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MECHANICS OF MACHINES

Description of the Fields of Vibration in 2D Latticed Structures with Triangular (Hexagonal) Cells

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Abstract—This article deals with the problem of describing the fields of vibration in string lattices with triangular (centered hexagonal) cells. The problems of this kind have not been adequately studied despite the fact that the required models find application in the dynamic analysis of various machines and structures as well as in crystallography and materials science. Equations of motion are provided. Problems of propagation of sinusoidal waves and induced oscillations under a random broadband force action are considered.

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(1) Latticed structures are widely used in the aircraft industry and shipbuilding, and in many other industries of modern machinery manufacturing, civil engineering, and agriculture. An important subclass of such systems is plane (2D) lattices. They determine the designs of a number of mining, screening, and grading machines; they are used to model numerous 2D objects, viz., membranes, plates, panels, and building structures. Moreover, they can be used to analyze waves in metamaterials, periodically reinforced composite materials, crystals, and nanostructured surface and near-surface layers of structural materials [1–7]. This list can be considerably extended. However, for a number of reasons, dynamic analysis of lattice structures has not found due wide application and works related to the dynamics of the lattice structures considering the collisions of their nodes are rather rare [8–10].

The dynamic behavior of the plane lattices depends to a considerable degree on their structural features, in particular, on the types of the lattice cells. This article deals predominantly with problems of the dynamics of string lattices with triangular (centered hexagonal) cells. Equations of motion are provided for discrete and continual models, and the problem of forced random oscillations of the lattice that vibrates near the arrester under the action of random broadband forces is analyzed on the assumption of the applicability of the Markovian diffusion process methods.

(2) Let us consider a string lattice comprised of three families of elastic linear strings that form regular triangular cells (Fig. 1a). Let us select on the lattice the acute Cartesian coordinate system formed by axes x and y positioned at an angle of $\pi/3$ to each other (Fig. 1b). To identify the lattice nodes, obviously two current subscripts suffice. Let denote them by $k = 0, 1, 2, ..., N_1$ and $q = 0, 1, 2, ..., N_2$ assuming that subscript k marks the numbers along axis x and subscript q marks the numbers along axis y.

Let the nodes be perfectly solid bodies with masses $m_{kq} \equiv m$. We search for their displacements (deflections) along the axes positioned perpendicularly to the lattice plane.

The string elements of the cells are assumed to be inertia-free. The fixations of the strings at the nodes are considered perfectly rigid, and the tensions are so high that their changes under linear oscillations can be neglected.

Let us denote the tension of each string span by τ, and the lengths of the sides of the triangle, by a. Then, the coefficient of elasticity is $c = \tau a^{-1}$.

Each node inside the lattice has six equivalent neighboring nodes; therefore, the system can be interpreted as a hexagonal centered lattice [11, 12]. The Lagrangian function of the lattice has the form

$$L(u_{kq},\dot{u}_{kq}) = T_*(\dot{u}_{kq}) - U_*(u_{kq}) = \frac{1}{2}m\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\dot{u}_{kq}^2 - \frac{1}{2}c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6}D_n^2(u_{kq}), \quad (1)$$

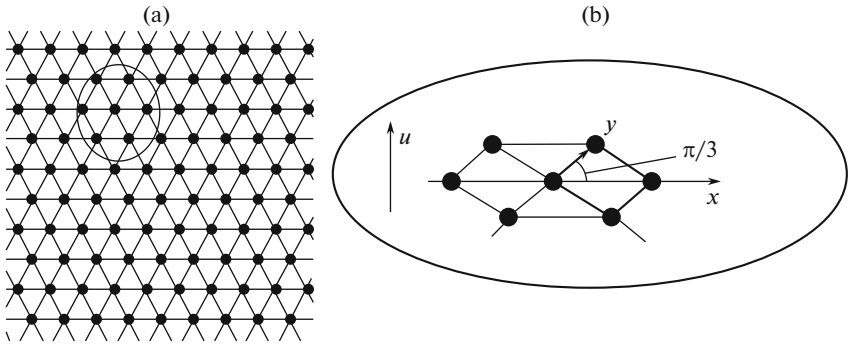

Fig. 1.

