ISSN 1052-6188, Journal of Machinery Manufacture and Reliability, 2017, Vol. 46, No. 5, pp. 442–447. © Allerton Press, Inc., 2017.
Original Russian Text © M.A. Il'gamov, 2017, published in Problemy Mashinostroeniya i Nadezhnosti Mashin, 2017, No. 5, pp. 30–35.

MECHANICS
OF MACHINES

On the Effect of Ambient Pressure on the Bending of a Plate

M. A. Il'gamov[a, b]

[a] *Blagonravov Institute of Mechanical Engineering, Russian Academy of Sciences, Moscow, Russia*
[b] *Institute of Mechanics, Ufa Scientific Center, Russian Academy of Sciences, Ufa, Russia*
e-mail: ilgamov@anrb.ru

Received May 24, 2017

Abstract—From preset pressure values on both surfaces of a plate, its bending is determined depending not only on the pressure difference but also on the product of the average pressure and the curvature of the middle surface. The latter component of this action is determined according to the models of Kirchhoff and Timoshenko for the case of a cylindrical linear bending of a plate. It is shown that taking into account the average pressure leads to an increase in the effective flexural rigidity. The value of deflection according to the models of Kirchhoff and Timoshenko is compared with a classical result. A criterion is established for the situation where the influence of the ambient pressure on the plate bending can be significant. The effect of the average pressure exerted on the longitudinal stability of the plates is determined.

DOI: 10.3103/S1052618817050089

INTRODUCTION

The analysis of the effect of ambient pressure on the bending of thin-walled elastic elements such as rods, plates, and shells is important in the case of high pressure values. This effect exerted on structures is not taken into account because of its insignificance under normal operating conditions. It also depends on the relative thickness of a plate.

In this paper, we consider static cylindrical bending of a plate. It is assumed that the ratio of the length L to the thickness h of the plate $L/h \sim 10^{1/2}$ and greater. In this connection, we start from the Timoshenko model [1] for bending of a plate, which takes into account the deformation of transverse shear amounting to $dw/dx - \psi$, where $w(x)$ is the deflection and $\psi(x)$ is the angle shown in Fig. 1. The result should be compared with bending according to the Kirchhoff model [1, 2].

It is assumed that an excess pressure p_1 acts on the lower surface of the plate, and an excess pressure p_2 acts on the upper one. Upon the bending of the plates, the pressure values p_1 and p_2 remain unchanged. A plate long in the direction y is pinned along the edges $x = 0$, L, and the pressure does not act on these edges. In the case of a gaseous medium, the difference in gas densities below and above the plate is not taken into account. Taking them into account gives an additional distributed transverse force. Such effects are considered by the authors of [3, 4]. In the case of a dropping liquid, its density should be taken into account. The direction of the z axis and that of the deflection w upwards are considered as positive.

The expressions for the bending torque and intersecting force have the following form [1, 2]:

$$M = D\frac{d\psi}{dx}, \quad Q = -B\left(\frac{dw}{dx} - \psi\right), \quad D = \frac{Eh^3}{12}, \quad B = Gh. \tag{1}$$

Here, the elastic moduli E and G are taken with the allowance for the Poisson coefficient. In the composition of G, a coefficient amounting to 0.833 [1, 2] is taken into account.

In the equations for torques and transverse forces

$$\frac{dM}{dx} - Q = 0, \quad -\frac{dQ}{dx} - \gamma h + p' = 0, \tag{2}$$

γ and p' are the specific weight of the plate material and the pressure drop across its surface, respectively. According to the Kirchhoff model, the second equation of (2) contains p instead of p'.

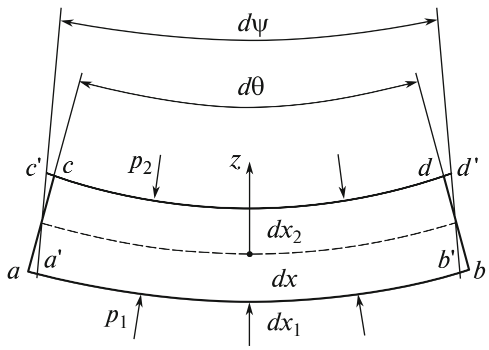

Fig. 1.

