ISSN 0278-6419, Moscow University Computational Mathematics and Cybernetics, 2021, Vol. 45, No. 2, pp. 81–84. © Allerton Press, Inc., 2021.
Russian Text © The Author(s), 2021, published in Vestnik Moskovskogo Universiteta, Seriya 15: Vychislitel'naya Matematika i Kibernetika, 2021, No. 2, pp. 39–42.

On the Limiting Distribution of Studentized Intermediate Order Statistics

V. I. Pagurova*

Department of Computational Mathematics and Cybernetics, Moscow State University, Moscow, 119991 Russia

Received March 15, 2021; in final form, March 29, 2021; accepted March 29, 2021

Abstract—Conditions are considered under which studentization does not change the limiting distribution of the normalized intermediate order statistics. A similar problem is considered by Berman as applied to a limiting distribution of extreme order statistics.

Keywords: intermediate order statistics, studentization, weak convergence.

DOI: 10.3103/S0278641921020047

INTRODUCTION

Let independent and identically distributed random variables $X_1, ..., X_n$ have a common distribution function (DF) $F(x) = \mathbf{P}\{X \leqslant x\}$ and distribution density $f(x) = F'(x)$. We denote by $X_k^{(n)}$ the kth order statistic in variational series $X_1^{(n)} \leqslant ... \leqslant X_n^{(n)}$ constructed using the variables $X_1, ..., X_n$, $\bar{X} = \sum\limits_{i=1}^{n} X_i/n$, and $S^2 = \sum\limits_{i=1}^{n}(X_i - \bar{X})^2/n$. For some $c_n > 0$ and d_n. Below, we study the asymptotic distribution (as $n \to \infty$) of the quantity

$$((X_k^{(n)} - \bar{X})/S - d_n)/c_n, \tag{1}$$

where

$$k = k(n) \to \infty, \quad \lambda_{k,n} = k/n \to 0, \quad n \to \infty. \tag{2}$$

In [1], a similar problem was considered as applied to a limiting distribution of extremal order statistics.

1. MAIN RESULT

Theorem. *Let*

$$\mathbf{E}X_1 = 0, \quad \mathbf{E}X_1^2 = 1, \quad 0 < \mathbf{E}X_1^4 < \infty. \tag{3}$$

If, for some $c_n > 0$ and d_n, the quantity

$$(X_k^{(n)} - d_n)/c_n \tag{4}$$

has a limiting DF $H(x)$ *as $n \to \infty$ and the limit relation*

$$d_n/(c_n\sqrt{n}) \to 0 \tag{5}$$

holds, quantity (1) *has the same limiting* DF $H(x)$.

*e-mail: pagurova@yandex.ru.

Proof. We write (1) in the form

$$\left(\frac{X_k^{(n)}}{S} - d_n\right)\frac{1}{c_n} - \frac{\bar{X}}{c_n S}.$$

It follows from conditions (3) that, as $n \to \infty$,

$$S \xrightarrow{P} 1, \quad \frac{1}{S} = \frac{1}{1-(1-S)} = 1 + (1-S) + o_p(1-S),$$

expression (1) is equivalent to the expression

$$\frac{X_k^{(n)} - d_n}{c_n} + \frac{X_k^{(n)}}{c_n}(1-S)(1+o_p(1)) - \frac{\bar{X}}{c_n S}. \tag{6}$$

Let us show that quantity (6) has the same limit as quantity (4). It follows from conditions (3) that quantities $\sqrt{n}\bar{X}$ and $\sqrt{n}(1-S)$ as $n \to \infty$ are asymptotically normal.

We consider the expression

$$\frac{X_k^{(n)}}{c_n\sqrt{n}} = \frac{X_k^{(n)} - d_n}{c_n\sqrt{n}} + \frac{d_n}{c_n\sqrt{n}}. \tag{7}$$

Since both terms on the right-hand side of relation (7) converge in probability to zero as $n \to \infty$ by the hypotheses of the theorem, we have

$$\frac{X_k^{(n)}}{c_n\sqrt{n}} \xrightarrow{P} 0. \tag{8}$$

It thus follows that the second term in expression (6) converges to zero in probability as $n \to \infty$. Let us show that as $n \to \infty$,

$$1/(c_n\sqrt{n}) \to 0. \tag{9}$$

Let us consider two cases.

1. Dstribution F is unbound from the left; i.e., $F(x) > 0$ for all x. As $n \to \infty$, we then have $X_k^{(n)} \xrightarrow{P} -\infty$, and condition (8) implies condition (9).

