G. Let $x_k(v, C_1, C_2, \dots, C_m)$ be the general solution of system (10). Then the partial derivatives $\frac{\partial x_k}{\partial C_j}$ exist and are continuous in the domain G; moreover, the determinant of the matrix $\left\{\frac{\partial x_k}{\partial C_j}\right\}$ is not equal to zero.

In the sequel, we always assume that, for a fixed value of t, the values of $v_1, \dots, v_d$ belong to a neighborhood of the point $\overline{W}(t)$. Thus, the problem on finding solutions of system (1) of SDEs reduces to construction of $\overline{\varphi}$ and application of Theorem 1. The general scheme for finding solutions of systems of SDEs is as follows. With the help of chain (6)–(9), we consecutively revise the form of the functions $\overline{\varphi}$. We transform each of systems (6)–(9) to a normal system of ODEs. Solving the entire chain of systems of ODEs, we completely describe $\overline{\varphi}(t, \overline{v})$.

We consider system (6) as an autonomous system of ODEs with respect to the variable v_1 and regard the other variables as parameters. Taking into account Remark (a), we find the general solution φ_i of the system

$$\varphi_i(t, \overline{v}) = \varphi_i(t, v_1 + C_1^{(1)}, C_2^{(1)}, \dots, C_n^{(1)}), \ \ i = 1, 2, \dots, n, \tag{12}$$

of ODEs, where $C_1^{(1)}, \dots, C_n^{(1)}$ are arbitrary functions of the variables

$$t, v_2, \dots, v_d, \text{ i.e., } C_j^{(1)} = C_j^{(1)}(t, v_2, \dots, v_d).$$

We substitute functions (12) into the subsequent system of ODEs in the chain. Taking derivatives with respect to v_2, we obtain the following system of linear equations with respect to $(C_j^{(1)})'_{v_2}$:

$$\Big\{(\varphi_k)'_{v_2} = (\varphi_k)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{v_2} + \dots + (\varphi_k)'_{C_n^{(1)}}\big(C_n^{(1)}\big)'_{v_2} = \sigma^{k2}, \ \ k = 1, 2, \dots, n. \tag{13}$$

By Remark (b), the matrix $\Big\{(\varphi_k)'_{C_l^{(1)}}\Big\}$ is nonsingular. Hence, there exists a unique solution of system (13). Moreover, this solution can be found with the help of Cramer's rule. We obtain

$$\Big\{(C_k^{(1)})'_{v_2} = \Psi_k^{(1)}\big(t, C_1^{(1)}, C_2^{(1)}, \dots, C_n^{(1)}\big), \ \ k = 1, 2, \dots, n, \tag{14}$$

where each function $\Psi_k^{(1)}$ is known. There exists a unique solution of system (14) if, for example, the partial derivatives of each function $\Psi_k^{(1)}$ are continuously differentiable. Since the functions of the form $\Psi_k^{(1)}$ are found with the help of Cramer's rule, it suffices to require that the second partial derivatives of each function φ_k be continuously differentiable. This conditions holds if each function σ^{ij} is twice continuously differentiable (i.e., under our assumptions). In view of our assumptions on the diffusion matrix, system (13) cannot be a homogeneous system of linear equations; hence, the solution of (13) cannot be trivial.

Notice that system (14) is again an autonomous system of ODEs with respect to the variable v_2 (we assume that the other variables are fixed). Recall that there is a nonzero function of the form $\Psi_k^{(1)}$. Let, for example, $\Psi_2^{(1)}$ be such a function. By Remark (a), the general solution of system (14) has the form

$$\Big\{C_k^{(1)} = C_k^{(1)}\big(t, v_2 + C_1^{(2)}, C_2^{(2)}, \dots, C_n^{(2)}\big), \ \ k = 1, 2, \dots, n, \tag{15}$$

where each $C_k^{(2)}$ is an unknown function of the variables $t, v_3, \dots, v_d$.