Figure 1 shows a plate element having length $dx \approx Rd\theta \approx R'd\psi$ of the middle surface. Here R and R' are the radii of curvature formed upon bending according to the Kirchhoff and Timoshenko models. According to the first of these, the lengths dx_1 and dx_2 of the extreme fibers are represented by segments ab and cd, whereas according to the Timoshenko model they are represented by segments $a'b'$ and $c'd'$. They are, respectively,

$$dx_1 = \left(R + \frac{h}{2}\right)d\theta, \quad dx_2 = \left(R - \frac{h}{2}\right)d\theta, \quad dx_1' = \left(R' + \frac{h}{2}\right)d\psi, \quad dx_2' = \left(R' - \frac{h}{2}\right)d\psi. \tag{3}$$

The pressure drop across the plate is determined from the equalities $pdx = p_1dx_1 - p_2dx_2$ and $p'dx = p_1dx_1' - p_2dx_2'$. Since dx_1, dx_2 and dx_1', dx_2' differ in these models, the corresponding values of the difference should also be different:

$$p = p_1 - p_2 + \frac{h(p_1 + p_2)}{2R}, \quad p' = p_1 - p_2 + \frac{h(p_1 + p_2)}{2R'}.$$

In the first of these expressions, the curvature R^{-1} can be replaced by d^2w/dx^2. Since $dx = Rd\theta = R'd\psi$, in the second expression, $1/R' = d\psi/Rd\theta = d\psi/dx$.

The expressions for p and p' take the form

$$p = p_1 - p_2 + \frac{h(p_1 + p_2)}{2}\frac{d^2w}{dx^2}, \quad p' = p_1 - p_2 + \frac{h(p_1 + p_2)}{2}\frac{d\psi}{dx}. \tag{4}$$

If we proceed from the Kirchhoff hypotheses, then, instead of (1), $M = Dd^2w/dx^2$, and according to (2), $Q = Dd^3w/dx^3$. The bending equation $d^2M/dx^2 = -\gamma h + p$ takes the form

$$\frac{d^4w}{dx^4} - \frac{h(p_1 + p_2)}{2D}\frac{d^2w}{dx^2} = -\frac{p_e}{D}, \quad p_e = \gamma h - p_1 + p_2. \tag{5}$$

According to the Timoshenko model, in accordance with (1)–(4), we have

$$D\frac{d^2\psi}{dx^2} + B\left(\frac{dw}{dx} - \psi\right) = 0, \quad B\left(\frac{d^2w}{dx^2} - \frac{d\psi}{dx}\right) = p_e. \tag{6}$$

Expressing dw/dx from the first equation of (6), we have

$$\frac{dw}{dx} = \psi - \frac{D}{B}\frac{d^2\psi}{dx^2}; \tag{7}$$

the second equation of (6), taking (4) into account, can be reduced to the form

$$\frac{d^3\psi}{dx^3} - \frac{h(p_1 + p_2)}{2D}\frac{d\psi}{dx} = -\frac{p_e}{D}.$$

In this case, the expressions for the bending torque and the intersecting force have the form

$$M = D\frac{d\psi}{dx}, \quad Q = D\frac{d^2\psi}{dx^2}. \tag{8}$$

The system (6) can also be reduced to an equation with respect to the function $w(x)$. From the second equation of (6) and expression (7), we have

$$\frac{d\psi}{dx} = \left(\frac{d^2w}{dx^2} - \frac{p_e}{B}\right)\left(1 - \frac{h(p_1 + p_2)}{2B}\right)^{-1}. \tag{9}$$

