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PROBLEMS OF ASTRONOMY

Dynamics of Expansion of the Universe in the Models with Nonminimally Coupled Dark Energy[1]

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Abstract—The dark energy model with barotropic equation of state, which interacts with dark matter by gravitation and by other force, which causes the energy-momentum exchange between them, is considered. Both components are described in approximation of ideal fluid, which are parameterized by density, equation of state and effective sound speed parameters. The three types of interactions between dark components are considered: interaction independent from their densities, interaction proportional to energy density of dark energy, and interaction proportional to energy density of dark matter. The equations that describe the expansion dynamics of homogeneous and isotropic Universe and evolution of densities of both components for different values of interaction parameter are obtained on the bases of the general covariant conservation equations and Einstein's ones. For three kinds of interactions, the existing of the range of values of parameters of dark energy for which the densities of dark components are negative was shown. The conditions of positivity of energy density of dark energy and dark matter were written for which the constraints on the value of parameter of interaction were derived. The dynamics of expansion of the Universe with these interactions of dark energy and dark matter is analyzed.

Keywords: nonminimally coupled dark energy, dark matter, expansion dynamics of the Universe

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INTRODUCTION

One of the most popular explanations of accelerated expansion of the Universe is that it is caused by some scalar field, for which the strong condition of energy domination is not satisfied: $\varepsilon_{de} + 3p_{de} < 0$. Different models of scalar field are considered: quintessence, phantom, quintom, tachyon, and other more exotic models of scalar field (see for example, [8, 5, 15] and citations in them). For description of its impact on expansion of the Universe or other components, the hydrodynamics approximation is used—the field represented by some ideal fluid with equation of state $p_{de} = w_{de}\rho_{de}$, where w_{de} is called the parameter of equation of state. In the simplest case, the quantity $w_{de} < -1/3$ is a constant, but more general models, where parameter of equation of state changes with time exist. Beside this, there are models in which scalar field nongravitationally interacts with other fields [2, 3, 5, 7, 9, 12, 13, 23, 24]. In the total Lagrangian of scalar field and fields with which it interacts, an additional term appears that has both field of dark energy and fields of other types of matter. In the general case, this Lagrangian can be written in the form: $L = L_\varphi(X, \varphi) + L_{int} + L_\psi(\psi_n, \partial_i\psi_n)$, where $X = -1/2g^{ik}\partial_i\varphi\partial_k\varphi$ is kinetic term of field, φ is the variable of scalar field, which is the dark energy, and ψ_n is the fields of other types of matter. In general, in which form to take L_{int} is unknown. Still, there are not any physical principle or experimental data from which it could be possible to derive Lagrangian of interaction of scalar field of dark energy with other fields. Now the attempts to build unified theory of field are ongoing, and the scalar field somehow must be coupled with other fields. On the possibility of such interaction and observational data that points to it, see works [1, 3, 4, 7, 10, 11, 13, 14, 19, 20, 23].

Below we analyze the impact of nongravitational interaction between dark energy and dark matter on dynamics of the expansion of the Universe and on evolution of densities of these components. As we do not know the form of Lagrangian of scalar field, we shall describe its behavior phenomenologically. Evolution of homogeneous and isotropic Universe and its components shall be described by system of Ein-

[1] The article was translated by the authors.

stein's equations and by equations that express the conservation laws of energy and momentum considering additional interaction between components. We shall assume that dark energy nongravitationally interacts only with dark matter, and with baryon matter or relativistic matter interacts only gravitationally. All components are described by ideal fluid approximation. As the basis, we took the model of dark energy [15–18] that had been studied as minimally coupled, which gives the possibility to discriminate the impact of interaction between dark components on dynamics of the expansion of the Universe and evolution of their densities.

1. EINSTEIN'S EQUATIONS AND CONSERVATION LAWS FOR NONMINIMALLY COUPLED COMPONENTS OF THE UNIVERSE

Such properties of the observational Universe have generally been accepted: homogeneity, isotropy, flatness of 3-space, and accelerated expansion at large scales. The metrics of 4-space of such a world are the Friedmann-Lemaitre-Robertson-Walker metrics (FLRW), which have such a form in comoving spherical coordinates in conformal representation:

$$ds^2 = g_{ik}dx^i dx^k = a^2(\eta)\left[d\eta^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{1}$$

where g_{ik} is the diagonal metric tensor with only one unknown function $a(\eta)$, which is a scale factor, which describes the expansion of the Universe. Here and further indices i, k,... take the values 0, 1, 2, 3, where x^0-component will always be a time variable. In the model of the Universe with a zero curvature of 3-space (flat space), it is convenient to normalize it by 1 at the present epoch: $a(\eta_0) = 1$. The first and second derivative of this factor on time describe the rate and acceleration of expansion of the Universe, i.e., the dynamics of such expansion. The variable η is a conformal time, which is related with physical cosmological time t by the simple differential relation: $cdt = a(\eta)d\eta$; c—the speed of light. The large set of independent observational data (see chapter 1 in book [15]) indicates that $da/dt > 0$ and $d^2a/dt^2 > 0$, or in conformal time $\dot{a}/a > 0$ and $\ddot{a}/a - (\dot{a}/a)^2 > 0$, where—here and further—the dot denotes the derivative $d/d\eta$. Traditionally in cosmology, the expansion rate of the Universe is described by relative quantity H, which is called the Hubble parameter, and the acceleration by the dimensionless quantity q, which is called the deceleration parameter:

$$H \equiv \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a^2}, \quad q \equiv -\frac{1}{aH^2}\frac{d^2a}{dt^2} = -\frac{\ddot{a}}{a^3H^2} + 1.$$

The Einstein's equations

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik}, \tag{2}$$

where R_{ik} is the covariant curvature Ricci tensor, R is the scalar of 4-space curvature, and T_{ik} is the total covariant energy-momentum tensor of the components of the Universe, give the equations for them. We specify the energy-momentum tensor in the following way: consider that the Universe is uniformly filled by nonrelativistic matter—baryon matter (b) and dark matter (dm); relativistic (r)—cosmic microwave background and relict neutrino; and also dark energy (de). Each of it will be described by energy-momentum tensor of ideal fluid,

$$T^k_{i(N)} = \left(c^2\rho_{(N)} + p_{(N)}\right)u_i u^k - p_{(N)}\delta^k_i, \tag{3}$$

where $\rho_{(N)}$ is the density of N-component, $p_{(N)}$ is its pressure, and u_i is 4-vector of velocity. Then the Einstein's equations give the equations for H and q

$$H^2 = \frac{8\pi G}{3}\sum_N \rho_{(N)}, \tag{4}$$

$$qH^2 = \frac{4\pi G}{3}\sum_N \left(\rho_{(N)} + 3p_{(N)}\right), \tag{5}$$

called the Friedmann's equations. Here and further we used the variables in which $c = 1$. For solving those equations, it is necessary to give the equations of state of each component and nongravitational interaction between them. Lets take the equation of state in the form $p_{(N)} = w_{(N)}\rho_{(N)}$, with the given known parameters of equation of state for nonrelativistic and relativistic components ($w_{dm} = w_b = 0$, $w_r = 1/3$) and unknown for dark energy $w_{de} = w(a)$. We shall find it for scalar-field dark energy from the condition that the relation ratio of variation of pressure to energy density is constant and specific form of nongravitational interaction with dark matter.