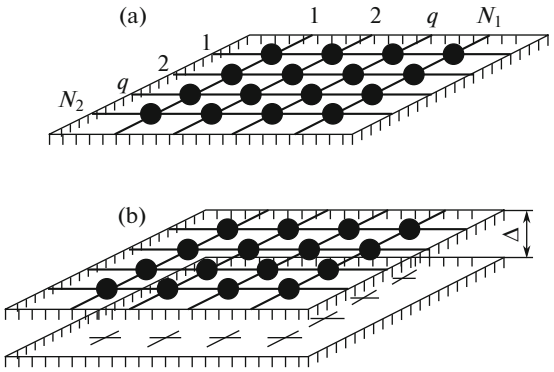

Fig. 2.

where $T_*(\dot{u}_{kq})$ and $U_*(u_{kq})$ are the kinetic and potential energies; $D_n(u_{kq})$ are changes in the lengths of the string spans between node (k, q) and its six neighbors ($n = 1, ..., 6$).

Let external forces representable by operator functions $g_{kq}(d/dt; t, u_{kq})$ be applied to the nodes. Then, we obtain for the bending oscillations of the lattice the equations of motion as

$$m\ddot{u}_{kq} + c(6u_{kq} - u_{(k-1,q)} - u_{(k+1,q)} - u_{(k,q-1)} - u_{(k,q+1)} - u_{(k-1,q-1)} - u_{(k+1,q+1)}) = g_{kq}(d/dt; t, u_{kq}). \quad (2)$$

When considering the boundary value problem, e.g., for the case of a lattice in the form of a parallelogram and having the strings to be restrained at the ends (not shown in Fig. 1), we assume that

$$u_{kq} = 0, \quad k = 0; N_1, \quad q = 0; N_2. \quad (3)$$

The restraint pattern may have another structure. The notation of Eq. (3) will be preserved; however, if the structure of the restraint pattern is complicated, apparent auxiliary relations may be added to the notation. If necessary, it may be supplemented by the initial conditions.

The system under consideration is similar to a lattice structure with square (rectangular) cells (Fig. 2a) [8–10]. The equations of motion for the deflections of the lattice with square cells have the form

$$m\ddot{u}_{kq} + c(4u_{kq} - u_{(k-1,q)} - u_{(k+1,q)} - u_{(k,q-1)} - u_{(k,q+1)}) = g_{kq}(d/dt; t, u_{kq}); \quad c = a\tau^{-1}, \quad (4)$$

where a is the length of the square cell side. The form of boundary conditions (3) is preserved.

(3) We go over to the continual analog of problems (2) and (3). Let us assume that the number of the strings that form the lattice is great; i.e., its local structural characteristic, the length of the triangular cell side a, is a small quantity. At the same time, the averaged lattice characteristics such as the surface density $\rho = m/a^2$ (the mass of the unit area) and elasticity of the spans c are not small. We postulate that there

exists a required number of times the continuously differentiable function $u(x, y, t)$ coincides at the lattice nodes with the discrete physical variable $u_{kq} = u(x, y, t)$. In addition, accurate to infinitely small ($O(a^2)$),

$$u_{(k\pm1,q)} = u(x \pm a, y, t); \quad u_{(k,q\pm1)} = u(x, y \pm a, t); \quad u_{(k\pm1,q\pm1)} = u(x \pm a, y \pm a, t). \tag{5}$$

However, we can rewrite as $g_{kq}(p; t, u_{kq}) = g[x, y; p; t, u(x, y, t)]$. Substituting (5) into (2), we can make obvious power series expansions and obtain

$$\begin{gathered}\rho u_{tt} - c[u_{xx}(x, y, t) + u_{xy}(x, y, t) + u_{yy}(x, y, t)] \\ - \frac{ca^2}{12}[u_{4x}(x, y, t) + u_{2x2y}(x, y, t) + u_{4y}(x, y, t)] + O(a^6) = g_0[x, y; \partial/\partial t; t, u(x, y, t)],\end{gathered} \tag{6}$$

where $g_0 = ga^{-2}$.

Boundary conditions (3) go into the conditions

$$u(x_0, y_0, t) = 0, \ (x_0, y_0) \in \mathbf{P}, \tag{7}$$

where **P** is the contour—a parallelogram or any other restraint contour—on which restrained boundary lattice nodes are positioned.

