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RELIABILITY, STRENGTH, AND WEAR RESISTANCE OF MACHINES AND STRUCTURES

Problems on Comparing Analytical and Numerical Estimations of Stressed-Deformed State of Structure Elements

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Abstract—A methodical approach to estimating the accuracy of the stressed-deformed state of structural elements has been presented that consists of the reverse analysis of errors in analytical and numerical solutions. The novelty of the approach lies in defining the problems of comparing analytical and numerical results jointly with the developed method for estimating the computational errors. The approach makes it possible to increase the validity of computation results according to the applied computation procedures, which makes it possible to increase the reliability and safety of technical objects.

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Computation models, including the description of geometry and boundary conditions (computation procedure), as well as analytical expressions that make possible to obtain close solutions for calculating unknown values (deformations and stresses), are widely used to analyze the stressed-deformed state of load-carrying elements of technical objects. These models are generated using several assumptions, simplifications, and hypotheses. As a result, the analytical solutions obtained by using these models are characterized by certain, sometimes unknown inaccuracies.

In the present paper, by *analytical solution,* we mean analytical expressions for calculating approximate solutions according to simplified computation models.

To estimate the inaccuracies, it is reasonable to compare analytical and numerical (as a rule, finite-elemental) solutions obtained for the same computation procedures. However, numerical solutions are calculated with errors due to many factors depending on the selected numerical method, ranging from the finite dimensionality of the discrete space in which the accurate solution is approximated, to the round-off error.

The level of errors depends on the parameters of the finite-element model, i.e., the performances of the used finite elements and mesh domains, and can be controlled and estimated to further select the required accuracy.

Therefore, there is uncertainty with respect to the adequacy and accuracy of alternative (analytical or numerical) variants of the solution, which makes it difficult to properly choose methods for their practical application. In several cases, when logically analyzing the correspondence between analytical and numerical solutions and real object, we can suppose that one method is preferable. However, it is reasonable to quantitatively estimate the accuracy of the solution to increase the validity of the choice.

The disagreement between analytical and numerical solutions, which hypothetically contains errors of a different nature and the necessity to eliminate this uncertainty, make it possible to accept that the approach is promising if, according to the results on comparing numerical and analytical solutions for the same computation procedure, it enables the following:

—estimates of the accuracy of analytical and numerical solutions and decisions on whether its use is reasonable;

—conclusions regarding the real values of unknown parameters based on considering both numerical and analytical solutions.

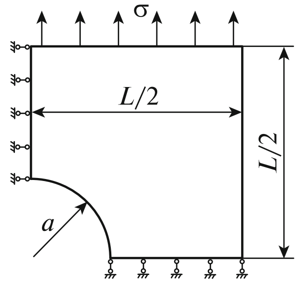

Fig. 1.

In the following two cases, this approach is especially reasonable:

(1) to calculate the structural elements of important technical objects if there are strict requirements for minimizing the safety factor;

(2) to investigate and design the computation models of unique structure elements.

This approach consists of obtaining analytical and numerical solutions of the same computation procedure; formulating a hypothesis based on its adequacy, i.e., correspondence to an accurate value; adding it via the system analysis of errors and inaccuracies; and to identifying the influence of factors on its values. A supposedly more adequate (analytical or numerical) solution is accepted as the basic solution for comparative analysis.

The following definitions of the problem are presented in this approach:

1. Estimate the inaccuracies of the numerical solution and reveal the factors in their decrease. This definition of the problem can be used under the assumption of the high adequacy of the analytical solution.

2. Estimate the inaccuracies of the analytical solution and reveal the factors in their decrease. This definition of the problem is used under the assumption of high adequacy of a numerical solution if the solution converges over the final element mesh.

3. Estimate the interval that contains an accurate solution according to the results of analytical and numerical solutions. This definition of the problem is used if there is no reason to accept one of the solutions (analytical or numerical) as the most adequate, or if both of solutions are characterized by a certain inaccuracy and we assume that the solution converges over the final element mesh.

Depending on the features of the computation procedure and the requirements for the results of investigations, it is possible to use one or several of the supposed definitions of the problem in different combinations. A feature of the first and second definitions of the problem is that, depending of the disagreement between analytical and numerical solutions, we accept that both solutions can be used or conclude that only the basic variant can be used. When solving the problem in third definition, we suppose to obtain in explicit form the quantitative estimations for truncation errors in order to consider them when determining the boundaries of the intervals. For this purpose, it is possible to use the procedure to estimate the error of the numerical solution of a set of linear algebraic equations with the stiffness matrix of the finite element model implemented by solving the auxiliary inverse problem on error analysis [1] according to the information obtained during calculations in the ASNSYS environment.