We use the subsequent equation in the chain and differentiate with respect to v_3. We obtain the following system of linear equations with respect to $\big(C_j^{(2)}\big)'_{v_3}$:

$$\left\{(\varphi_k)'_{v_3} = \sum_{l=1}^{n}\left[\sum_{m=1}^{n}(\varphi_k)'_{C_m^{(1)}}\big(C_m^{(1)}\big)'_{C_l^{(2)}}\right]\big(C_l^{(2)}\big)'_{v_3} = \sigma^{k3}, \ \ k = 1, 2, \dots, n. \right. \tag{16}$$

It is convenient to pass to the matrix notation. We denote

$$\mathbf{A}_2 = \big\{a_{ij}^{(2)}\big\}, \ \text{ where } \ a_{ij}^{(2)} = (\varphi_i)'_{C_j^{(1)}}, \ \ i, j = 1, \dots, n.$$

For $m = 2, \dots, d$, we introduce the Jacobians

$$\mathbf{D}_{m-1,m} = \big\{d_{ij}^{(m)}\big\}, \ \text{ where } \ d_{ij}^{(m)} = \big(C_i^{(m-1)}\big)'_{C_j^{(m)}}, \ \ i, j = 1, \dots, n,$$

and define column matrices

$$\mathbf{C}_m = \big\{c_i^{(m)}\big\}, \quad \mathbf{B}_m = \big\{b_i^{(m)}\big\},$$

where

$$c_i^{(m)} = \big(C_i^{(m-1)}\big)'_{v_m}, \ \ b_i^{(m)} = \sigma^{im}, \ \ i = 1, \dots, n.$$

Put

$$\mathbf{A}_m = \mathbf{A}_2\mathbf{D}_{1,2}\dots,\mathbf{D}_{m-2,m-1}, \ \ m = 3, \dots, d.$$

We rewrite system (16) in the matrix form as follows: $\mathbf{A}_3\mathbf{C}_3 = \mathbf{B}_3$. By Remark (b), the matrices $\mathbf{A}_j$, $j = 2, 3$, are nonsingular; hence, there exists a unique solution of system (16). Notice that this solution is nontrivial. We again obtain an autonomous system of ODEs with respect to $\big(C_j^{(2)}\big)'_{v_3}$, $j = 2, \dots, n$. In view of our assumptions, it is possible to solve this system by the method from Remark (p).

We continue this process. For $k = d$, we obtain a system of linear equations whose matrix form is as follows:

$$\mathbf{A}_d\mathbf{C}_d = \mathbf{B}_d;$$

moreover, the matrix $\mathbf{A}_d$ is nonsingular in view of Remark (b). We solve this system of linear equations and the corresponding autonomous systems of ODEs. We find the unknown functions $C_i^{(d-1)}(t, v_d)$ up to a tuple of arbitrary constants

$$\overline{C}^{\,(d)}(t) = \big(C_1^{(d)}(t), \dots, C_n^{(d)}(t)\big)$$

that depend on the variable t. Notice that *the functions φ_i regarded as functions of the variables t, $\overline{v}$, and $\overline{C}^{\,(d)}$ are deterministic.*

We use the latter relation in (9) and arguments that are similar to those presented above. We obtain the Cauchy problem for a normal system of ODEs with respect to $C_i^{(d)}(t)$, $i = 1, \dots, n$. Since the expressions on the right-hand sides of the ODEs depend on realizations of the Wiener process, the solution of the normal system of ODEs is formed by stochastic functions $C_i^{(d)}$. Moreover, the initial conditions $\eta_i(t) = \eta_i^0$ for the initial SDE are reduced to the initial conditions for the stochastic adapted functions $C_j^{(d)}(t) = C_j^{(d)}(t, \omega)$, $j = 1, \dots, n$.

Thus, we have constructed the functions φ_i, $i = 1, \ldots, n$, with the help of chain (6)–(9). Hence, the solution of the initial system of SDEs has the form

$$\eta_i(t) = \varphi_i\big(t, \overline{W}(t)\big), \ \ i = 1, \ldots, n,$$

and is a *deterministic vector-valued function of the Wiener process* $\overline{W}(t)$ *and smooth stochastic adapted functions* $\overline{C}^{(d)}$.

We present a series of remarks.

Remarks. (a) The method for solving systems of SDEs presented above is a "pathwise" method. Therefore, it can be modified for solving systems of SDEs with stochastic coefficients $B^i = B^i(t, \overline{\eta}, \overline{x}, \omega)$ and $\sigma^{ij} = \sigma^{ij}(t, \overline{\eta}, \omega)$ in an obvious way.

(b) We may solve a system of SDEs with the help of a chain of ODEs using a different order of steps. Then we may obtain a solution in a different form. However, if there exists a unique solution of the initial system of SDEs then the probability that all constructed solutions coincide is equal to 1.