The term that contains derivatives of higher order in Eq. (6) allows the consideration of the specific nature of the initial lattice structure—it is proportional to lattice parameter a—and is important for consideration of high-frequency oscillations. It is frequently reasonable to neglect such terms and simplify the problem replacing Eq. (6) by the equation

$$\rho u_{tt} - c[u_{xx}(x, y, t) + u_{xy}(x, y, t) + u_{yy}(x, y, t)] = g_0[x, y; \partial/\partial t; t, u(x, y, t)]. \tag{8}$$

Let us assume that the lattice is not restrained and $g_0 = 0$. Let us represent the solution to Eq. (6) or Eq. (8) in the form of sinusoidal wave packets as

$$u(x, y, t) = B\exp[i(K_1x + K_2y - \omega t)]; \quad B = \text{const} > 0, \tag{9}$$

where $\{K_1, K_2\}$ are components of the wave vector and ω is the frequency. Hence, we can obtain the dispersion relations for system (8) as

$$\rho\omega^2 = c(K_1^2 + K_2^2 + K_1K_2); \tag{10}$$

and for system (6) we get

$$\rho\omega^2 = c(K_1^2K_2^2 + K_1K_2) + \frac{ca^2}{12}(K_1^4 + K_2^4 + K_1^4K_2^4). \tag{11}$$

Obviously, if the dispersion relation accounts for a larger number of terms that depend on parameter a, its order will increase by two units every time.

We should note that the long-wave approximations for equations of motion (4) of a lattice with square cells yield, instead of Eqs. (8) and (6), the respective membrane equations as

$$\rho u_{tt} - c[u_{xx}(x, y, t) + u_{yy}(x, y, t)] = g_0[x, y; \partial/\partial t; t, u(x, y, t)]; \tag{12}$$

$$\begin{gathered}\rho u_{tt} - c[u_{xx}(x, y, t) + u_{yy}(x, y, t)] - \frac{ca^2}{12}[u_{4x}(x, y, t) + u_{4y}(x, y, t)] + O(a^6) \\ = g_0[x, y; \partial/\partial t; t, u(x, y, t)].\end{gathered} \tag{13}$$

The dispersion relations assume the form

$$\rho\omega^2 = c(K_1^2 + K_2^2), \quad \rho\omega^2 = c(K_1^2 + K_2^2) + \frac{ca^2}{12}(K_1^4 + K_2^4). \tag{14}$$

For the basic discrete model of a triangular lattice, assuming that

$$u_{kn}(t) = B\exp[i(K_1k + K_2n - \omega t)]; \quad \text{B} = \text{const} > 0, \tag{15}$$

we find, after a series of transformations that follow from entering (15) into Eq. (2), the dispersion relation in the form

$$m\omega^2 = 4\sin^2\frac{1}{2}K_2 + 4\sin^2\frac{1}{2}K_1 - 4\sin^2\frac{1}{2}(K_1 + K_2)^2. \tag{16}$$

Fig. 3.

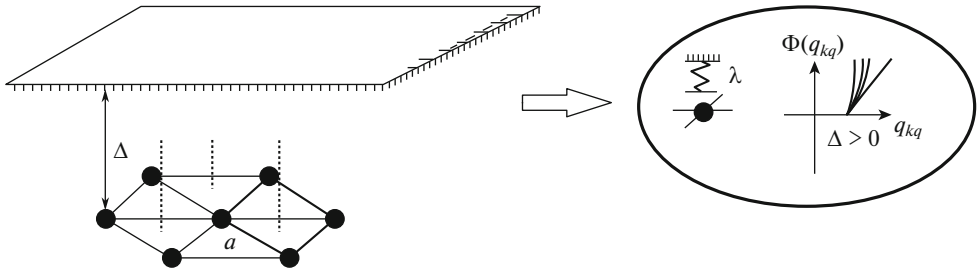

Fig. 4.

For long waves, $K_1, K_2 \to 0$, and from Eq. (16), we find

$$m\omega^2 = 2K_1^2 + 2K_2^2 + 2K_1 K_2,$$

which coincides accurate to the multiplier with the principal part of dispersion relation (11).

Let us consider how the models of the lattices with triangular and square cells relate to each other. Let us take the models that satisfy Eqs. (8) and (12). They can be combined by introducing before the mixed derivative a certain dimensionless multiplier, which we denote by σ. In the square bracket contained on the left-hand side of Eq. (8), the term $\sigma u_{xy}(x, y, t)$ appears. Let $\sigma \in [0,1]$. If $\sigma = 1$, we have model (8); at $\sigma \to 0$, we come to model (12).

If we introduce the same parameter σ before the differences $(u_{kq} - u_{(k+1,q+1)})$ and $(u_{kq} - u_{(k-1,q-1)})$, the model of the lattice with triangular cells (Eq. (2)) will be transformed into the model with square cells (Eq. (4)).