The way to estimate the truncation errors is as follows. It is known that the maximal number of truncation errors that forms the error of numerical solution by the finite element method takes place when we solve the resolving set of linear algebraic equations $Ax = b$, where A is the sparse matrix of coefficients (stiffness matrix), x is the unknown vector of nodal displacements, and b is the right part vector (vector of nodal forces). To estimate the error norm for the set of linear algebraic equations with stiffness matrix, two sets of linear algebraic equations are examined with the same matrix of coefficients A and with different right-hand sides, which are characterized by the same norm. At the first stage, vector b^* is formed so that its norm is equal to the norm of vector b ($\|b^*\| = \|b\|$) and set $Ax = b^*$ has exact solution x^*. Then, by numerically solving set $Ax = b^*$, we determine the vector of nodal displacements x_{num}. The error of nodal displacements can be written as follows:

$$\|x^* - x_{\text{num}}\|. \quad (1)$$

Let us examine how to solve problems according to each of the definitions.

1. In the computation procedure (Fig. 1), we examine a problem on stretching an infinite plate with circular cut (Kirsch problem). The problem has the exact analytical solution in the elasticity theory. The main aspects of the performed numerical analysis of this computation procedure [2] are as follows.

By considering the condition of symmetry, we examine one-fourth of the plate (Fig. 1). Since it is impossible to simulate an infinite plate, size L is set much higher (L = 100–1000 mm) than the cut radius (R = 10 mm), and it is examined as one of the factors that influence to the inaccuracy of the result.

The finite-element model is generated using PLANE82 elements from the ANSYS package with a switch-on option for the flat stressed state. We use the regular mesh of finite elements in the form of equilateral triangles with side h in the range of 0.1–5 mm. The bottom boundary is accepted by considering the dimensionality of the problem and the time needed to solve it; the top boundary is accepted from the condition that there is no visual distortion of the cut contour during the generation of the mesh.

According to the results of the simulation, we determine the calculated coefficient of stress concentration k_r. The disagreement between the calculated value and the exact value $k_t = 3$ at different ratios between step h and size L is in the range of 5.4–12%.

The obtained results can be explained by the following factors of errors formation under numerical solution.

If the size of the plate L is increased, the following occurs:

—the error and boundary effects decrease, i.e., the area of tensile forces superposition moves off the cut and the finite element model becomes more adequate to the computation procedure for the infinite plate, for which we obtain the analytical result;

—the dimensionality of the problem increases and, as a result, the accumulated truncation error rises.

If step h of finite elements mesh is increased, the following occurs:

—the discretization errors increase, i.e., the cut contour that consists of segments differs more from a circular arc;

—the dimensionality of the problem decreases and, as a result, the accumulated computation error drops.

Sometimes, the superposition of the aforementioned factors accelerates error accumulation, and sometimes errors compensate each other. Here, the advantages of the analytical solution are evident, the factors that form errors of the numerical solution are revealed, and their values are estimated. The analytical solution is boundedly acceptable at a high accuracy of the results.

2. The analytical solution for a cylindrical shell with a circular cut subjected to internal pressure is known [3, 4]. However, the variety of cut sizes and accepted assumptions and simplifications result in different definitions of scientific problems. For the corresponding small cuts, the basic and disturbed stressed states are investigated according to the geometric linear problem on the concentration of stresses, while for large cuts, the homogenous problem of shell theory in a bi-connected domain has been solved. In this case, additional fairly rigid constraints for the cut shape and boundary conditions along its contour are introduced in the problem. Based on the assumption that the cut contour is flat and under the involute of the shell, the circumference is related to the restrictions on the cut shape. At high sizes of the cut, these restrictions can introduce great error into the result.

Boundary conditions along the cut contour correspond to the assumption that there is a specially structured cup that only transmits pressure to the wall of the cut in the form of discontinuous force, which makes it possible to save the cylindrical form of the shell and secures the geometric linearity of the problem.

The aforementioned restrictions decrease the accuracy of the analytical solution and, as a result, "the exit solutions are devalued by errors contained in its formulation" [5]. Here, there is a reason to suppose a priori that the numerical solution is more accurate for the procedure of calculating a cylindrical shell with aperture under the conditions of investigation and mesh convergence.