To find the dependencies of densities on time η (or on scale factor a), the equations (4)–(5) must be complemented by the conservation equations for each component. In general, if other kinds of interactions exist between components, except gravitational interaction, the equations of conservation of energy and momentum will be written in such a form:

$$T^{k(N)}_{i;k} = J^{(N)}_i, \quad \sum_n J^{(N)}_i = 0, \tag{6}$$

where $J^{(N)}_i$ is the 4-vector of change of density flow of energy-momentum to/from component N as the result of interaction with other components and ";" denotes the covariant derivative with respect to coordinate x^k. For minimally coupled components, $J^{(N)}_i = 0$. The second equation in (6) is the result of Bianchi identities. Further, we will consider that nongravitational interaction is only between dark energy and dark matter; thus, the nonzero are only $J^{(de)}_i$ for dark energy and $J^{(dm)}_i$ for dark matter. From the second equation in (6), it follows, that

$$J^{(de)}_i = -J^{(dm)}_i = J_i. \tag{7}$$

Then, the conservation laws for dark energy and dark matter will be written in such a form:

$$T^{k(de)}_{i:k} = J_i, \quad T^{k(dm)}_{i;k} = -J_i. \tag{8}$$

For baryon matter and relativistic matter, the conservation laws will be with zero flow of energy momentum, caused by interaction, because we shall analyze the dynamics of expansion of the Universe in the epoch after recombination, when their motion is free and interact only gravitationally:

$$T^{k(b)}_{i;k} = 0, \quad T^{k(r)}_{i;k} = 0. \tag{9}$$

The conservation equations (8) and (9) for homogeneous isotropic Universe contain only the equations of discontinuity, which, in the metrics (1), have the form

$$\dot{\rho}_{de} + 3\frac{\dot{a}}{a}\rho_{de}(1+w) = J_0, \tag{10}$$

$$\dot{\rho}_{dm} + 3\frac{\dot{a}}{a}\rho_{dm} = -J_0. \tag{11}$$

Whereas the Universe is homogeneous and isotropic and the perturbations are absent on large scales, the 4-vector of energy-momentum flow J_i has only one nonzero component J_0—the change of energy density per unit of time.

Analogical to (10) and (11), the equations for baryon and relativistic components with zero right parts give the well-known expressions: $\rho_b(a) = \rho^{(0)}_b a^{-3}$, $r(a) = \rho^{(0)}_r a^{-4}$, where up or down index (0) here and further will mean the quantity at the present epoch.

Now we have to consider the dependence of parameter of equation of state w on scale factor a. For this purpose, we shall introduce the new parameter—the adiabatic speed of sound $c_a^2 = \dot{p}_{de}/\dot{\rho}_{de}$. Then, using equation (10), we shall obtain such differential equation for $w(a)$:

$$\frac{dw}{da} = \frac{3}{a}(1+w)\left(w - c_a^2\right) - \frac{J_0}{\rho_{de}a^2H}\left(w - c_a^2\right). \tag{12}$$

In general, the quantity c_a^2 can depend on time, which must be given or obtained from known or given other physical properties of dark energy. In this work, we shall put it as constant: $c_a^2 = \mathrm{const} \leqslant 0$ [17]. The equation (12) must be solved together with equations (10) and (11); however, the J_0 must be given or derived from some considerations (see the reviews [5, 6, 8]). As we do not know anything about such interaction, it is natural to suggest that it is the function of energies of these two components:

$$J_0 = aHf(\rho_{de}, \rho_{dm}). \tag{13}$$

For small values of densities, it can be represented as

$$J_0 = -3aH(\alpha + \beta\rho_{de} + \gamma\rho_{dm}), \tag{14}$$

where α, β, γ are the constants that give the strength and sign of interaction. Consider that interaction between dark energy and dark matter must not depend on the expansion rate of the Universe and conformal time η apparently. Therefore, we took the expression for interaction J_0 in the form (13) and (14) with the multiplier aH present in it, which also remove the apparent dependence of interaction between dark components on the Hubble parameter H and the conformal time η, as such multiplier is in the right part of equations (10) and (11). This is seen well, if in these equations, we go from the differentiation on η to the differentiation on a:

$$\frac{\dot{a}}{a} \to aH, \quad (\dot{\ }) \equiv \frac{d}{d\eta} \to a^2 H \frac{d}{da}.$$

The change of dark components' energy densities is described in this case by an integral-differential equation, which can be solved numerically. In this article, we consider only some partial cases of such interaction, for which the analytical solutions exist:

$$\beta = 0,\ \gamma = 0: \ \ J_0 = -3\alpha \mathrm{a} \mathrm{H} \rho_{cr}, \tag{15}$$

$$\alpha = 0,\ \gamma = 0: \ \ J_0 = -3\beta \mathrm{a} \mathrm{H} \rho_{de}(a), \tag{16}$$

$$\alpha = 0,\ \beta = 0: \ \ J_0 = -3\gamma \mathrm{a} \mathrm{H} \rho_{dm}(a), \tag{17}$$

where $\rho_{cr} = 3H_0^2 / 8\pi G$ is the critical density at present epoch. The component's densities at the present epoch are convenient to represent in the critical units by the dimensionless parameter of density Ω_N : $\rho_N^{(0)} = \Omega_N \rho_{cr}$.

In literature, the other kinds of interactions are also considered, in particular, $J_0 = Q\dot{\varphi}\rho_{dm}$, where φ is the scalar field, which is the dark energy [2, 19, 21]. It can be rewritten in the form $J_0 = -3\gamma(a) aH\rho_{dm}$, where γ now depends on a. The models with such interaction and interaction (17) are considered in [3, 5].

Further, we shall consider the three cases of interactions (15), (16) and (17) between dark energy and dark matter (further the DE-DM interaction), which are reduced to analytical solutions of conservation equations (10) and (11) for energy densities DE and DM and the equation (12) for equation of state parameter of dark energy.

Henceforth, we shall call the dark energy as quintessence if its density decreases in the process of expansion of the Universe and the dark energy whose density increases in the process of expansion of Universe as phantom.

2. DE-DM INTERACTION, INDEPENDENT FROM THE DENSITIES OF DARK COMPONENTS

Consider the interaction (15), which does not depend on the densities of dark components. In this case, from equations (10) and (12), we obtain for arbitrary J_0

$$\rho_{de} = \rho_{de}^{(0)} \frac{w_0 - c_a^2}{w - c_a^2}, \tag{18}$$

and we obtained such equation for w as a result:

$$\frac{dw}{da} = \frac{3}{a}\left(w - c_a^2\right)\left(1 + w + \alpha \frac{\rho_{cr}}{\rho_{de}^{(0)}} \frac{w - c_a^2}{w_0 - c_a^2}\right).. \tag{19}$$

This is Riccati's equation, which has the partial solution: $w = c_a^2$. With its help, we find the general solution:

$$w(a) = \frac{\left(1 + c_a^2\right)\left[(1 + w_0)\Omega_{de} + \alpha\left(1 - a^{3\left(1+c_a^2\right)}\right)\right]}{(1 + w_0)\Omega_{de} - \left(w_0 - c_a^2\right)\Omega_{de} a^{3\left(1+c_a^2\right)} + \alpha\left(1 - a^{3\left(1+c_a^2\right)}\right)} - 1. \tag{20}$$