(c) We have constructed solutions of (1) under conditions (A)–(C). These conditions can be weakened. Indeed, the problem on existence (but not uniqueness!) of a solution of (1) reduces, in fact, to two problems: on the existence of solutions of each constructed system of linear equations and on the existence of solutions of each normal chain of ODEs above. As is known, the latter problem is solved with the help of a version of the Peano theorem (for a normal system of ODEs) which only requires that the functions on the right-hand sides of the equations be continuous.

(d) The problem on numerical simulation of solutions of systems of SDEs is essentially simplified. Indeed, this problem is reduced to solving (by analytical or numerical methods) a chain of normal systems of ODEs, where only the latter one contains a realization of a Wiener process on the right-hand side. Therefore, for constructing a numerical model of a solution of a system of SDEs, it suffices to construct a model of a Wiener process. The latter problem is quite simple.

3. We present an example. We consider the following system of SDEs:

$$\begin{cases} d\eta_1(t) = \sigma^{11} * dW_1(t) + \sigma^{12} * dW_2(t) + B^1\, dt, \\ d\eta_2(t) = \sigma^{21} * dW_1(t) + \sigma^{22} * dW_2(t) + B^2\, dt, \end{cases} \qquad \eta_i(0) = x_i, \ \ i = 1, 2, \tag{17}$$

where $\sigma^{ij} = \sigma^{ij}\big(t, \eta_1(t), \eta_2(t)\big)$, $i, j = 1, 2$, and $B^i = B^i\big(t, W_1(t), W_2(t), \eta_1(t), \eta_2(t)\big)$, $i = 1, 2$. We introduce certain restrictions. Namely, we assume that

$$\sigma^{11} \neq 0, \ \ \sigma^{12} \neq 0, \ \ \Delta = \sigma^{11}\sigma^{22} - \sigma^{12}\sigma^{21} \neq 0. \tag{18}$$

The case in which $\Delta = 0$ is mentioned below. We find a solution of system (17) of the form $\eta_i(t) = \varphi_i\big(t, W_1(t), W_2(t)\big)$, $i = 1, 2$.

STEP 1. The first system of ODEs in the chain is

$$\begin{cases} (\varphi_1)'_{v_1}(t, v_1, v_2) = \sigma^{11}\big(t, \varphi_1(t, v_1, v_2), \varphi_2(t, v_1, v_2)\big), \\ (\varphi_2)'_{v_1}(t, v_1, v_2) = \sigma^{21}\big(t, \varphi_1(t, v_1, v_2), \varphi_2(t, v_1, v_2)\big). \end{cases} \tag{19}$$

Taking into account assumptions (18), we rewrite this system in the following form:

$$\frac{d\varphi_2}{d\varphi_1} = \frac{\sigma^{21}(t, \varphi_1, \varphi_2)}{\sigma^{11}(t, \varphi_1, \varphi_2)}, \quad dv_1 = \frac{d\varphi_1}{\sigma^{11}(t, \varphi_1, \varphi_2)}, \tag{20}$$

where t and v_2 are regarded as parameters. We integrate the first equation. Since the expression on the right-hand side of the equation is independent of v_1, we obtain $\varphi_2 = \widehat{\varphi}_2\big(t, \varphi_1, C_2^{(1)}(t, v_2)\big)$. We substitute φ_2 into the second equation. We obtain a separable differential equation. By the implicit function theorem, the solution can be represented in the form

$$\varphi_1 = \varphi_1^*\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big). \tag{21}$$

We substitute φ_1 into the expression for φ_2. We obtain

$$\begin{aligned}\varphi_2 &= \widehat{\varphi}_2\big(t, \varphi_1, C_2^{(1)}(t, v_2)\big)\\ &= \widehat{\varphi}_2\Big(t, \varphi_1^*\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big), C_2^{(1)}(t, v_2)\Big).\end{aligned} \tag{22}$$

Thus, integrating the first equation in the chain, we have described the functions φ_1 and φ_2 up to functions $C_1^{(1)}$ and $C_2^{(1)}$. It remains to describe the latter functions.