Let us substitute (9) in the first equation of (6), writing it in the form

$$\frac{D}{B}\frac{d^3\psi}{dx^3} + \frac{d^2w}{dx^2} - \frac{d\psi}{dx} = 0.$$

Taking into account that only the case of constant values of p_1 and p_2 over the length of the plate is considered, we can write the equation

$$\frac{d^4w}{dx^4} - \frac{h(p_1 + p_2)}{2D}\frac{d^2w}{dx^2} = -\frac{p_e}{D},$$

which coincides with Eq. (5) according to the Kirchhoff model. It should be noted that there are no coincidences in the case of p_1 and p_2 values variable with respect to x.

As one can see from (8) and (9), the expressions for the bending torque and shear force differ from $M = Dd^2w/dx^2$ and $Q = Dd^3w/dx^3$

$$M = D\left(\frac{d^2w}{dx^2} - \frac{p_e}{B}\right)\left(1 - \frac{h(p_1 + p_2)}{2B}\right)^{-1}, \quad Q = D\frac{d^3w}{dx^3}\left(1 - \frac{h(p_1 + p_2)}{2B}\right)^{-1}. \tag{10}$$

In the case of p_1 and p_2 variable with respect to x, the expression for Q in (10) is more complicated.

ANALYSIS AND COMPARISON OF EXISTING MODELS

In order to compare the solutions for the Kirchhoff and Timoshenko models, here we directly use system (6). Let us take the conditions at $x = 0$, L in the form $M = 0$, $w = 0$. These conditions are satisfied by the functions

$$w = W\sin(\pi x/L), \quad \psi = \Psi\cos(\pi x/L). \tag{11}$$

Approximation (11) is sufficient for a qualitative estimation of the solution. From the first equation of (6) and functions (11), we find that

$$\Psi = \frac{\pi W}{L(1 + \beta)}, \quad \beta = \frac{\pi^2 D}{L^2 B}. \tag{12}$$

The second equation of (6) and expressions (4) and (12) give

$$B\left[\left(\frac{\pi}{L}\right)^2 - \frac{2B - h(p_1 + p_2)}{2(D + B(\pi / L)^2)}\right]W\sin\frac{\pi x}{L} = -p_e.$$

Multiplying this equation by $\sin(\pi x/L)$, after integration over the range from 0 to L, we obtain

$$W = -\frac{4p_eL^4(1 + \beta)}{\pi^5 D(1 + \alpha)}, \quad \alpha = \frac{hL^2(p_1 + p_2)}{2\pi^2 D}. \tag{13}$$

After the integration of Eq. (5) taking into account (11), we can find the deflection amplitude according to the Kirchhoff model

$$W = -\frac{4p_eL^4}{\pi^5 D(1 + \alpha)}. \tag{14}$$

Since the direction of the z axis upward is positive, there is a negative sign in (13) and (14).

Thus, in (13), the coefficient

$$\beta = \frac{\pi^2 E}{12G}\left(\frac{h}{L}\right)^2 \tag{15}$$

determines the influence of transverse shear upon bending, which leads to an increase in deflection. For many materials, $E/G \approx 2.5$–2.7 [5]; therefore, $\beta \approx 2(h/L)^2$. In (13) and (14), taking into account the aver-

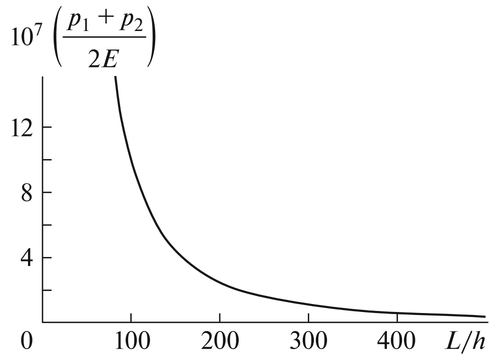

Fig. 2.