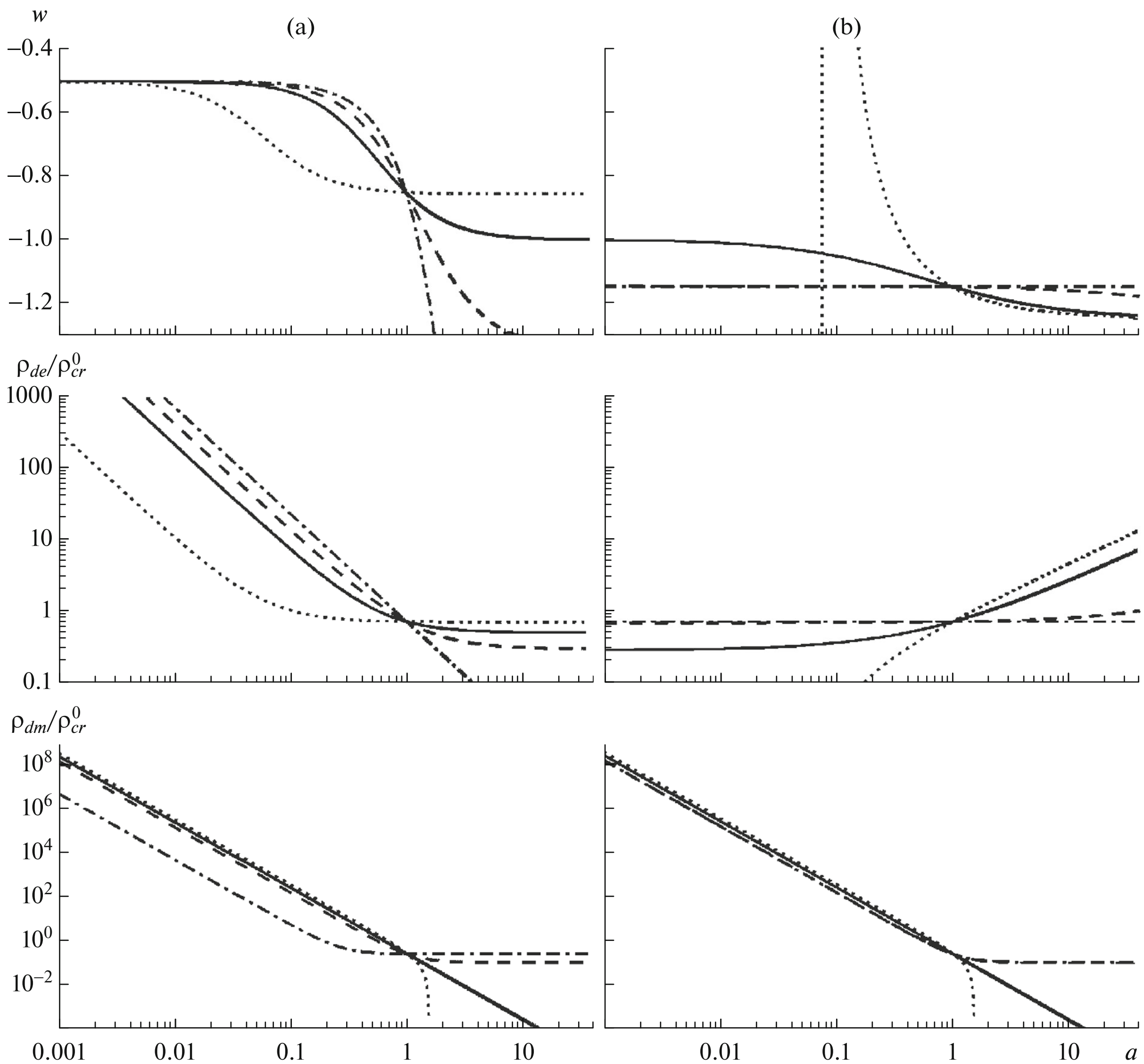

Fig. 1. Evolution of equation of state parameter of dark energy w, the density of dark energy ρ_{de}, and the energy density of dark matter ρ_{dm} for the different values of interaction parameter α (dotted line is for $\alpha = -0.1$, solid line is for $\alpha = 0$, dashed line is for $\alpha = 0.1$, dash-dotted line is for $\alpha = \alpha_u$): (a) quintessence dark energy ($w_0 = -0.85$, $c_a^2 = -0.5$), (b) phantom ($w_0 = -1.15$, $c_a^2 = -1.25$). Here, ρ_{cr}^0 is critical density at present epoch ($a = 1$), $\Omega_{de} = 0.7$, $\Omega_{dm} = 0.25$.

It is seen that the character of evolution of state parameter depends on the values of all parameters of dark energy and parameter of interaction α. Moreover, if $\alpha \to 0$, then w goes to $w^{(mc)}$—the equation of state parameter of minimally coupled dark energy [17]:

$$w^{(mc)}(a) = \frac{\left(1 + c_a^2\right)\left(1 + w_0\right)}{1 + w_0 - \left(w_0 - c_a^2\right)a^{3(1+c_a^2)}} - 1.$$

As is seen, the introduction already of the simplest interaction has changed the character of evolution of state parameter significantly, which is seen from the comparison of Fig. 1 with Fig. 1 in [17] and Fig. 1 in [18]. In particular, the quintessence dark energy can take such properties in the future, in which $w < -1$, while remaining the quintessence, namely such that density of which decreases in the process of expansion of the Universe, while vice versa for phantom. Considering (18) and (20), the conservation equations (10) and (11) have exact analytical solutions:

$$\rho_{de}(a) = \rho_{de}^{(mc)} - \alpha\rho_{cr}\frac{1 - a^{-3\left(1+c_a^2\right)}}{1 + c_a^2}, \tag{21}$$

$$\rho_{dm}(a) = \rho_{dm}^{(0)}a^{-3} - \alpha\rho_{cr}\left(1 - a^{-3}\right), \tag{22}$$

where $\rho_{de}^{(mc)}$ is the known solution for minimally coupled dark energy ($\alpha = 0$)

$$\rho_{de}^{(mc)}(a) = \rho_{de}^{(0)}\frac{(1 + w_0)a^{-3\left(1+c_a^2\right)} - w_0 + c_a^2}{1 + c_a^2}, \tag{23}$$

which is the smooth function of a for the arbitrary parameters w_0 and c_a^2 [17]. The solutions (21) and (22) are also the smooth functions for $0 < a < \infty$ and arbitrary parameters of dark energy. They allow the negative values of energy density of dark components for certain values of parameters of dark energy and the parameter of interaction. The condition of positivity of energy density of dark matter for all a is the condition that $0 \leqslant \alpha \leqslant \Omega_{dm}$. The condition of positivity of energy density of quintessence dark energy for all a is the condition $w_0 < c_a^2$, $\alpha \leqslant \Omega_{de}(c_a^2 - w_0)$, and that of phantom is $w_0 < -1$, $\Omega_{de}(c_a^2 - w_0) \leqslant \alpha \leqslant -\Omega_{de}(1 + w_0)$. Thereby, the interaction (15) provides the positive values of energy density of dark components only in the range of values of parameter of interaction

$$0 \leqslant \alpha \leqslant \min(\Omega_{dm}, \Omega_{de}(c_a^2 - w_0)),\ w_0 < c_a^2, \tag{24}$$

$$\max\left(0, \Omega_{de}\left(c_a^2 - w_0\right)\right) \le \alpha \le \min\left(\Omega_{dm}, -\Omega_{de}(1 + w_0)\right), \quad w_0 < -1, \tag{25}$$

in the models with quintessence and phantom dark energy accordingly. It is interesting that, in the case of quintessence dark energy, the expansion of the Universe goes to exponential and the constant energy densities of both dark components, while, in the case of phantom, it goes to the singularity of Big Rip at constant values of energy density of dark matter.

Let us analyze the behavior of the quantities w and ρ_{de} with the expansion of the Universe considering the conditions (24) and (25). For the quintessence dark energy ($1 + c_a^2 > 0$), when $a \to 0$, we have $w \to c_a^2$, $\rho_{de} \to \infty$, and when $a \to \infty$, we have

$$w \to \frac{\alpha c_a^2 - \left(w_0 - c_a^2\right)\Omega_{de}}{\alpha + \left(w_0 - c_a^2\right)\Omega_{de}},$$

$$\rho_{de} \to \frac{\left(c_a^2 - w_0\right)\Omega_{de} - \alpha}{1 + c_a^2}\rho_{cr}.$$

For the phantom dark energy ($1 + c_a^2 < 0$), when $a \to 0$, we have

$$w \to \frac{\alpha c_a^2 - \left(w_0 - c_a^2\right)\Omega_{de}}{\alpha + \left(w_0 - c_a^2\right)\Omega_{de}},$$

$$\rho_{de} \to \frac{\left(c_a^2 - w_0\right)\Omega_{de} - \alpha}{1 + c_a^2}\rho_{cr}.$$

and when $a \to \infty$ $w \to c_a^2$, $\rho_{de} \to \infty$. In Fig. 1, the dependencies of quantities w, ρ_{de}, and ρ_{dm} on a in the range of its values $[10^{-3}, 40]$ are given. We see that, with values of parameter α at the upper bound for the model with quintessence dark energy ($\alpha_u = \Omega_{de}(c_a^2 - w_0)$) when $a \to \infty$, we have $w \to -\infty$, $\rho_{de} \to 0$. For the dark energy, with the parameters of phantom, for the upper bound ($\alpha_u = -\Omega_{de}(1 + w_0)$) when $a \to \infty$, we have $w = \text{const}$, $\rho_{de} = \text{const}$; namely, the dark energy in this case is similar to the cosmological constant. We also see that, at negative α, the energy density of dark matter ρ_{dm} in the process of expansion of the Universe at first is positive and then becomes negative.