To ensure the high accuracy of the result (we suppose that equations of 3D theory of elasticity are potentially more accurate with respect to equations of shell theory) and to take into consideration the features of the distribution of stresses over the thickness of the shell in the cut area, the numerical solution of this problem (Fig. 2) is obtained under 3D definition using the 10-node tetrahedral element of the second order SOLID187 with three translational degrees of freedom at each node [6]. A cut with radius R is varied at fixed values of L = 500 mm, a = 100 mm, and t = 1 mm. In the investigation of the numerical model, the discretization level ranges greatly. We consider the equivalent stresses reduced to the nodes of the mesh to be averaged (nodal approximation of the results) and not averaged (elemental approximation of the results) between the final elements of the neighbor. The recommended [7] agreement between nodal and elemental approximation is achieved. We see asymptotic behavior of the relationship between calculated

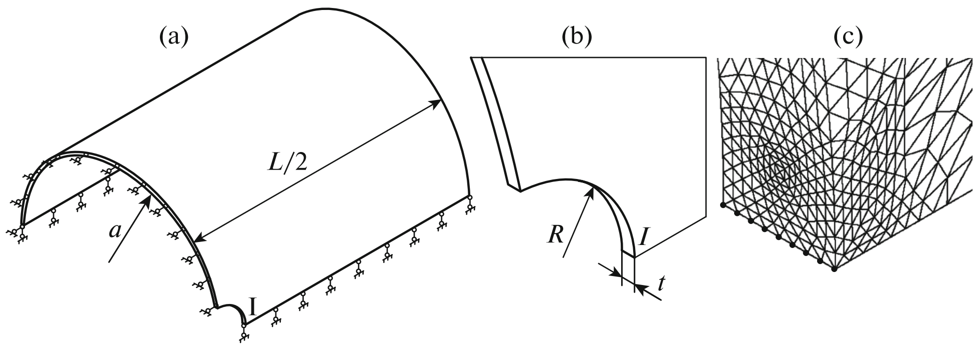

Fig. 2. Computation procedure for the model problem: (a) boundary conditions for one-fourth of the shell; (b) area of the circular cut; (c) fragment of the finite element mesh in area I of the greatest stresses.

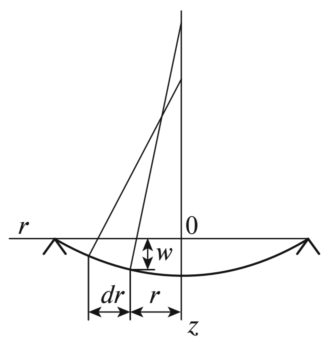

Fig. 3.

stresses and discretization step. As a result, we use a mesh that yields nine nodes over the thickness of the wall t of the shell aperture in the most stressed area (Fig. 2c). This secures the convergence of the mesh and the stability of the solution for all R/a, as well as sufficient accuracy to consider the components of bending stress along the aperture contour.

We analyze the results and reveal the following.

The geometric nonlinearity of the problem is revealed. At $R/a > 0.15$, the bending of the shell in the area of the cut is higher than the thickness of the shell and, according to [4], it is a sign of the geometric nonlinearity of the deformation.

We compare the results of numerical solution with Vandyke data [8], which makes it possible to estimate how the special cup for aperture influences the maximal stresses over its contour. For $0 \le R/a \le 0.45$, the disagreement between the results of the numerical solution and Vandyke data obtained under the assumption that the cup is 25%. Here, the solution of the problem with the free edge (without cup) based on considering the deviation of the shell's shape from cylindrical shows that the solution without considering the geometric nonlinearity of the problem causes the undervalue of maximal stresses.

Hereby, we verify the supposed advantages of the numerical solution. It has been shown that, since the analytical solution does not consider geometric nonlinearity and stress bending components in the cut area, the error of stress estimations is 25%. That is why it is impossible to use the analytical solution in high-accuracy problems.

3. One of the classical problems on deformed solid mechanics, which is widely used in engineering, is that on symmetrical bending of circular plate. The numerical procedure (Fig. 3) was generated by S.P. Timoshenko [9] based on the differential equation for the symmetrical bending of a circular plate as follows:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{rw}{dr}\right)\right]\right\} = \frac{q}{D},$$

where $D = \dfrac{Et^3}{12(1-\nu^2)}$ is the rigidity of the plate, E is the modulus of elasticity, q is the lateral load intensity, t is the thickness of the plate, ν is the Poisson ratio, and a is the radius of the plate.

