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The Edgeworth–Pareto Principle in the Case of a Type-2 Fuzzy Preference Relation

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Abstract—The Edgeworth–Pareto principle is extended to the class of multicriteria choice problems, in which the preference relation of the decision maker is described by a type-2 fuzzy binary relation. The necessary condition for its fulfillment is the adoption of two assumptions: the Pareto axiom and axiom of exclusion of dominated variants. Brief information about fuzzy sets and relations of types 1 and 2 precedes the main results of the paper. In order to justify the Edgeworth–Pareto principle a fuzzy set of nondominated variants is introduced in the case of a type-2 fuzzy preference relation.

Keywords: multicriteria choice, axiomatic approach, Edgeworth–Pareto principle, Pareto set reduction, type-2 fuzzy relation

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INTRODUCTION

Since the time of the economists Francis Edgeworth (1845–1926) and Wilfredo Pareto (1848–1923) it has been believed that the "best" solutions (variants) of multicriteria problems should be sought within the Pareto set, whose elements (Pareto-optimal solutions) are characterized by the fact that it is impossible to improve on any of the available criteria without deteriorating the values for at least one other criterion. In other words, until the beginning of the 21st century, researchers, without proper justification, adhered to the point of view that the "best" solutions to multicriteria problems are contained in the Pareto set. This principle of choice [1] was called the "naive" Edgeworth–Pareto principle by analogy with "naive" set theory, which does not rely on axioms; therefore, the boundaries of the reasonable application of this theory are not quite clear. Over time, such methods for solving multicriteria problems began to appear, which allowed the existence of the "best" solutions (so-called satisfactory solutions) outside the Pareto set. In this regard, an urgent need arose for a logical substantiation of the Edgeworth–Pareto principle; in [1] an axiomatic version of this principle for numerical criteria was proposed, which required expanding the set of basic objects of a multicriteria problem (a vector criterion and the set on which it is specified) by adding the binary preference relation of the decision maker (DM) and the introduction of the concept of a set of selected variants (vectors), which is the solution to this problem. It turned out that the application of the Edgeworth–Pareto principle will be justified if the decision maker accepts certain axioms of "reasonable" choice. If at least one of these axioms is rejected the "best" solution does not have to be Pareto optimal.

Subsequently, the class of multicriteria choice problems for which the application of the Edgeworth–Pareto principle is justified gradually expanded. It became clear that the multicriteria choice problem, which includes as the main objects a vector numerical criterion, a set of feasible variants, and an asymmetric binary preference relation of the decision maker, has features similar to the so-called general choice problems [2]. The Edgeworth–Pareto principle was substantiated in terms of the choice function [3, 4]. The universality of the Edgeworth–Pareto principle also manifested itself in the fact that it turned out to be possible to use it not only in problems of multicriteria choice with numerical criteria taking values in quantitative scales of ratios, differences, and intervals, but also when the values of the criteria are measured only in ordinal scales or when there are no criteria at all, and instead of them only a set of some "individual" binary preferences is given. In this regard, the term "generalized Edgeworth–Pareto principle" appeared [4].

In [5], the development of this principle was traced from the moment of its substantiation in 2002 up to 2006. It was found that the requirement of transitivity of the preference relation, which guides the decision maker in the selection process, is not mandatory when using the Edgeworth–Pareto principle. Thus, the number of axioms that justify the application of this principle was reduced to two. This is the Pareto axiom, according to which "the larger the values of the criteria are, the better it is for the decision maker," as well as the axiom of excluding dominated variants, which states that the variant that is less preferable in comparison with some other should not become "best" within the original set variants. Both of these axioms are quite capable of being called "reasonable," since they not only do not contradict, but fully correspond to the common sense of the "reasonable" behavior of the decision maker in the process of choice. This can explain the fact that until now the overwhelming number of researchers in solving multicriteria problems are guided by this principle.

Further, the class of multicriteria choice problems for which the Edgeworth–Pareto principle is valid was expanded by including fuzzy multicriteria choice problems containing a set of irreflexive, transitive and weakly connected binary relations on the corresponding sets (instead of criteria), a fuzzy set of initial variants, and the fuzzy (type-1) asymmetric binary preference relation of the decision maker [6–8].

In 2015, the development of the Edgeworth–Pareto principle was continued [9] in the direction of its extension to the so-called k-effective points, which at $k = 1$ coincide with effective (i.e., Pareto-optimal) points, and at the other extreme value $k = \mathrm{m}$, with weakly effective (Slater optimal) points. Thus, in essence, a number of principles were formulated and axiomatically substantiated, from the Edgeworth–Pareto principle to the Slater principle. In addition, in [9], these principles were extended to the class of problems with a fuzzy type-1 decision maker's preference relation, when the values of the fuzzy relation membership function are ordinary (crisp) numbers.