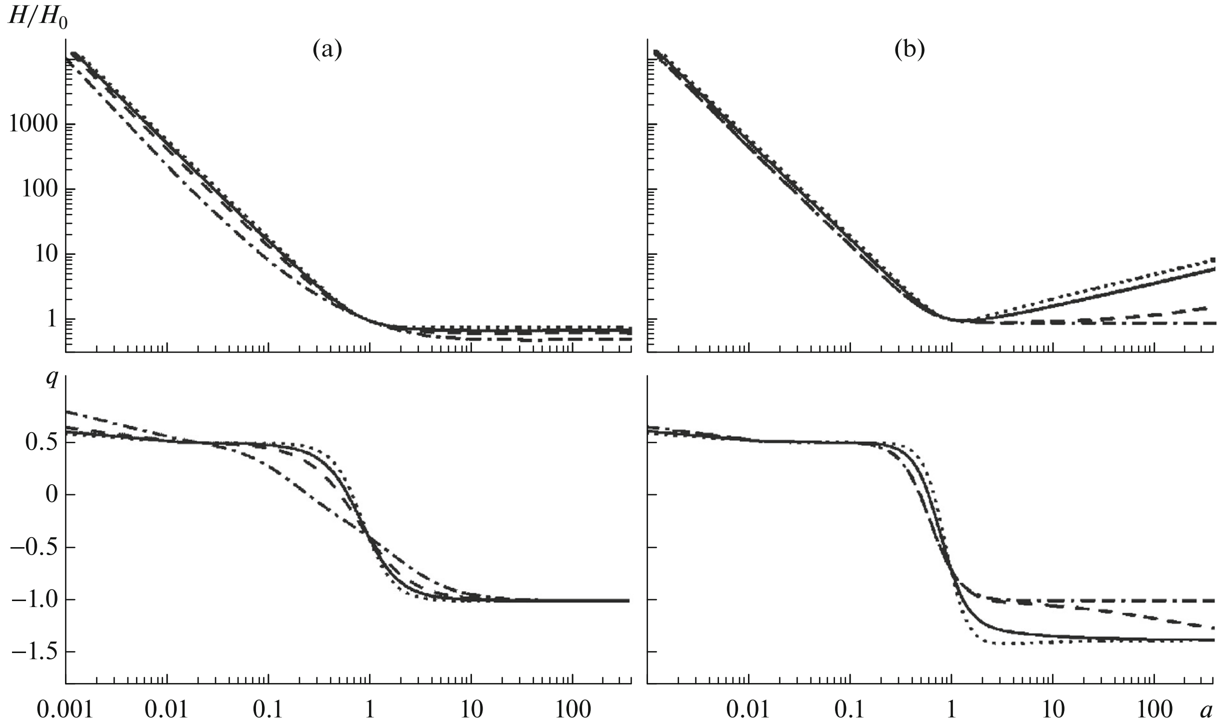

Fig. 2. Evolution of Hubble parameter H and deceleration parameter q at different values of interaction parameter α (dotted line is for $\alpha = -0.1$, solid line is for $\alpha = 0$, dashed line is for $\alpha = 0.1$, dash-dotted line is for $\alpha = \alpha_u$): (a) model with quintessence dark energy, (b) with the phantom. The parameters of models are the same as in Fig. 1.

Consider now the impact of interaction (15) on the dynamics of expansion of the Universe. For this, we substitute into the Friedmann's equations (4) and (5) the found expressions for energy densities of dark energy (21) and dark matter (22), and also the expressions for the energy density of baryon matter $\rho_b = \rho_b^{(0)}a^{-3}$ and relativistic matter $\rho_r = \rho_r^{(0)}a^{-4}$. Calculate the quantities H/H_0 and q for the standard cosmological model with $\Omega_{de} = 0.7$, $\Omega_{dm} = 0.25$, $\Omega_b = 0.05$, $\Omega_r = 4.17 \times 10^{-5}/h^2$, and $h = H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$. The results are shown in Fig. 2. The dependencies of quantities H/H_0, q on a are given for the diapason of its values $[10^{-3}, 400]$. Here, at $\alpha = -0.1$, 0.0, +0.1 for the quintessence dark energy, when $a \to 0$, then $H/H_0 \to \infty$, $q \to 1$, and when $a \to \infty$, then

$$H/H_0 \to \left(\frac{\left(c_a^2 - w_0\right)\Omega_{de} + \alpha c_a^2}{1 + c_a^2}\right)^{1/2}, \quad q \to -1.$$

For the phantom dark energy, when $a \to 0$, then $H/H_0 \to \infty$, $q \to 1$; when $a \to \infty$, then $H/H_0 \to \infty$, $q \to 1/2 + (3/2)c_a^2$. For the upper bounds of interaction parameter α_u for the quintessence dark energy, we have $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to (\Omega_{de}(c_a^2 - w_0))^{1/2}$, $q \to -1$ when $a \to \infty$. For the dark energy with the upper bound α_u with the parameters of phantom, we have the dark energy with constant w and ρ_{de}. For it $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to (-w_0\Omega_{de})^{1/2}$, $q \to -1$ when $a \to \infty$.

3. DE-DM INTERACTION, PROPORTIONAL TO THE DENSITY OF DARK ENERGY

Let the expression for density of energy flow J_0 be such as in (16). Then, substituting it in the equation (12) for w, we shall obtain

$$\frac{dw}{da} = \frac{3}{a}(1 + w)\left(w - c_a^2\right) + \frac{3\beta}{a}\left(w - c_a^2\right). \tag{26}$$

This equation has the exact analytical solution

$$w_{de}(a) = \frac{\left(1 + c_a^2 + \beta\right)(1 + w_0 + \beta)}{1 + w_0 + \beta - \left(w_0 - c_a^2\right)a^{3\left(1 + c_a^2 + \beta\right)}} - 1 - \beta, \quad (27)$$

where $w_0 \equiv w(1)$ is the initial condition. Now, substituting the expression (27) for w in expression (18) for energy density ρ_{de}, we obtain the exact analytical solution of equation (10) for $\rho_{de}(a)$:

$$\rho_{de}(a) = \rho_{de}^{(0)} \frac{(1 + w_0 + \beta)a^{-3\left(1 + c_a^2 + \beta\right)} - w_0 + c_a^2}{1 + c_a^2 + \beta}. \quad (28)$$

As is seen, the dependencies $w(a)$ and $\rho_{de}(a)$ have three parameters w_0, c_a^2, β, which give the general properties and type of dark energy. We see on the behavior of w and ρ_{de}, when $a \to 0$ i $a \to \infty$. If $1 + c_a^2 + \beta > 0$, then when $a \to 0$ we have $w \to c_a^2$, $\rho_{de} \to \infty$; when $a \to \infty$, we have $w \to -1 - \beta$, $\rho_{de} \to \rho_{de}^{(0)}\left(c_a^2 - w_0\right)/\left(1 + c_a^2 + \beta\right)$. If $1 + c_a^2 + \beta < 0$, then when $a \to 0$ we have $w \to -1 - \beta$, $\rho_{de} \to \rho_{de}^{(0)}\left(c_a^2 - w_0\right)/\left(1 + c_a^2 + \beta\right)$; when $a \to \infty$, we have $w \to c_a^2$, $\rho_{de} \to \infty$. At the values of parameters w_0, c_a^2 and β, for which the inequalities $w_0 > c_a^2$ and $\beta < -(1 + w_0)$ or $w_0 < c_a^2$ and $\beta > -(1 + w_0)$ are valid, the dependence of the equation of state parameter of dark energy w on a has the singularity of the second kind. From the definition of w it is clear that in that case the always smooth dependence $\rho_{de}(a)$ changes, the sign in the process of expansion of the Universe, namely $\rho_{de}(a_{\rho=0}) = 0$ at the point

$$a_{\rho=0} = \left(\frac{1 + w_0 + \beta}{w_0 - c_a^2}\right)^{1/3\left(1 + c_a^2 + \beta\right)}.$$

Namely, the solution (28) also allows $\rho_{de} < 0$. We consider that the energy densities of the dark components of the Universe can be only positive as during its whole past, at the present epoch, and always in the future.