When we generate the numerical procedure, we assume the following:

—The relationships obtained for pure bending between the bending moments and curvature of the bent surface are retained.

—We do not consider how tangent stresses that act in the normal cross sections of the plate normal to the meridians influence the bending.

—We do not consider the small difference between cutting forces over two opposite faces of the plate element cut by two cylindrical and two diameter cross sections.

—We do not consider several values of infinitesimal order.

As the integral performance of the deformed state, we examine the vertical bending in the plate center. If the plate is restrained over the contour, it is

$$u_{\max} = qa^4/64D. \quad (2)$$

According to the given reasons, the error in the analytical solution, i.e., analytical expressions for calculating approximate solutions according to simplified computation models, can be high. However, the disagreement between numerical solution obtained using 2D finite elements (plates and shells) and the exact solution can be also high. It follows from alternative models of deformed plates and shells generated by using Kirchhoff, Timoshenko, Mindlin, and the Reissner hypotheses. If these models are used to generate finite elements, in the general case, the results are in disagreement. Here, we cannot choose a priori which solution, i.e., numerical or analytical, is better.

Let us examine the analytical and numerical solutions for the examined computation procedure. Let us set the following values for the plate restrained over a contour: $E = 2.1 \times 10^5$ MPa, $\nu = 0.3$, $q = 1$ N/mm^2, $t = 1$ mm, $a = 10$ mm. We obtained an analytical solution according to (2), i.e., $u_{\max} = 8.125 \times 10^{-3}$ mm.

Numerical solutions are obtained in the ANSYS environment using triangle and tetragonal (three-node and four-node, respectively) SHELL181, triangle and tetragonal of the second order (six-node and eight-node, respectively) SHELL281 finite elements with six degrees of freedom at each node. We have performed several calculations by varying step h (length of the edge of the finite element) of regular mesh. In this case, we calculate the maximal bending of the design u_{num} and estimate the computation error $\|u^* - u_{\text{num}}\|$ according to (1), as well as make the bottom $u_{\text{num}} - \|u^* - u_{\text{num}}\|$ and top $u_{\text{num}} + \|u^* - u_{\text{num}}\|$ estimates for bending based on considering the error. In all cases, the mesh convergence is secured at $h = 0.05$ mm. The results are presented in the table.

We analyze the analytical and numerical solutions and reveal that, for all cases, the numerical estimates for bending u_{num} are higher than the analytical estimates; values u_{num} and the truncation error $\|u^* - u_{\text{num}}\|$ are varied in a small range depending on the finite element used. The precision of numerical estimations obtained at different basic functions for finite elements of different order, if we have convergence over mesh, small value of computation error makes it possible to conclude that the result of numerical solutions is preferable. In this case, among four numerical solutions, it is reasonable to choose the one that corresponds to the highest interval containing the unknown exact bending value. This solution is obtained using the tetragonal elements of the second order SHELL281. Here, the basis for this is that the exact banding value lies in the interval [8.171×10^{-3}; 8.821×10^{-3}] mm. The difference between the bottom boundary of this interval 8.171×10^{-3} and the analytical result 8.125×10^{-3} is 4.6×10^{-5} can be taken as the bottom estimate of the error for the analytical solution obtained according to (2).

Here, we present and practically test the definitions of the problem of the comparative analysis of analytical and numerical solutions, which makes it possible to estimate and improve the grounding and reliability of the calculation results according to the used calculation procedures, which raises the reliability and safety of the technical objects.

Calculated estimation, mm	SHELL181		SHELL281	
	triangle	tetragonal	triangle	tetragonal
u_{num}	8.667×10^{-3}	8.496×10^{-3}	8.496×10^{-3}	8.496×10^{-3}
$\|u^* - u_{num}\|$	2.087×10^{-5}	2.820×10^{-4}	1.610×10^{-4}	3.247×10^{-4}
$u_{num} - \|u^* - u_{num}\|$	8.646×10^{-3}	8.214×10^{-3}	8.335×10^{-3}	8.171×10^{-3}
$u_{num} + \|u^* - u_{num}\|$	8.688×10^{-3}	8.778×10^{-3}	8.657×10^{-3}	8.821×10^{-3}

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