age pressure on the plate via coefficient α results in a decrease in deflection. This effect is the same for both models, although the expressions for p and p' according to (4) differ from each other. The approximate value of α is

$$\alpha \approx \frac{p_1 + p_2}{2E}\left(\frac{L}{h}\right)^2. \tag{16}$$

As one can see from (15) and (16), the correction introduced by taking into account the transverse shear decreases with the relative thickness as $(h/L)^2$, while taking into account the average pressure on the plate surfaces gives an increase in the correction as $(L/h)^2$. It should be noted that the E/G ratio varies little for different materials, whereas the ratio between the average pressure $(p_1 + p_2)/2$ and the elastic modulus E can vary over a wide range.

The effect of transverse shear on the bending of beams, plates, and shells is well studied, which cannot be said about taking into account the average pressure on the surfaces of a plate.

When the excess pressures p_1 and p_2 have negative values, according to (13) and (16), the value of α is negative too. Consequently, in this case, the deflection of the plate exceeds the corresponding deflection according to the classical theory.

In Fig. 2, the curve is plotted for the value of $\alpha = 10^{-2}$ in (16), or $10^2(p_1 + p_2)/(2E) = (h/L)^2$, which means a one-percent correction to the value of the deflection amplitude according to formulas (13) and (14). In the area below the curve, the influence of the average pressure on the bending of the plate is small, whereas above the curve this effect becomes noticeable. If, for example, $L/h = 300$, then in the area lower than the ratio $(p_1 + p_2)(2E) \approx 10^{-7}$ there is no influence of the average pressure on bending.

At $\alpha \ll 1$, formulas (13) and (14) for the deflection amplitude coincide with the classical ones; at $\alpha \gg 1$, formula (14) has the form

$$\frac{W}{h} = -\frac{8p_e}{\pi^3(p_1 + p_2)}\left(\frac{L}{h}\right)^2. \tag{17}$$

In this limiting case, the bending of the plate is determined not by its rigidity, but by the average pressure of the surrounding medium (the first of these factors is proportional to h^3, whereas the second one is proportional to h).

In this case, the term with the coefficient D can be omitted from Eqs. (5) and (6). From the first equation of (6), it follows that $\psi = dw/dx$, which means that there are no differences in the Kirchhoff and Timoshenko models. From the second equation of (6), we obtain $p = \gamma h$, which, together with (4), coincides with Eqs. (5) when the term with bending rigidity D can be omitted therein. Thus, the limiting case under consideration corresponds to the equilibrium between the self-weight of the plate and the excess pressures p_1 and p_2. It should be noted that the exact solution of Eq. (5) without the first term, satisfying the conditions $w = 0$ $(x = 0, L)$, has the form

$$w = -\frac{p_e x(L - x)}{h(p_1 + p_2)}. \tag{18}$$

As follows from (18), the bending occurs with a constant curvature over the entire length of the plate amounting to $2p_eh^{-1}(p_1 + p_2)^{-1}$. The approximate solution (11) gives an amplitude value (17) and a curvature that varies in a sinusoidal way. For $x = L/2$, the amplitude of exact solution (18) is

$$\frac{W}{h} = -\frac{p_e}{4(p_1 + p_2)}\left(\frac{L}{h}\right)^2. \tag{19}$$

Instead of the figure 4 in (21), there is $\pi^3/8 \approx 3.88$ in the approximate value of (17).

RESULTS AND DISCUSSION

Let us consider change in pressures p_1 and p_2 exerted on the plate connected with the specific weights $\gamma_1 = g\rho_1$ and $\gamma_2 = g\rho_2$ of incompressible media. Here, g is the gravitational acceleration, and ρ_1 and ρ_2 are the densities of the lower and upper liquids. Let us confine ourselves to considering the problem according to Kirchhoff's model.