For the equation of state parameter and the density of dark energy, we see here the new behavior, different from the case $\beta = 0$, studied in detail in the works [17, 18]. The phantom divide line is shifted on the quantity of interaction parameter β and becomes equal to $w_{phd} = -1 - \beta$, if the quintessence dark energy is defined as that whose density decreases ($1 + c_a^2 + \beta > 0$), and the phantom dark energy as that whose density increases ($1 + c_a^2 + \beta < 0$) in the process of expansion of the Universe.

The condition that $\rho_{de} \geqslant 0$ at arbitrary $a \geqslant 0$, for quintessence dark energy is such:

$$c_a^2 \geq w_0, \quad \beta > -1 - w_0, \quad (29)$$

and that for phantom dark energy

$$c_a^2 \leq w_0, \quad \beta > -1 - w_0, \quad (30)$$

Now, we shall find the dependence of energy density of dark matter ρ_{dm} on a. Substituting the expression (28) in the equation (11), we shall obtain

$$\frac{d\rho_{dm}}{da} + \frac{3}{a}\rho_{dm} = \frac{3}{a}\beta\rho_{de}^{(0)}\left(Aa^{-3\left(1 + c_a^2 + \beta\right)} - B\right), \quad (31)$$

where

$$A = \frac{1 + w_0 + \beta}{1 + c_a^2 + \beta}, \quad B = \frac{w_0 - c_a^2}{1 + c_a^2 + \beta}. \quad (32)$$

Note that the right part of equation (31) is the regular function for an arbitrary finite value $0 < a < \infty$ and arbitrary values of parameters w_0, c_a^2, and β. In the case of $1 + c_a^2 + \beta = 0$, when $\rho_{de} = \rho_{de}^{(0)} = \text{const}$ and $w = w_0 = \text{const}$, the solution of (31) is

$$\rho_{dm}(a) = \left(\rho_{dm}^{(0)} - \beta\rho_{de}^{(0)}\right)a^{-3} + \beta\rho_{de}^{(0)}. \quad (33)$$

The condition that $\rho_{dm} \geqslant 0$ for any $0 < a < \infty$ is the restriction of the range of values of the interaction parameter:

$$0 \le \beta \le \frac{\Omega_{dm}}{\Omega_{de}}. \tag{34}$$

Thus, in this particular case, $\rho_{de} = \text{const}$, the interaction between the dark matter and dark energy of the form (16) is such that the energy flows from the dark energy to dark matter. The flow rate decreases in the process of expansion of the Universe and goes to an asymptotic regime $J_0 \propto -\beta a H_0 \rho_{de}^{(0)}$, such that $\rho_{dm} \to \beta\rho_{de}^{(0)}$ when $a \to \infty$.

The general solution of equation (31) is

$$\rho_{dm}(a) = \rho_{dm}^{(0)} a^{-3} + \beta\rho_{de}^{(0)} \times \left[\left(\frac{A}{c_a^2+\beta} + B\right)a^{-3} - \frac{A}{c_a^2+\beta} a^{-3\left(1+c_a^2+\beta\right)} - B\right]. \tag{35}$$

It is easily seen that the solution is regular at all of interval $0 < a < \infty$ for arbitrary values of parameters w_0, c_a^2, and β. If $w_0 = c_a^2$, then $w = \text{const}$, and the expressions for energy densities of dark components (28) and (35) coincide with the corresponding expressions in the work [1]. In the particular case, $c_a^2 + \beta = 0$, which is the case for quintessence dark energy,

$$\rho_{de}(a) = (1 + w_0 + \beta)\rho_{de}^{(0)} a^{-3} - (w_0 + \beta)\rho_{de}^{(0)},$$

the expression (35) is simplified:

$$\rho_{dm}(a) = \left(\rho_{dm}^{(0)} + \beta(\beta + w_0)\rho_{de}^{(0)}\right)a^{-3} - \beta(\beta + w_0)\rho_{de}^{(0)}. \tag{36}$$

It is easy to see that, in this case, the condition $\rho_{dm} \geqslant 0$ for an arbitrary $0 < a < \infty$, together with the condition $\rho_{de} \geqslant 0$ (29), is satisfied at

$$\max(0, -1 - w_0) \le \beta \le -w_0. \tag{37}$$

and the realistic values of equation of state parameter $w_0^2 < 4\Omega_{dm}/\Omega_{de}$ [22].

Let us find the range of values β at which $\rho_{dm} \geqslant 0$ for an arbitrary $0 < a < \infty$ and arbitrary parameters of dark energy. In the case of quintessence dark energy with $\rho_{de} \geqslant 0$, the value of energy density of dark matter ρ_{dm} always will be positive with the condition

$$\max(0, -1 - w_0) \le \beta \le -\frac{c_a^2\Omega_{dm}/\Omega_{de}}{1 + w_0 - c_a^2 + \Omega_{dm}/\Omega_{de}}, \quad \beta \ne -1 - w_0, \tag{38}$$

while, in the case of phantom, that will be with condition

$$0 \le \beta \le \min\left(-1 - w_0, -\frac{c_a^2\Omega_{dm}/\Omega_{de}}{1 + w_0 - c_a^2 + \Omega_{dm}/\Omega_{de}}\right), \quad \beta \ne -1 - w_0, \tag{39}$$

Here, the conditions of positivity of energy density of dark energy (29) and (30) are taken into account. The upper bound of values of β from the inequalities (38) and (39), we shall further denote β_u.

On Fig. 3, the dependencies $w(a)$, $\rho_{de}(a)$, and $\rho_{dm}(a)$ are represented for the model of the Universe with quintessence (left) and phantom (right) dark energy and the three values of interaction parameter $\beta = -0.1$, 0.0, +0.1, and its upper value $\beta = \beta_u$. The parameters are chosen the same as in Fig. 1. The impact of value of interaction parameter β on the evolution of equation of state parameter is equivalent to the shift of phantom divide line ($w = -1$) on the quantity β. In the case of $\beta > 0$, when the energy flows from dark energy to dark matter, the energy density of quintessence dark energy decreases faster, and that of phantom increases slower than in the case without interaction ($\beta = 0$). In the case of quintessence dark energy, the energy density of dark matter goes to the constant value, while that in the case of phantom slowly goes to infinity. We also see, that for the upper value of interaction parameter $\beta_u = -1 - w_0$ for the dark energy with parameters of phantom, we have the case that we have considered earlier, in which the quantities w and ρ_{de} are constant. In the case of $\beta < 0$, when the energy flows from dark matter to dark energy, the energy density of quintessence dark energy decreases slower and that of phantom increases faster than in the case without interaction. The energy density of dark matter in this case decreases quickly, going to zero