The correct description of the transformation of the lattice upon varying parameter σ is nontrivial since, despite the fact that the triangular and square cells are topologically equivalent [13] (see Fig. 3) and can be continuously deformed one into the other, imparting an unambiguous physical meaning to the intermediate values of parameter $\sigma \in (0,1)$ without postulating complementary hypotheses is problematic. The relevant matters are not considered.

(4) The considerations set forth below refer only to model (2) and the case of forced random oscillations that occur in a viscous medium. We assume that near each of the nodes of the lattice with the clearance $\Delta \geq 0$, generally speaking, a compliant arrester is installed (Fig. 4, one hexagonal cell of the lattice shown). Let the oscillations be excited by random forces that act on all nodes; these forces are described by stationary standard white noises with zero mean. The interaction of the lattice nodes with the arresters is assumed to be elastic. A vibro-impact variant of model (4) (Fig. 2b) was considered earlier in [8–10].

We use canonical variables $u_{kq} \equiv q_{kq}$ and $p_{kq} = m\dot{u}_{kq}$. Let $\lambda\Pi(q_{kq})$ be the potential energy of the force of the elastic impact interaction determined by the impact hypothesis, and let $\lambda \gg 1$ be a large parameter. Considering that the adopted interaction (impact) hypotheses are equal in all impact pairs, we can write

$$\lambda\Pi(q_{kq}) = \lambda \int_0^{q_{kq}} \Phi(z)dz, \tag{17}$$

where $\lambda\Phi(q_{kq})$ is the force of interaction.

The statement of the impact hypothesis is a motivated task of representing the interaction force. For example, if the impact pair is unidirectional and nonsymmetrical, the threshold function $\lambda\Phi_{kq}(q_k)$ that corresponds to it (Fig. 4) and by means of which, strictly speaking, the impact hypothesis is stated in the

impact pair in question will also be nonsymmetrical. The class of threshold functions $\{\Phi\}_\Delta$ for the problem under consideration is determined as

$$\Phi_{kq}(q_{kq}) \in \{\Phi\}_\Delta \equiv \{\Phi(q_{kq}); \Phi_{kq}(q_{kq}) = \alpha(q_{kq} - \Delta)\eta(q_{kq} - \Delta);$$
$$\Delta \geq 0; \alpha(q_{kq} - \Delta) \equiv 0; q_{kq} \geq 0\}.$$

Here, we assume that the function $\alpha(q_{kq})$ is continuously differentiable over the entire number axis, increases monotonically, and is convex at $q_{kq} \geq 0$ (Fig. 4) and $\eta(x)$ is a unit function. If the interaction is bilateral, the interaction function is a two-threshold function. After the initial work [14], a similar approach was further developed in [15] and in [9, 15, etc.] for vibro-impact systems with numerous impact pairs, in particular, for 2D systems.

At $\lambda \to \infty$, the hypothesis of interaction becomes Newton's hypothesis for a perfectly elastic impact. This means that the configuration space of system $\Re$ is set by the inequalities

$$\Re = \{q_{kq} \leq \Delta\}; \quad k = 0, \ldots, N_1; \quad q = 0, \ldots, N_2. \tag{18}$$

Upon collisions at certain points of time t_α : $p_{kq}(t_\alpha - 0) = -p_{kq}(t_\alpha + 0)$.

Assuming that the lattice is placed into a viscous medium that creates resistance proportional to the absolute velocity of the node with a constant of proportionality equal to $2b$, we write, considering also the transition to canonical variables, the representation for the forces contained in equations of motion (2) as

$$g_{kq} \equiv -2bm^{-1}p_{kq}\lambda\Phi_{kq}(q_k) + \xi_{kq}(t).$$

Here, according to the condition, $\xi_{kq}(t)$ are random forces described by standard white noises with equal intensities $2S$: $\langle \xi_{kq}\xi_{rs} \rangle = 2S\delta_{kq}\delta_{rs}\delta(t - t')$; δ_{kq} and δ_{rs} are the Kronecker deltas ($k, r = 1, 2, \ldots N_1$; $q, s = 1, 2, \ldots, N_2$)—the angle parentheses indicate the operation of statistical averaging; and $\delta(t) - \delta$ is a function.