2. There exists a finite number x_0, such that $d_n \to x_0$ and $X_k^{(n)} \xrightarrow{P} x_0$ as $n \to \infty$, and the condition $\mathbf{E}X_1 = 0$ implies that $x_0 < 0$. Again, condition (9) follows from condition (8); consequently, as $n \to \infty$, the third term in expression (6) converges in probability to zero. The theorem is proved.

2. APPLICATIONS

Suppose that condition (2) holds. A necessary and sufficient condition for asymptotic normality as $n \to \infty$ of statistics $T_n = (X_k^{(n)} - d_n)/c_n$ for some $c_n > 0$ and d_n is that the following relation is satisfied for any x [2, 3]:

$$\lim_{n\to\infty}(F(c_n x + d_n) - \lambda_{k,n})\sqrt{k}/\lambda_{k,n} = x. \tag{10}$$

For absolutely continuous distributions, quantities c_n and d_n are determined using relations

$$F(d_n) = \lambda_{k,n}, \quad c_n = \sqrt{k}/(nf(d_n)).$$

In [4, 5], it was shown that under condition (2) and as $n \to \infty$, probable limiting distributions of statistics T_n are normal and lognormal distributions. The joint asymptotic distribution of intermediate order statistics was studied in [6, 7].

We assume below that $z_F = \inf\{x : F(x) > 0\}$, $k = [n^\alpha]$, $0 < \alpha < 1$, and $[x]$ denotes the integral part of number x, and we consider typical classes of distributions widely used in statistical applications.

Class B_1 :

$$F(x) \sim a|x|^\gamma \exp(-b|x|^\Delta), \quad f(x) \sim ab\Delta|x|^{\gamma+\Delta-1}\exp(-b|x|^\Delta) \quad \text{for } x \to -\infty, \quad a, b, \Delta > 0,$$

$$d_n = -\left(\frac{(1-\alpha)\ln n}{b}\right)^{1/\Delta}\left(1+\frac{\gamma\ln\ln n+\ln c}{\Delta^2(1-\alpha)\ln n}\right), \quad c=\left(\frac{1-\alpha}{b}\right)^{\gamma} a^{\Delta},$$

$$c_n = \left(\frac{(1-\alpha)\ln n}{b}\right)^{(1-\Delta)/\Delta}\frac{1}{\Delta b n^{\alpha/2}}\,.$$

Class B_2 :

$$F(x)\sim a|x|^{-\Delta}, \quad f(x)\sim a\Delta|x|^{-\Delta-1} \quad \text{for} \ \ x\to-\infty, \quad a,\Delta>0,$$

$$d_n = -a^{1/\Delta}n^{(1-\alpha)/\Delta}, \quad c_n = a^{1/\Delta}/(\Delta n^{-(1-\alpha)/\Delta+\alpha/2}).$$

Class B_3 :

$$F(x)\sim a(x-z_F)^{\Delta}, \quad f(x)\sim a\Delta(x-z_F)^{\Delta-1} \quad \text{for} \ \ x\to z_F, \quad -\infty<z_F<\infty, \quad a,\Delta>0,$$

$$d_n = z_F+(an^{1-\alpha})^{-1/\Delta}, \quad c_n = 1/(\Delta a^{1/\Delta}n^{(1-\alpha)/\Delta+\alpha/2}).$$

It is easy to show that relation (10) holds for all three classes B_1, B_2, and B_3, and the limiting distribution of the statistics T_n as $n\to\infty$ is the standard normal distribution. For classes B_1 and B_2, condition (5) is satisfied,while for class B_3 condition (5) is satisfied under the constraint $\Delta>2$. Conditions (3) require additional constraints on the parameters for all three classes. For class B_3, it follows from (3) that $z_F<0$.

Examples of the distributions from class B_1 that satisfy the hypotheses of the theorem are the standard normal distribution and the Laplace distribution with density $f(x)=\exp(-\sqrt{2}|x|)/\sqrt{2}$, $|x|<\infty$.