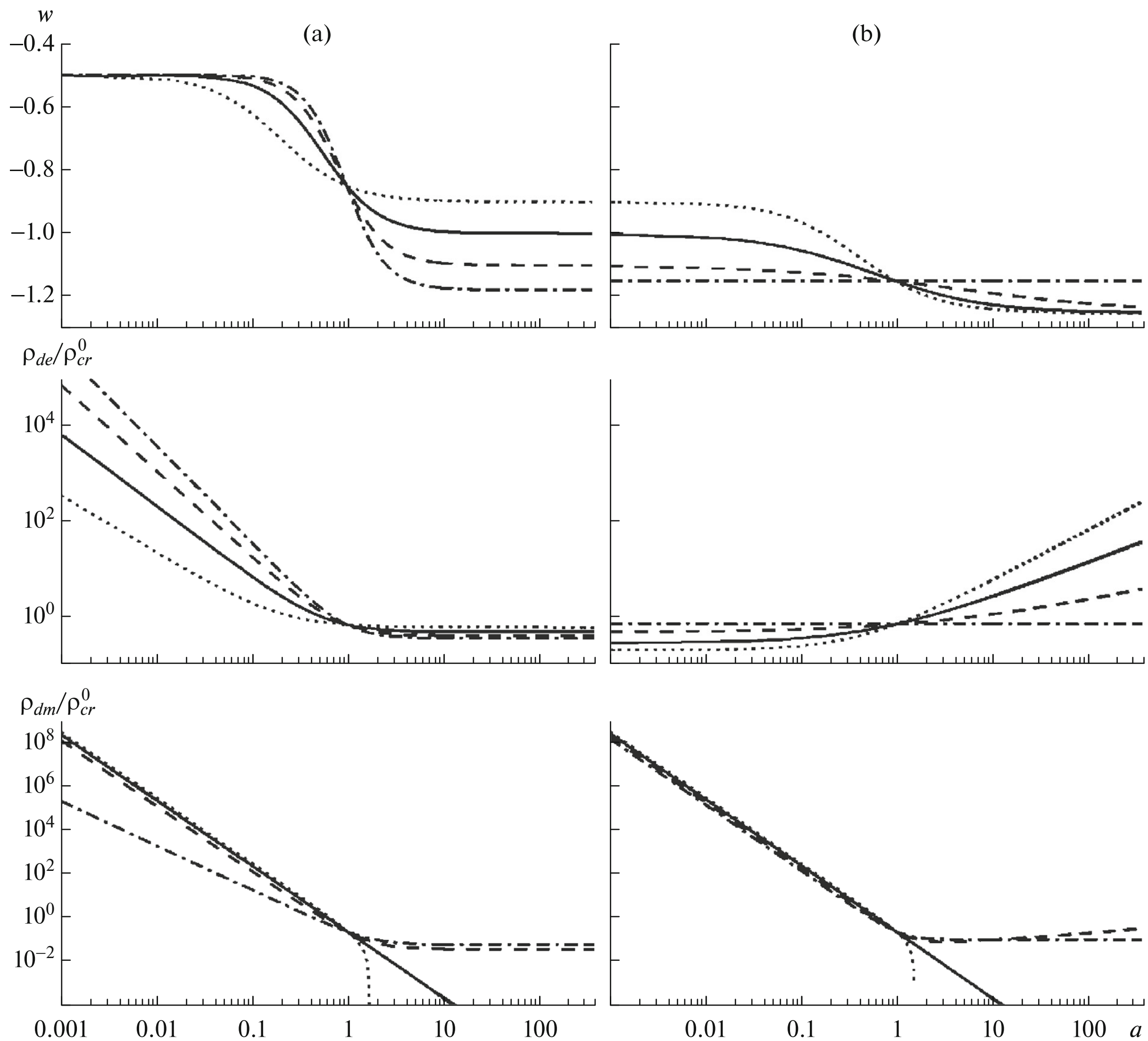

Fig. 3. Evolution of equation of state parameter of dark energy w, the density of dark energy ρ_{de}, and the energy density of dark matter ρ_{dm} at the different values of interaction parameter β (dotted line is for $\beta = -0.1$, solid line is for $\beta = 0$, dashed line is for $\beta = 0.1$, dash-dotted line is for $\beta = \beta_u$): (a) quintessence dark energy ($w_0 = -0.85$, $c_a^2 = -0.5$), (b) phantom ($w_0 = -1.15$, $c_a^2 = -1.25$). Here ρ_{cr}^0 is the critical density at present epoch ($a = 1$), $\Omega_{de} = 0.7$, $\Omega_{dm} = 0.25$.

and to the negative values in the future, which we consider nonphysical. This means that the interaction of type (16) can take place only when $\beta > 0$.

Consider now the impact of interaction (16) on the dynamics of expansion of the Universe. Substituting the expressions for energy density of all components in Friedmann's equations (4) and (5), we shall obtain the dependence of quantities H/H_0 and q on a. The results are represented in Fig. 4 for the same parameters as in Fig. 2: left is the model with quintessence dark energy and right is that with phantom. On the all panels of both figures are also the curves that correspond to the upper bounds of values β from inequalities (38) and (39). Here we see that, for the quintessence dark energy ($1 + c_a^2 + \beta > 0$) $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to -\Omega_{de}B(1 + \beta)$, $q \to -1$ when $a \to \infty$. For the phantom $1 + c_a^2 + \beta < 0$) $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to \infty$, $q \to 1/2 + 3/2(c_a^2 + \beta)$ when $a \to \infty$. For the upper bound of interaction parameter ($\beta_u = -1 - w_0$), for the dark energy with constant w and ρ_{de} with parameters of

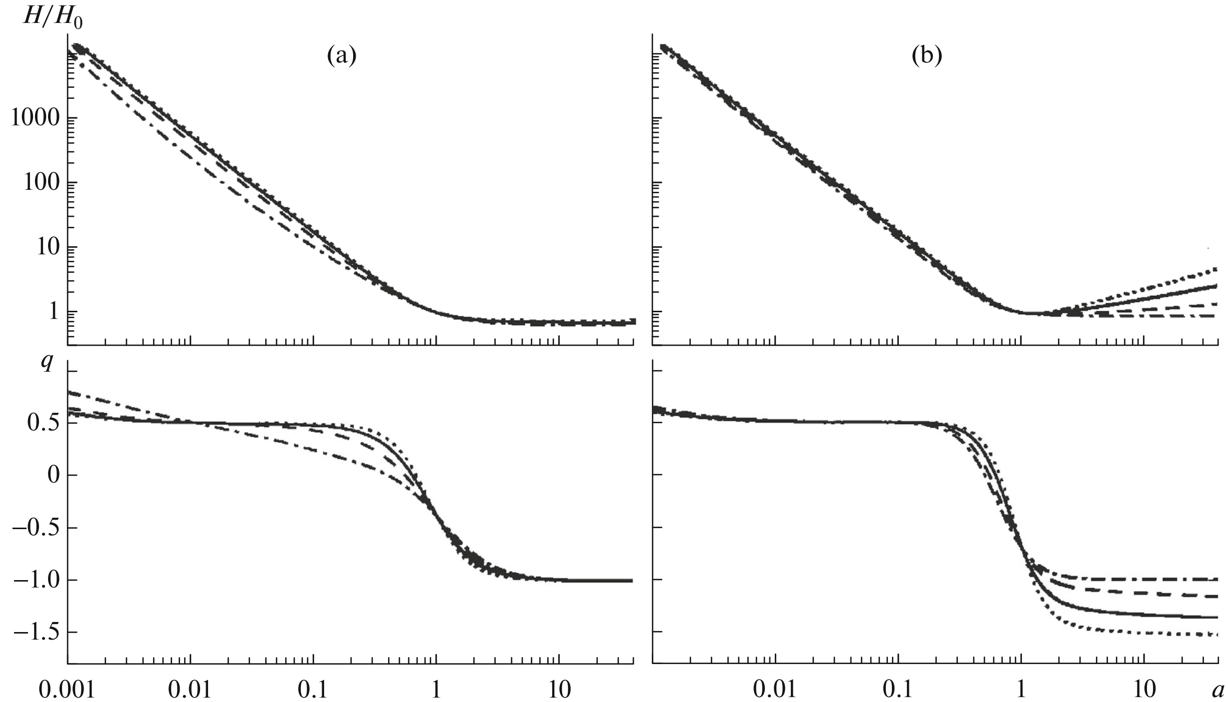

Fig. 4. Evolution of Hubble parameter H and deceleration parameter q at the different values of interaction parameter β (dotted line is for $\beta = -0.1$, solid line is for $\beta = 0$, dashed line is for $\beta = 0.1$, dash-dotted line is for $\beta = \beta_u$): (a) quintessence dark energy, (b) phantom. The parameters of models are the same as in Fig. 1.

phantom $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to (-w_0\Omega_{de})^{1/2}$, $q \to -1$ when $a \to \infty$. As we see, such interaction with the values considered values of parameter β has impact on the dynamics of expansion of the Universe in the past and present epochs (in the figure, $a \leqslant 1$), but such impact is not large. This means, that for the setting of observational constraints of quantity β the high precision data is needed.