Let us assume that, at $g = 0$ and $w = 0$, the pressures p_1 and p_2 on the plate surface from below and from above are p_0. Then, for nonzero values of g and w,

$$p_1 = p_0 - \gamma_1\left(w - \frac{h}{2}\right), \quad p_2 = p_0 - \gamma_2\left(w + \frac{h}{2}\right). \tag{20}$$

Taking into account dx_1 and dx_2 according to (3), we determine the effect exerted on the plate from the side of the media

$$p = \frac{p_1dx_1 - p_2dx_2}{dx} = p_1\left(1 + \frac{h}{2R}\right) - p_2\left(1 - \frac{h}{2R}\right).$$

Substituting here expressions (20) and $R^{-1} = d^2w/dx^2$, disregarding nonlinear terms, we obtain

$$p = \frac{h}{2}(\gamma_1 + \gamma_2) - (\gamma_1 - \gamma_2)w + \frac{h}{2}\left(2p_0 + \frac{h}{2}(\gamma_1 - \gamma_2)\right)\frac{d^2w}{dx^2}. \tag{21}$$

If the difference between dx_1 and dx_2 is ignored, then instead of (21) we have $p = (h/2)(\gamma_1 + \gamma_2) - (\gamma_1 - \gamma_2)w$.

From the equation $Dd^4w/dx^4 = -\gamma h + p$ and expression (21), it follows that

$$D\frac{d^4w}{dx^4} - h\left(p_0 + \frac{h}{4}(\gamma_1 - \gamma_2)\right)\frac{d^2w}{dx^2} + (\gamma_1 - \gamma_2)w = -h\left(\gamma - \frac{\gamma_1 + \gamma_2}{2}\right). \tag{22}$$

Taking the approximate solution as before in the form of (11), after the integration of (22), we obtain the expression for the deflection amplitude

$$W = -\frac{4p_eL^4}{\pi^5D(1 + \alpha)}, \quad p_e = h\left(\gamma - \frac{\gamma_1 + \gamma_2}{2}\right), \quad \alpha = \frac{1}{\pi^4D}\left[\pi^2L^2\left(hp_0 + \frac{h^2}{4}(\gamma_1 - \gamma_2)\right) + L^4(\gamma_1 - \gamma_2)\right], \tag{23}$$

which coincides with (14), but p_e and α appearing in (23) differ from expressions (5) and (13). They coincide at $g = 0$ and $p_1 = p_2 = p_0$.

It follows from (23) that, if the weight γh per unit length of the plate is greater than the corresponding Archimedean force $(\gamma_1 + \gamma_2)h/2$, then the plate bends downward (negative sign). Otherwise, there occurs a bending upward. The average pressure p_0 always leads to a decrease in deflection. This was established above for the case of different pressures p_1 and p_2. At $\alpha > 0$, which takes place for $\gamma_1 > \gamma_2$ and at any value of p_0, the deflection is less than in the classical result (23) (when $\alpha = 0$).

If $\alpha < 0$, which could be, for example, at $p_0 = 0$ and $\gamma_2 > \gamma_1$ (the upper liquid is heavier than the bottom one), the deflection is greater than it should be according to the classical formula. When α tends to -1, linear theory predicts an unlimited increase in deflection. The value of $\alpha = -1$ can be called a critical value, which is achieved when the specific weight (or density) of the upper liquid exceeds the specific weight of the lower liquid, amounting to

$$\gamma_2 - \gamma_1 = \frac{\pi^2}{L^4}\left(\pi^2D + hL^2p_0\right). \tag{24}$$

Here, the term $(\pi h/2L)^2$ is omitted as a small one compared with unity, which corresponds to neglecting the last term in (21) as compared with the term $(\gamma_1 - \gamma_2)w$ (the same is done in (22) and (23)).