For the conservative part of the system, the Hamiltonian function $H(q_{kq}, p_{kq})$ is easy to find based on the form of Lagrangian function (1) and Eq. (17), as well as the assumption that the energy losses upon collisions in each impact pair are not taken into consideration

$$H(q_{kq}, p_{kq}) = T_*(p_{kq}) + U_{0*}(q_{kq}) = \frac{1}{2}\sum_{k=1}^{N_1}\sum_{q=1}^{N_2} p_{kq}^2 + \frac{1}{2}c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6} D_n^2(q_{kq}) + \lambda\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\Pi(q_{kq}). \tag{19}$$

At any values of numbers N_1 and N_2 that determine the dimensions of the lattice under consideration, the latter satisfies the Hamiltonian equations in the form

$$\frac{dq_{kq}}{dt} = \frac{\partial H}{\partial p_{kq}}; \quad \frac{dp_{kq}}{dt} = -\frac{\partial H}{\partial q_{kq}} - 2bp_{kq} + \xi_{kq}. \tag{20}$$

Let us use methods of Markovian diffusion processes and write the Fokker–Planck–Kolmogorov equation for $w(t, \mathbf{q}_{kq}, \mathbf{p}_{kq})$, which is the joint probability density of a multidimensional random process $\{\mathbf{q}_{kq}(t), \mathbf{p}_{kq}(t)\}$ [16–19] that determines the state of the lattice, as

$$\frac{\partial P}{\partial t} + \sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\left[\{H, w\}_{kq} - 2b\frac{\partial(p_{kq}w)}{\partial p_{kqi}} - 2S\frac{\partial^2 w}{\partial p_{kq}^2}\right] = 0; \quad \{A, B\}_{kq} = \frac{\partial A}{\partial p_{kq}}\frac{\partial B}{\partial q_{kq}} - \frac{\partial A}{\partial q_{kq}}\frac{\partial B}{\partial p_{kq}} \tag{21}$$

is the Poisson bracket for node (k, q), and A and B are arbitrary continuously differentiable functions. Each component of matrices $\mathbf{q}_{kq}(t)$ and $\mathbf{p}_{kq}(t)$ yields the corresponding value of canonical variables q_{kq} and p_{kq}.

As is well-known [16], for a stationary process, when $\frac{\partial P}{\partial t} = 0$, Eq. (6) has an accurate solution yielded by the canonical Gibbs distribution as

$$P(\mathbf{q}_{kq}, \mathbf{p}_{kq}) = C\exp\left[-\frac{2b}{S}H(\mathbf{q}_{kq}, \mathbf{p}_{kq})\right], \tag{22}$$

which can be directly verified by substituting Eq. (22) into Eq. (21) and using the stationarity condition; constant C is determined upon normalization.

It is well-known—and can be seen from Eqs. (19) and (22)—that the momentums and coordinates are statistically independent.

Upon integration over all coordinates, the Gibbs distribution yields a normal distribution over momentum variables (the Maxwell distribution): the kinetic energy in the exponent index depends only on the squared momentums as

$$P(\mathbf{p}_{kq}) = C_1 \exp\left\{-\frac{2b}{S}\sum_{k=1}^{N_1}\sum_{q=1}^{N_2} p_{kq}^2\right\}; \quad C_1 = \left[\frac{2b}{\pi S}\right]^{0.5N}, \quad N = N_1N_2. \tag{23}$$

To test its normality, it suffices to number the momentums and change from double subscripting to sequential subscripting. Using Eq. (23), one can obtain all necessary information on the momentums and force actions in the system [15, 19].

(5) Upon integration of distribution (22) over all generalized momentums, we turn to the Boltzmann distribution over the generalized coordinates as

$$w(\mathbf{q}_{kq}) = C_2 \exp\left\{-2bS^{-1}\left[\frac{1}{2}c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6} D_n^2(q_{kq}) + \lambda\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\Pi(q_{kq})\right]\right\}, \tag{24}$$

which considers the form of Hamiltonian function (19).

To find the probability density of lower measurements, it is necessary to perform the required number of integrations of density function (24) over the coordinates of no interest.

Usually, the characteristic of the loading in the interaction region is presented in the form of a power function [14, 15]. Then, the potential energy of the elastic interaction is expressed by a piecewise power function as

$$w(\mathbf{q}_{kq}) = C_2 \exp\left\{-bS^{-1}\left[c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6} D_n^2(q_{kq}) + \lambda\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}(q_{kq}-\Delta)^{\alpha}\eta(q_{kq}-\Delta)\right]\right\}, \quad \alpha \geq 1. \tag{25}$$

The coordinate distribution function is

$$W(\mathbf{q}_{kq}) = \int_{-\infty}^{\mathbf{q}_{kq}} w(\mathbf{z}_{kq})d\mathbf{z}_{kq} = \mathbf{P}\{\mathbf{q}_{*kq} < \mathbf{q}_{kq}\}, \tag{26}$$

in which an N_1N_2-fold integration over all coordinates is supposed and the symbol $d\mathbf{z}_{kq}$ is the product of N_1N_2-differentials and $\mathbf{P}\{\mathbf{q}_{*kq} < \mathbf{q}_{kq}\}$ is the probability of an event in which the above ordering relation (<) takes place component-by-component.