3. EXAMPLE OF THE DISTRUBUTION FROM CLASS B_2 SATISFYING THE HYPOTHESES OF THE THEOREM

Let us consider a distribution with density $f(x)=b_1/(x^6+b_2^6)$,$|x|<\infty$. Positive numbers b_1 and b_2 are determined later by using condition (3). We have

$$F(x)\sim\frac{b_1}{5|x|^5}, \quad f(x)\sim\frac{b_1}{x^6} \quad \text{as} \quad x\to-\infty,$$

$$1 = b_1\int_{-\infty}^{\infty}\frac{u^2du}{u^6+b_2^6} = \frac{b_1}{3b_2^3}\int_{-\infty}^{\infty}\frac{dt}{t^2+1} = \frac{b_1\pi}{3b_2^3};$$

hence, $b_2^3=b_1\pi/3$. Further,

$$1 = b_1\int_{-\infty}^{\infty}\frac{du}{u^6+(b_1\pi/3)^2} = b_1(b_1\pi/3)^{-5/3}\int_{-\infty}^{\infty}\frac{dt}{t^6+1}.$$

Since

$$\int_0^{\infty}\frac{dt}{t^6+1}=\frac{\pi}{3} \quad \text{(see [8, p. 165 of Russian translation])},$$

it follows that $b_1=6\sqrt{2}/\pi$ and $b_2^6=8$. We have

$$d_n = -\left(\frac{b_1}{5}\right)^{1/5}n^{(1-\alpha)/5}, \quad c_n = \left(\frac{b_1}{5}\right)^{1/5}\frac{1}{5}n^{(1-\alpha)/5-\alpha/2},$$

$$\frac{d_n}{c_n\sqrt{n}} = -5n^{-(1-\alpha)/2}\to 0 \quad \text{as} \quad n\to\infty.$$

The conditions of the theorem are met.

4. AN EXAMPLE OF THE DISTRUBUTION FROM CLASS B_3 SATISFYING THE HYPOTHESES OF THE THEOREM

Let $F(x) = a(x - z_F)^{\Delta}$, $f(x) = \Delta a(x - z_F)^{\Delta - 1}$, $x \in (z_F, b)$, and $a, \Delta > 0$. In light of condition $F(b) = 1$ and relations (3), we obtain

$$z_F = -\sqrt{\Delta(\Delta+2)}, \quad b = \sqrt{\Delta(\Delta+2)}, \quad a = \Delta^{\Delta/2}/((\Delta+2)^{\Delta/2}(\Delta+1)^{\Delta}).$$

Under constraint $\Delta > 2$, the conditions of the theorem hold.

5. EXAMPLE OF A DISTRUBUTION FROM CLASS B_3 NOT SATISFYING THE HYPOTHESES OF THE THEOREM

Let $F(x) = 1 - \exp(-(x+1))$, $\quad f(x) = \exp(-(x+1))$, $x > -1$. We assume that conditions (3) holds,

$$c_n = \frac{1}{n^{1-\alpha/2}}, \quad d_n = -1 + \frac{1}{n^{1-\alpha}}, \quad \frac{d_n}{c_n\sqrt{n}} \sim -n^{(1-\alpha)/2} \to -\infty,$$

and condition (5) is not satisfied. We have

$$(X_{[n^\alpha]}^{(n)} - d_n)/c_n \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty,$$

but since $X_{[n^\alpha]}^{(n)} \xrightarrow{P} -1$, $\sqrt{n}\bar{X}$ and $\sqrt{n}(1 - S)$ are asymptotically normal,

$$\frac{1}{c_n\sqrt{n}} \to \infty \quad \text{as} \quad n \to \infty,$$

the second and third terms in representation (6) grow infinitely in absolute magnitude.

REFERENCES

1. S. M. Berman, "Limiting distribution of the studentized largest observation," Scand. Actuarial J. **1962** (3–4), 155–161 (1962).
2. N. V. Smirnov, "Limit distributions for the terms of a variational series," Amer. Math. Soc. Transl. Ser. 1 **11** (67), 82–143 (1952).
3. N. V. Smirnov, "On convergence of terms of a variational series to the normal distribution law," Izv. Akad. Nauk Uzb. SSR **3**, 24–32 (1966).
4. D. M. Chibisov, "On limit distributions for order statistics," Theory Probab. Appl. **9** (1), 142–148 (1964).
5. C. Y. Wu, "The types of limit distributions for some terms of variational series," Sci. Sinica **15**, 749–762 (1966).
6. B. Cooil, "Limiting multivariate distributions of intermediate order statistics," Ann. Probab. **13** (2), 469–477 (1985).
7. V. I. Pagurova, "A limiting multidimensional distribution of intermediate order statistics," Moscow Univ. Comput. Math. Cybern. **41** (3), 130–133 (2017).
8. V. A. Ditkin and A. P. Prudnikov, *Integral Transforms and Operational Calculus* (Fizmatgiz, Moscow, 1961; Pergamon Press, Oxford, 1965).

Translated by I. Tselishcheva