STEP 2. We clarify the form of $C_1^{(1)}$ and $C_2^{(1)}$. We use the following relation from chain (6)–(9) of ODEs:

$$\begin{cases}(\varphi_1)'_{v_2}(t, v_1, v_2) = \sigma^{12}\big(t, \varphi_1(t, v_1, v_2), \varphi_2(t, v_1, v_2)\big),\\ (\varphi_2)'_{v_2}(t, v_1, v_2) = \sigma^{22}\big(t, \varphi_1(t, v_1, v_2), \varphi_2(t, v_1, v_2)\big).\end{cases} \tag{23}$$

We find the derivatives on the right-hand sides of (23). Taking into account (19) and (20), we obtain

$$\begin{aligned}(\varphi_1)'_{v_2} &= (\varphi_1^*)'_{v_1}\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big)\big(C_1^{(1)}\big)'_{v_2}(t, v_2)\\ &\quad + (\varphi_1^*)'_{C_2^{(1)}}\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big)\big(C_2^{(1)}\big)'_{v_2}(t, v_2)\\ &= \sigma^{11}\big(t, \varphi_1(t, v_1, v_2), \varphi_2(t, v_1, v_2)\big)\big(C_1^{(1)}\big)'_{v_2}(t, v_2)\\ &\quad + (\varphi_1^*)'_{C_2^{(1)}}\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big)\big(C_2^{(1)}\big)'_{v_2}(t, v_2),\\ (\varphi_2)'_{v_2} &= (\widehat{\varphi}_2)'_{\varphi_1}\Big(t, \varphi_1^*\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big), C_2^{(1)}(t, v_2)\Big)\\ &\qquad \times (\varphi_1^*)'_{v_2}\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big)\\ &\quad + (\widehat{\varphi}_2)'_{C_2^{(1)}}\Big(t, \varphi_1^*\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big), C_2^{(1)}(t, v_2)\Big)\big(C_2^{(1)}\big)'_{v_2}(t, v_2)\\ &= \frac{\sigma^{21}\big(t, \varphi_1(t, v_1, v_2), \widehat{\varphi}_2(t, v_1, v_2)\big)\sigma^{12}\big(t, \varphi_1(t, v_1, v_2), \widehat{\varphi}_2(t, v_1, v_2)\big)}{\sigma^{11}\big(t, \varphi_1(t, v_1, v_2), \widehat{\varphi}_2(t, v_1, v_2)\big)}\\ &\quad + (\widehat{\varphi}_2)'_{C_2^{(1)}}\Big(t, \varphi_1^*\big(t, v_1 + C_1^{(1)}(t, v_2), C_2^{(1)}(t, v_2)\big), C_2^{(1)}(t, v_2)\Big)\big(C_2^{(1)}\big)'_{v_2}(t, v_2).\end{aligned}$$

We substitute these derivatives into (23). We obtain the following system of linear equations with respect to $\big(C_1^{(1)}\big)'_{v_2}$ and $\big(C_2^{(1)}\big)'_{v_2}$:

$$\begin{cases}(\varphi_1^*)'_{v_1}\big(C_1^{(1)}\big)'_{v_2} + (\varphi_1^*)'_{C_2^{(1)}}(C_2)'_{v_2} = \sigma^{12},\\ (\widehat{\varphi}_2)'_{C_2^{(1)}}\big(C_2^{(1)}\big)'_{v_2} = \dfrac{\Delta}{\sigma^{11}}.\end{cases} \tag{24}$$

Since $(\widehat{\varphi}_2)'_{C_2^{(1)}} \neq 0$, there exists a unique solution of system (24); namely, we have

$$\begin{aligned}\big(C_1^{(1)}\big)'_{v_2} &= \frac{\sigma^{12}}{\sigma^{11}} - \frac{(\varphi_1^*)'_{C_2^{(1)}}\Delta}{(\widehat{\varphi}_2)'_{C_2^{(1)}}(\sigma^{11})^2},\\ \big(C_2^{(1)}\big)'_{v_2} &= \frac{\Delta}{\sigma^{11}(\widehat{\varphi}_2)'_{C_2^{(1)}}}.\end{aligned} \tag{25}$$

As above $\big($cf. solution of system (19)$\big)$, we rewrite (25) as follows:

$$\begin{aligned} \frac{dC_2^{(1)}}{dC_1^{(1)}} &= \frac{\sigma^{11}\Delta}{\sigma^{11}\sigma^{12}(\widehat{\varphi}_2)'_{C_2^{(1)}} - (\varphi_1^*)'_{C_2^{(1)}}\Delta}, \\ dv_2 &= \frac{(\widehat{\varphi}_2)'_{C_2^{(1)}}(\sigma^{11})^2 dC_1^{(1)}}{\sigma^{11}\sigma^{12}(\widehat{\varphi}_2)'_{C_2^{(1)}} - (\varphi_1^*)'_{C_2^{(1)}}\Delta}. \end{aligned} \tag{26}$$

The general solution of system (26) has the form

$$\begin{aligned} C_1^{(1)} &= C_1^{(1)}\big(t, v_2 + C_1^{(2)}, C_2^{(2)}\big), \\ C_2^{(1)} &= \widehat{C}_2^{(1)}\Big(t, C_1^{(1)}\big(t, v_2 + C_1^{(2)}, C_2^{(2)}\big), C_2^{(2)}\Big), \end{aligned} \tag{27}$$

where $C_1^{(2)}$ and $C_2^{(2)}$ are unknown functions. Thus, we have obtained the functions φ_1 and φ_2 of the form (21) and (22), where the functions $C_1^{(1)}$ and $C_2^{(1)}$ are determined by formulas (27).