4. DE-DM INTERACTION PROPORTIONAL TO THE DENSITY OF DARK MATTER

Now, let the expression for energy density flow J_0 be such as in (17). Substituting it in the equation (11) for ρ_{dm}, we shall obtain

$$\frac{d\rho_{dm}}{da} + \frac{3}{a}\rho_{dm} = \frac{3\gamma}{a}\rho_{dm}. \tag{40}$$

The general solution of this equation is the expression

$$\rho_{dm}(a) = \rho_{dm}^{(0)} a^{-3(1-\gamma)}. \tag{41}$$

Thus, the energy density of dark matter is always the smooth function, which takes only the positive values for an arbitrary $0 < a < \infty$ and γ. In the case of $\gamma > 0$ (the flow of energy from dark energy to dark matter), the energy density of dark matter decreases slower for $\gamma < 1$, than in the case of noninteracting components. Obviously, the observed large-scale structure of the Universe and the anisotropy of cosmic microwave background indicates that $|\gamma| \ll 1$. Using the formula (18), we obtain such ordinary differential equation for w:

$$\frac{dw}{da} = \frac{3}{a}\left(w - c_a^2\right)\left(1 + w + \gamma\frac{\Omega_{dm}a^{-3(1-\gamma)}}{\Omega_{de}}\frac{w - c_a^2}{w_0 - c_a^2}\right). \tag{42}$$

As in the previous cases, we have the Riccati's equation with partial solution $w = c_a^2$, by which can be easily found the general solution:

$$w(a) = \frac{\left[1 + w_0 + \gamma\left(1 + c_a^2\right)\frac{\Omega_{dm}}{\Omega_{de}}\frac{1 - a^{3\left(c_a^2+\gamma\right)}}{c_a^2 + \gamma}\right]\left(1 + c_a^2\right)}{1 + w_0 + \gamma\left(1 + c_a^2\right)\frac{\Omega_{dm}}{\Omega_{de}}\frac{1 - a^{3\left(c_a^2+\gamma\right)}}{c_a^2 + \gamma} - \left(w_0 - c_a^2\right)a^{3\left(1+c_a^2\right)}} - 1. \quad (43)$$

Substitute it in the expression for ρ_{de} (18) and obtain

$$\rho_{de}(a) = \rho_{de}^{(0)}\left[\frac{(1 + w_0)a^{-3\left(1+c_a^2\right)} + c_a^2 - w_0}{1 + c_a^2} + \gamma\frac{\Omega_{dm}}{\Omega_{de}}\frac{1 - a^{3\left(c_a^2+\gamma\right)}}{c_a^2 + \gamma}a^{-3\left(1+c_a^2\right)}\right]. \quad (44)$$

First of all, let us notice that the obtained solutions for the energy density of dark matter $\rho_{dm}(a)$ and dark energy $\rho_{de}(a)$ are the regular functions at all diapason of values of scale factor $0 < a < \infty$ for an arbitrary values of parameters of dark energy Ω_{de}, w_0, c_a^2, and interaction parameter γ. Indeed, the expression (44) is finite as when $1 + c_a^2 \to 0$, and when $c_a^2 + \gamma \to 0$. In the particular case of constant w, when $w_0 = c_a^2$, the expression (44) coincides with the expression for the energy density of dark energy given in [3].

Let us analyze the behavior of $w(a)$ and $\rho_{de}(a)$ when $a \to 0$ and $a \to \infty$ considering the condition $|\gamma| \ll 1$. If $1 + c_a^2 > 0$, then, when $a \to 0$, we have $w \to c_a^2$, $\rho_{de} \to \infty$; when $a \to \infty$, we have $w \to -1$, $\rho_{de} \to -\rho_{de}^{(0)}\frac{w_0 - c_a^2}{1 + c_a^2}$, and, when $c_a^2 \geqslant w_0$, this asymptotic value of the density of dark energy is positive. If then $1 + c_a^2 < 0$, then, when $a \to 0$, we have $w \to c_a^2$, $\rho_{de} \to \infty$, and, when $a \to \infty$, we have $w \to c_a^2$, $\rho_{de} \to \infty$. Thus, as before, when $1 + c_a^2 > 0$, we have the quintessence dark energy, and, when $1 + c_a^2 < 0$, we have the phantom.

Let us write now the conditions of positivity for the density of dark energy ρ_{de} considering the condition $\gamma \ll 1$ for the realistic values of parameters w_0, c_a^2, Ω_{de}, Ω_{dm}. The energy density ρ_{de} is always positive only if $\gamma \geqslant 0$ and $c_a^2 + \gamma < 0$. For the quintessence dark energy considering condition $w_0 > -1$, the conditions are such:

$$w_0 \le c_a^2, \quad 0 \le \gamma \ll 1. \quad (45)$$

For the phantom dark energy, when $w_0 < -1$, the conditions are such:

$$w_0 \ge c_a^2, \quad 0 \le \gamma \ll 1, \quad (46)$$

$$w_0 < c_a^2, \quad 0 < \gamma \ll 1. \quad (47)$$

From the conditions (46) and (47), when $\gamma = 0$, the phantom dark energy can always be positive only if $w_0 \geqslant c_a^2$.

From formula (43), we see that such models exist, for which w may cross the line -1, when the quantity ρ_{de} is always positive. We do not give here the conditions of positivity of energy density of dark energy for them because they are cumbersome and the parameter values region, which corresponds to them, is narrow.

In Fig. 5, the behavior of w and ρ_{de} for quintessence dark energy and phantom is shown for the same values of parameters as in Fig. 1, with the values of interaction parameter $\gamma = -0.1, 0.0, +0.1, +0.2$. For the positive values of γ, the quantity ρ_{de} is always positive, and that for the quintessence dark energy is the monotonically decreasing function to the constant value when $a \to \infty$, and that for the phantom decreases at first to some minimal value and then again increases to $+\infty$ when $a \to \infty$. For the negative value $\gamma = -0.1$, both for quintessence dark energy and for phantom, ρ_{de} increases from $-\infty$ to 0 and higher; namely, it is negative at first and in the process of expansion becomes positive. By changing the sign of ρ_{de}, on the contrary, as we see from the figures, w has the singularity of second kind.

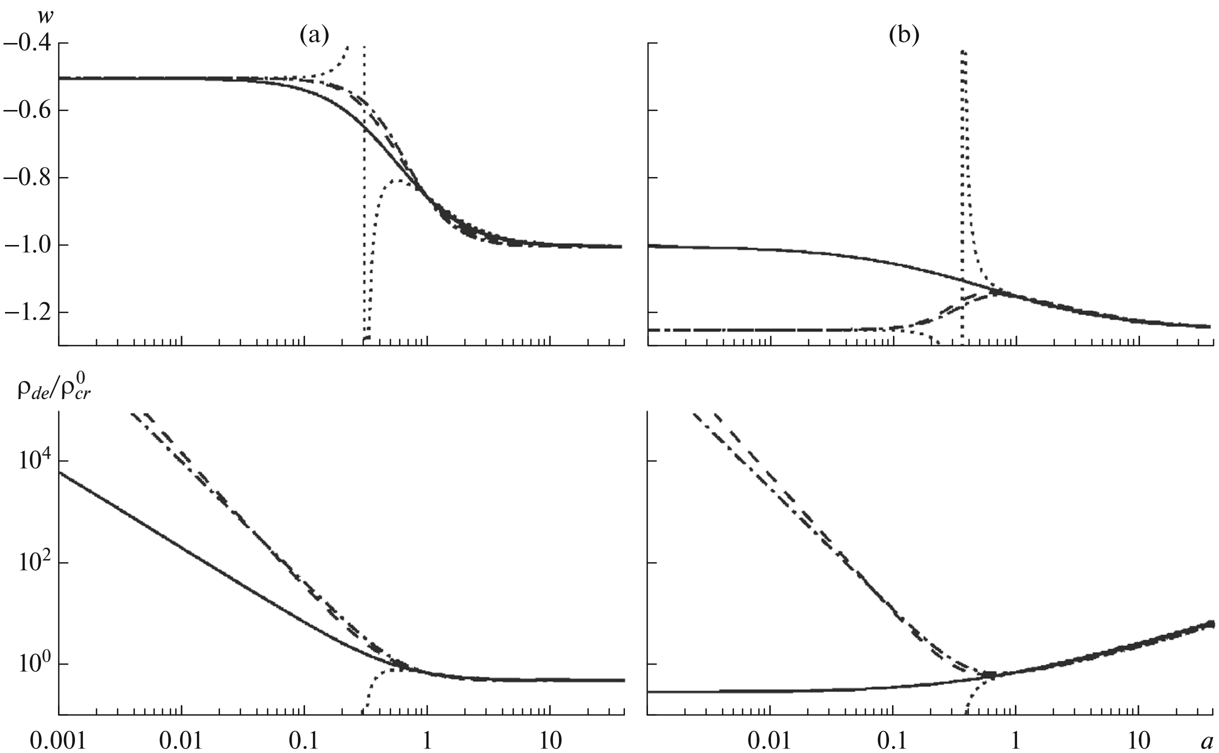

Fig. 5. Evolution of equation of state parameter of dark energy w, the density of dark energy ρ_{de} at the different values of interaction parameter γ (dotted line is for $\gamma = -0.1$, solid line is for $\gamma = 0$, dashed line is for $\gamma = 0.1$, dash-dotted line is for $\gamma = 0.2$): (a) quintessence dark energy, (b) phantom. The parameters are the same as in Fig. 1.