For preset values of L, h, D, γ_1, and γ_2, the critical value of p_0 on the surfaces of the plate according to (24) is

$$p_0 = -\left(\frac{\pi^2 D}{L^2 h} + \frac{L^2}{\pi^2 h}(\gamma_1 - \gamma_2)\right). \tag{25}$$

It is necessary to estimate the possibility for satisfying condition (25) for real parameters. Let $\gamma_1 = \gamma_2$, $h/L = 0.0005$, and $E = 2 \times 10^5$ MPa. Then $p_0 \approx -E(h/L)^2 = -0.05$ MPa $= -0.5$ bar. Thus, for thin plates, the pressure drop below atmospheric pressure causes an increase in deflections (as shown above, a positive overpressure leads to a decrease in deflections). These effects can be important for very thin plates (micro- and nanofilms).

If there is a longitudinal compressive force N per unit width of the plate, then a term Nd^2w/dx^2 appears on the left-hand side of Eq. (22). Accordingly, in the solution (23),

$$\alpha = \frac{1}{\pi^4 D}\left[\pi^2 L^2\left(hp_0 + \frac{h^2}{4}(\gamma_1 - \gamma_2) - N\right) + L^4(\gamma_1 - \gamma_2)\right]. \tag{26}$$

For $\alpha = -1$, the critical value of the compressive force follows from (26) (the small term mentioned above is also omitted)

$$N = \frac{\pi^2 D}{L^2} + \left(hp_0 + \frac{L^2}{\pi^2}(\gamma_1 - \gamma_2)\right).$$

Thus, the average ambient pressure p_0 and a greater specific weight of the lower medium comparing to the upper one $(\gamma_1 - \gamma_2 > 0)$ lead to an increase in the critical value of the compressive force. Otherwise ($p_0 <$ 0, $\gamma_1 < \gamma_2$), the critical value of N decreases. By the authors of [4], this problem was called an interaction of Euler and Rayleigh instabilities.

Up to now, it was assumed that there is no pressure of the media exerted on the end edges of the plate ($x = 0$, L). If the compressive force N is caused by the action of pressure p_0 of media with identical specific weights ($\gamma_1 = \gamma_2$) on the end sections of the plate ($x = 0$, L), then $N = p_0 h$. In this case, from (26), it follows that $\alpha = 0$. Therefore, in accordance with (23), it follows that the deflection amplitude

$$W = -\frac{4L^4 h}{\pi^5 D}(\gamma_1 - \gamma_2) \tag{27}$$

does not depend on the pressure of a homogeneous liquid. The absolute stability of an elastic band under all-round pressure was first established by the authors of [6] on the basis of the relationships of three-dimensional elasticity theory. When the specific weights of the plate material and the surrounding liquid are equal to each other ($\gamma = \gamma_1$), it follows from (27) that $W = 0$.

CONCLUSIONS

Thus, bending of an elastic plate depending on the excess pressures of media contacting the plate is revealed in this work. An increase in the average pressure leads to an increase in the effective flexural rigidity of the plate. In the case of a large average pressure and a small relative thickness of the plate, its bending is determined not only by the flexural rigidity but also by the average pressure.

REFERENCES

1. Timoshenko, S., Young, D.H., and Weaver, W., Jr., *Vibration Problems in Engineering,* New York: Wiley, 1974.
2. Vol'mir, A.S., *Nelineinaya dinamika plastin i obolochek* (Nonlinear Dynamics for Plates and Shells), Moscow: Nauka, 1972.
3. Il'gamov, M.A., *Static Problems of Hydroelasticity,* Moscow: Nauka, Fizmatlit, 1998.
4. Il'gamov, M.A., The interaction of instabilities in a hydroelastic system, *Appl. Math. Mech.*, 2017, vol. 80, pp. 400–408.
5. Pisarenko, G.S., Yakovlev, A.P., and Matveev, V.V., *Spravochnik po soprotivleniyu materialov* (Materials Strength. Handbook), Kiev: Naukova Dumka, 1988.
6. Ishlinskii, A.Yu., Research of elastic systems stability in terms of mathematical elastic theory, *Ukr. Mat. Zh.,* 1954, vol. 6, no. 2, pp. 140–146.

Translated by O. Polyakov