STEP 3. We find $C_1^{(2)}$ and $C_2^{(2)}$. We use the latter system of ODEs in (9). We calculate the partial derivatives and replace the variables v_1 and v_2 by the values $W_1(t)$ and $W_2(t)$ of the Wiener process. We obtain the following system of linear equations with respect to $(C_1^{(2)})'$ and $(C_2^{(2)})'$:

$$\begin{cases} A_{11}(C_1^{(2)})' + A_{12}(C_2^{(2)})' = B_1^{(1)}, \\ A_{21}(C_1^{(2)})' + A_{22}(C_2^{(2)})' = B_2^{(1)}, \end{cases} \tag{28}$$

where

$$\begin{aligned} A_{11} &= (\varphi_1^*)'_{v_1}\big(C_1^{(1)}\big)'_{v_2} + (\varphi_1^*)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{C_1^{(2)}} + (\varphi_1^*)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{C_1^{(2)}}, \\ A_{12} &= (\varphi_1^*)'_{v_1}\big(C_1^{(1)}\big)'_{C_2^{(2)}} + (\varphi_1^*)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{C_2^{(2)}}, \\ B_1^{(1)} &= B^1 - (\varphi_1^*)'_t - (\varphi_1^*)'_{v_1}\big(C_1^{(1)}\big)'_t - (\varphi_1^*)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_t - (\varphi_1^*)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_t, \\ A_{21} &= \big(\widehat{\varphi}_2\big)'_{\varphi_1^*}A_{11} + \big(\widehat{\varphi}_2\big)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{C_1^{(2)}} + \big(\widehat{\varphi}_2\big)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(2)}}, \\ A_{22} &= \big(\widehat{\varphi}_2\big)'_{\varphi_1^*}A_{12} + \big(\widehat{\varphi}_2\big)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_{C_2^{(2)}}, \\ B_2^{(1)} &= B^2 - \big(\widehat{\varphi}_2\big)'_{\varphi_1}B^1 - \big(\widehat{\varphi}_2\big)'_t - \big(\widehat{\varphi}_2\big)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_t - \big(\widehat{\varphi}_2\big)'_{C_2^{(1)}}\big(\widehat{C}_2^{(1)}\big)'_{C_1^{(1)}}\big(C_1^{(1)}\big)'_t. \end{aligned}$$

Solving system (28) of linear equations, we obtain the Cauchy problem for $C_1^{(2)}$ and $C_2^{(2)}$; moreover, the initial conditions have the form

$$\varphi_1^*\Big(0, W_1(0) + C_1^{(1)}\big(0, W_2(0)\big), C_2^{(1)}\big(0, W_2(0)\big)\Big) = x_1,$$

$$\varphi_2 = \widehat{\varphi}_2\Big(0, x_1, C_2^{(1)}\big(0, W_2(0)\big)\Big) = x_2,$$

where

$$C_1^{(1)}\big(0, W_2(0)\big) = C_1^{(1)}\big(0, W_2(0) + C_1^{(2)}(0), C_2^{(2)}(0)\big),$$

$$C_2^{(1)}\big(0, W_2(0)\big) = \widehat{C}_2^{(1)}\Big(0, C_1^{(1)}\big(0, W_2(0) + C_1^{(2)}(0), C_2^{(2)}(0)\big), C_2^{(2)}(0)\Big).$$

Remark. Let $\Delta = 0$, i.e., assume that the "diffusion matrix" is singular. Then equations (25) have the form

$$\big(C_1^{(1)}\big)'_{v_2} = \frac{\sigma_{12}}{\sigma_{11}}, \quad \big(C_2^{(1)}\big)'_{v_2} = 0.$$

Hence, the function C_2 is "degenerate," i.e., it is independent of v_2 $\big($we have $C_2(t, v_2) = C_2(t)\big)$.

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