Hence, one can see that the probability of existence of a configuration with threshold Δ exceeded for all lattice nodes is

$$\mathbf{P}\{\mathbf{q}_{kq} > \boldsymbol{\Delta}\} = 1 - W(\Delta), \tag{27}$$

where matrix $\boldsymbol{\Delta}$ corresponds to the achievement of the threshold configuration by all lattice nodes.

Entering Eq. (25) into (26) and then introducing the result into (29), we find

$$\mathbf{P}\{\mathbf{q}_{kq} > \boldsymbol{\Delta}\} = 1 - C_2\int_{-\infty}^{\boldsymbol{\Delta}} \exp\left\{-bcS^{-1}\left[c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6} D_n^2(z_{kq}) + \lambda\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}(z_{kq}-\Delta)^{\alpha}\eta(z_{kq}-\Delta)\right]\right\}d\mathbf{z}_{kq}. \tag{28}$$

At high λ values, the estimate of probability (28) is [23]

$$\mathbf{P}\{\mathbf{q}_{kq} > \boldsymbol{\Delta}\} = O(\lambda^{-N}); \quad N = N_1N_2; \quad \lambda \to \infty. \tag{29}$$

We can also show that for any fixed values of subscripts a and b, the probability of containment outside the threshold of the single lattice node $\mathbf{P}\{u_{ab} > \Delta\} = O(\lambda^{-\frac{1}{2}})$ is obtained for the pairs, triplets, and "ns" at any permissible values of subscripts at $\lambda \to \infty$ as $\mathbf{P}\{u_{ab} > \Delta; u_{cd} > \Delta\} = O(\lambda^{-1})$; $\mathbf{P}\{u_{ab} > \Delta; u_{cd} > \Delta; u_{cd} >\} = O(\lambda^{-\frac{3}{2}})$; $\mathbf{P}\{u_1 > \Delta; ...; u_n > \Delta\} = O(\lambda^{-\frac{n}{2}})$. Consequently, the implementation of the configurations accompanied by the containment of point bodies outside the threshold values is unlikely. These asymptotics can be refined and rewritten in the dimensional forms using general asymptotic estimate methods [21].

If $\lambda \to \infty$, potential barriers $\lambda\Pi(q_{kq})$ high at finite λ assume heights equal to the kinetic energies of each of the nodes and become overdense [9, 14]. The times of interactions become zero at the limit and the hypothesis of the impact becomes Newton's hypothesis for the elastic interaction.

In this case, the form of the Gibbs distribution will change. In representation (23), the term with the greater parameter in the exponent index vanishes and relations on the boundary and the impact conditions are added. In the case under consideration,

$$P(q_{kq}, p_{kq}) = C\exp\left[-\frac{2b}{S}H(q_{kq}, p_{kq})\right], \quad q_{kq} < \Delta; \quad p_{kq}(t_{0kq} - 0) = -p_{kq} + 0 \tag{30}$$

with the last two relations taking place for any (k, q) node and t_{0kq} being a random time point of the node's collision.

Maxwell distribution (23) for random generalized momentums preserves its form and meets the above conditions of the elastic impact. For the analysis, all kinds of characteristics of random impact momentums $J_{kq} = 2|p_{kq}|$ are important. Upon calculations, we obtain for a multidimensional random process, using the data of [8, 22], a truncated normal distribution for random process J_k

$$P(J_{kqi}) = \left[\frac{2b}{\pi S}\right]^{0.5N}\exp\left(-\frac{2b}{S}\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}J_{kq}^2\right); \quad J_{kq} \geq 0, \quad N = N_1N_2. \tag{31}$$

Distribution (31) determines the force effects related to the influence of the impacts on the dynamics of the multidimensional structure.

For odd and even jth moments of one-dimensional quantities J_{kq}, the following formulas can be obtained [15, 19]

$$m^{(2j+1)}(J_{kq}) = \sqrt{\frac{2b}{\pi S}}\frac{j!}{2(b/S)^{j+1}}; \quad m^{(2j)}(J_{kq}) = (2j-1)!![Sb^{-1}]^n.$$

(6) When considering random profiles in vibro-impact systems with a great number of degrees of freedom, one of the basic problems is the determination of the rate of occurrence of configurations accompanied by the interaction of the bodies that comprise the system with the arresters.