Now, having the expressions for energy densities of components, Friedmann's equations can be used for the analysis of impact of such model of interaction between the dark components on the dynamics of expansion of the Universe. We take the following expressions for energy densities of baryon and relativistic matters: $\rho_b = \rho_b^{(0)} a^{-3}$, $\rho_r = \rho_r^{(0)} a^{-4}$.

In Fig. 6, the evolution of quantities H/H_0 and q is shown for the same values of parameters as in Fig. 1. Considering the condition $|\gamma| \ll 1$ for the interaction parameters $\gamma = -0.1, 0.0, +0.1, +0.2$, we see that, for the quintessence dark energy, $H/H_0 \to \infty$, $q \to 1$ when $a \to 0$, and

$$H/H_0 \to \left(-\Omega_{de} \frac{w_0 - c_a^2}{1 + c_a^2} \right)^{1/2}, \quad q \to -1$$

when $a \to \infty$, and for the phantom—$H/H_0 \to \infty$, $q \to 1$ when $a \to 0$ and $H/H_0 \to \infty$, $q \to 1/2 + 3/2c_a^2$ when $a \to \infty$. Under the above conditions of positivity of ρ_{de}, the quantity H^2 is always positive. It is seen that this is true for the values of interaction parameter $\gamma = 0.0, +0.1, +0.2$. For $\gamma = -0.1$, despite the fact that ρ_{de} becomes negative, the total density of components always remains positive; therefore, the quantity H^2 is always positive, and q smoothly goes from +1 in the early Universe to −1 in the future in the case of quintessence dark energy or $(1 + 3c_a^2)/2$ in the case of phantom.

CONCLUSIONS

The dynamics of expansion of the Universe in the cosmological model with dynamical dark energy that interacts with dark matter gravitationally and nongravitationally was analyzed. The three types of interaction that cause the energy-momentum exchange between them were examined: independent on densities of components, proportional to the density of dark energy, and proportional to the density of dark matter. For all cases, we obtained the analytical dependencies $w(a)$, $\rho_{de}(a)$, and $\rho_{dm}(a)$, which are the exact solutions of the conservation equations of energy for the dark components. It was shown that, in all cases, the

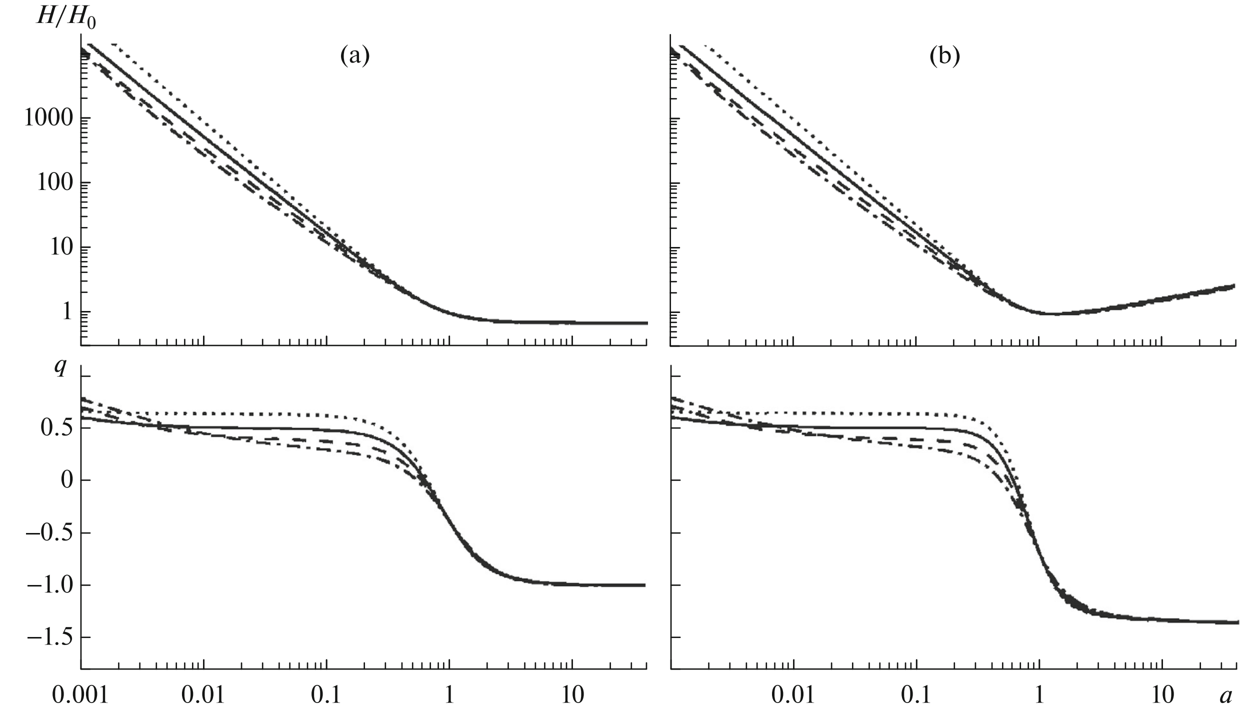

Fig. 6. Evolution of Hubble parameter H and deceleration parameter q at the different values of interaction parameter γ (dotted line is for $\gamma = -0.1$, solid line is for $\gamma = 0$, dashed line is for $\gamma = 0.1$, dash-dotted line is for $\gamma = 0.2$): (a) quintessence dark energy, (b) phantom. The parameters of models are the same as in Fig. 1.

dependencies of energy densities on a are the smooth function, which can take negative values at the certain values of parameters of dark energy and the interaction parameter. For each case, the range of values of parameters at which the energy density of dark components are positive for an arbitrary a was found. Common for all cases is the condition of positivity of interaction parameter. In the representation (15)–(17), this means that the density of dark matter is always positive only in the case of interaction at which is the flow of energy from the dark energy to the dark matter, and the density of dark energy is positive if the value of interaction parameter not exceed some quantity found for the each model. The other common feature of such models are the nonzero asymptotic values of densities of dark components when $a \to \infty$ if the dark energy is the quintessence. If the dark energy is phantom, then the asymptotic value of its energy density when $a \to 0$ is constant for all acceptable values of interaction parameter in models (15) and (16). In model (17), when the energy flow is proportional to the energy density of dark matter, the variants of special behavior of the phantom dark energy are possible: $\rho_{de} \to \infty$ when $a \to 0$ and $a \to \infty$, or $\rho_{de} = \text{const}$ when w_0, $c_a^2 < -1$ in the models (15) and (16). The interaction that causes the flow of energy from the dark matter to the dark energy always leads to fast decrease of density of dark matter to zero and transition in the negative values, which we consider the nonphysical solution. Figures 1, 3, and 5 confirm these conclusions.

The type of interaction and the value of interaction parameter between the dark components, as seen from Figs. 2, 4, and 6, have influence on the dynamics of expansion of the Universe—the Hubble parameter and the deceleration parameter, which can be used for identifying the type and the strength of interaction, or at least the upper limits of values of interaction parameter.

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