Considering random configurations that occur when implementing vibro-impact modes in the lattice, we take into consideration the low probability of "over-the-threshold" configurations (see (29)). At $\lambda \to \infty$, upon integration over the momentums, we find from (24) and (30) the Boltzmann distribution as

$$w(\mathbf{q}_{kq}) = C_2\exp\{-2bS^{-1}U_*(q_{kq})\}; \quad U_*(q_{kq}) = \left[\frac{1}{2}c\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}\sum_{n=1}^{6}D_n^2(q_{kq})\right]; \quad q_{kq} < \Delta, \tag{32}$$

where the meaning of the denotation becomes obvious from relations (1) and (19) and it is assumed that subscripts k and q take all permissible values.

Having obtained representations (30) and (32), one can completely describe the "configuration" characteristics of the systems, viz., the types of random profiles, their statistic characteristics, etc.

Let us assume that there is a possibility of recording in some way any randomly implemented profile $\mathbf{q} = \{q_{kq}\}$. We fix it as $\mathbf{q} = \mathbf{q}^0 \equiv \{q_{kq}^0\}$ and determine the mean frequency of its occurrence $(\Omega_0\{q_{kq}^0\})$ as the mathematical expectation of the lattice nodes crossing the corresponding surface in the configuration space at a positive velocity. As the configuration $\{q_{kq}^0\}$ is fixed, the value of the potential energy $U_*(q_{kq}^0)$ is also fixed, and, consequently, we can find

$$\Omega_0\{q_{kq}^0\} = C\exp\left[\left(-\frac{2b}{S}\right)U_*(q_{kq}^0)\right]\int_0^\infty\ldots\int_0^\infty\ldots\int_0^\infty\prod_{k=1}^{N_1}\prod_{q=1}^{N_2}p_{kq}\exp\left[\left(-\frac{2b}{S}\right)\sum_{k=1}^{N_1}\sum_{q=1}^{N_2}p_{kqi}^2\right]\mathbf{dp}. \tag{33}$$

One can easily see that, when the system under consideration is functioning, the equilibrium configuration $\mathbf{q}^0 \equiv \mathbf{0}$ will be recorded most frequently.

Since $U_*(\mathbf{0}) = 0$, upon integration, we find from Eq. (33)

$$\Omega_0\{\mathbf{0}\} = C\left(\frac{S}{16b}\right)^{N_1N_2}. \tag{34}$$

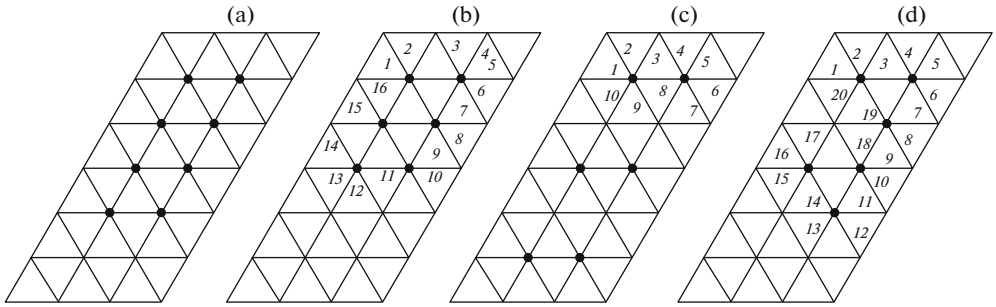

Fig. 5.

Let us consider random lattice profiles upon the occurrence of which at certain points of time the nodes are either in the state of equilibrium or thrown onto an arrester. Let us denote the arbitrary configuration as $\mathbf{q}(\Delta;\beta) = \{q_{kq}(\Delta;\beta)\}$, where index $\beta \in \mathbf{B}$ numbers the described profiles and the number of the elements of set **B** obviously equals $2^{N_1N_2}$.

From Eqs. (33) and (34), we find

$$\Omega_0\{q_{kq}(\Delta;\beta)\} = \Omega_0\{\mathbf{0}\}\exp\left[\left(-\frac{2b}{S}\right)U_*\{q_{kq}(\Delta;\beta)\}\right]. \quad (35)$$

Let us denote the relative rate of occurrence of the configuration in question as $\omega_\beta = \omega_\beta\{q_{kq}(\Delta;\beta)\} = \Omega_0\{q_{kq}(\Delta;\beta)\}/\Omega_0\{\mathbf{0}\}$. It follows from Eq. (35) that

$$\omega_\beta\{q_{kq}(\Delta;\beta)\} = \exp\left[\left(-\frac{2b}{S}\right)U_*\{q_{kq}(\Delta;\beta)\}\right]. \quad (36)$$

Thus, like in other similar cases, the profiles that have lower potential energy can occur relatively more frequently. Consequently, the fewer sharp bends the profile has, the more frequently it will be recorded. The exponent present in Eq. (36) shows that the growth of the potential energy of the profile makes very soon the recording of any "exotic" configurations with heavily broken profiles practically impossible.

The profiles in question correspond to two characteristic values of the generalized coordinates, that is, being in the equilibrium position and reaching the arresters: $\mathbf{q}(\Delta;\beta) = \{q_{kq}(\Delta;\beta)\}$. They are obviously rather characteristic of the lattices of this type and are convenient to record using the diagrams shown in Fig. 5 that present a lattice in the form of a parallelogram with 28 nodes including those on the boundary. We mark the nodes positioned near the arrester as shown in Fig. 5a. When in the equilibrium position, the node is not marked. In this way, we can obtain diagrams that correspond to the configurations under consideration with the help of which the values of the corresponding potential energy present in Eq. (36) can be assessed.

A detailed description of the potential configurations for different lattices can be provided using methods from combinatorial geometry [22], which is an independent task. As an example, let us select in Fig. 5 the configurations that correspond to six nodes that can be thrown to the arrester.

The lowest value of the potential energy will correspond to the configuration with a single marked point, since in this case only six elastic couplings appear to be engaged. In the example in question, one must compare the configurations in which six nodes reach the level of the arrester as shown in Figs. 5b, 5c, and 5d.

It can be seen that the higher the "compactness" of the colliding nodes, the lower the corresponding level of the potential energy. From the diagrams, one can see that the elastic elements of the neighboring interacting nodes do not become deformed. Consequently, the neighborhood of two marked or, on the contrary, unmarked nodes does not increase the value of the potential energy of the configuration. However, the neighborhood of a marked and an unmarked node increases the potential energy by a value of $\sim c\Delta^2$. As a result, for the configuration of diagram *b* we have 16 deformations and $U_* \sim 16c\Delta^2$; for diagram *c*, we have $U_* \sim 30c\Delta^2$; and for a certain intermediate configuration *d*, we have $U_* \sim 20c\Delta^2$. Consequently,

the relative rate of occurrence of configuration b is the highest for the combination with six colliding nodes.

(7) Below, we provide concluding remarks.

(7.1) It has been noted that the configurations comprised of a single colliding node have the lowest potential energy values. Simple calculations show that in the profiles in which $Q \geq 2$ nodes collide, those nodes have the lowest potential energy that comprise a convex polygon. At $Q = 2$, the polygon degenerates into a segment.

(7.2) Within the framework of the model in question, instead of inequalities (18) a configuration space of a more complicated structure as

$$\begin{gathered}\Re = \{\Delta_{2_{kq}} \leq q_{kq} \leq \Delta_{1_{kq}}\}; \quad k = 0, \ldots, N_1; \quad q = 0, \ldots, N_2; \\ -\infty \leq \Delta_{2_{kq}} \leq 0; \quad 0 \leq \Delta_{1_{kq}} \leq \infty\end{gathered} \tag{37}$$

can be considered.

This means that every node can be fitted with its own arrester. The infinite values correspond to the case when the node in question (k, q) has no arrester. Distributions (30) and (31) are preserved, and the constraints on the state change their form according to inequalities (37). The structure of Eq. (36) is preserved.

(7.3) For lattices with square (rectangular) cells [8], all qualitative conclusions hold true. With respect to one-dimensional chains [19], the profiles with single, double, triple, etc., throws onto the arrester are recorded relatively frequently. In this case, either the lattice that has a single node or all nodes lie in the same segment. This is the fundamental distinctive feature of the one-dimensional model.

The above models can be used to investigate two-dimensional discrete–continuous systems [22]. The systems considered can be used to study self-similar lattice structures [23].

(7.4) The engineering significance of the problems considered is pointed out in **Section 1**. They are not only of the model value but also of interest for design engineers since they offer the possibility of getting certain tentative ideas when performing dynamic analysis of a series of particular 2D structures that function under collisions of their elements. In this case, the problems become more complicated and their solution requires the use of numerical methods [24].

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