It is known [10] that along with the usual fuzzy sets, which are called type-1 fuzzy sets, there are type-2 fuzzy sets; the values of their membership functions are fuzzy values, i.e., numerical functions specified on the same segment [0,1] as their values. According to the statement from [11], type-2 fuzzy sets allow the use of linguistic estimates of the values of the membership function, thereby contributing to the successful presentation of fuzzy knowledge. Fuzzy relations of type-2 were also introduced; with their use it is possible to successfully simulate vague ideas about the unknown preference relation of the decision maker. A type-2 fuzzy relation is considered as one of the ways to increase the fuzziness of a binary relation, and, according to [12], "increased fuzziness in the description means an increased ability to cope with inaccurate information in a logically correct form."

The aim of this work is to formulate and substantiate the Edgeworth–Pareto principle for multicriteria choice problems with a type-2 fuzzy preference relation. In the case where a type-2 fuzzy relation degenerates into a type-1 fuzzy relation, the results obtained here coincide with those established earlier. For convenience of perceiving these results and fixing the terminology, the main material of this article is preceded by brief information from the theory of fuzzy sets. A fuzzy set of nondominated variants is then introduced, with its use the Edgeworth–Pareto principle is established.

1. NECESSARY INFORMATION FROM THE THEORY OF FUZZY SETS

Let A be some nonempty (universal) set. *Fuzzy set* (type-1) X in A is given by the membership function $\lambda_X : A \to [0,\ 1]$. For each element $x \in A$ the number $\lambda_X(x) \in [0,\ 1]$ is interpreted as the degree of membership of an element x to the set X. Often, when it is said that a certain fuzzy set is given, only its membership function is mentioned, since this set is uniquely determined by it. Two fuzzy sets are equal to each other if they have the same membership functions. A *support* of a fuzzy set consists of those elements of the set A whose membership degree is positive. A fuzzy set is called *normal* if the exact upper bound of its membership function is equal to one, otherwise it is called *subnormal*. By *fuzzy value* we usually understand a fuzzy set defined on a set (subset) of real numbers $A \subset \mathbb{R}$, and a *fuzzy number* is a normal convex fuzzy value. A *convex* fuzzy value, by definition, has a convex membership function $\lambda_X(\cdot)$, i.e., $\lambda_X(\alpha x + (1-\alpha)y) \geq \min\{\lambda_X(x), \lambda_X(y)\}$ for all $x, y \in A$ and all $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y \in A$.

The inclusion relation, as well as the operations of union, intersection, and complement of type-1 fuzzy sets X and Y in terms of their membership functions are defined as follows:

$$X \subset Y \Leftrightarrow \lambda_X(x) \leq \lambda_Y(x) \text{ for all } x \in A,$$

$$\lambda_{X \cup Y}(x) = \max\{\lambda_X(x); \lambda_Y(x)\} \text{ for all } x \in A,$$

$$\lambda_{X \cap Y}(x) = \min\{\lambda_X(x); \lambda_Y(x)\} \text{ for all } x \in A,$$

$$\lambda_{\bar{X}}(x) = 1 - \lambda_X(x) \text{ for all } x \in A.$$

Triangular, interval, and trapezoidal fuzzy numbers have become widespread; their names are associated with the shape of the graph of their membership function. Thus, a *triangular* fuzzy number has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \leq x < b, \\ \dfrac{c-x}{c-b}, & b \leq x < c, \\ 0, & x \geq c \end{cases}$$

and is characterized by three parameters: interval $(a;c)$ represents the support of this number and b is responsible for the maximum point of the membership function. Therefore, triangular fuzzy numbers are often denoted by a triple of numbers (a, b, c). An *interval* fuzzy number is defined by a pair of numbers $(a;b)$:

$$\mu_A(x) = \begin{cases} 0, & x < a, \\ 1, & a \leq x \leq b, \\ 0, & x > b. \end{cases}$$

A *trapezoidal* fuzzy number is a generalization of triangular and interval fuzzy numbers:

$$\mu_B(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x < b, \\ 1, & b \le x < c, \\ \dfrac{d-x}{d-c}, & c \le x < d, \\ 0, & x \ge d. \end{cases}$$

Such a fuzzy value is given by four numbers $(a;b;c;d)$. All of the listed fuzzy numbers are normal and convex.

If P and Q are fuzzy quantities with membership functions λ_P and λ_Q, respectively, then their minimum and maximum in terms of membership functions are determined by the formulas:

$$\lambda_{\min\{P;Q\}}(x) = (\lambda_P \wedge \lambda_Q)(x) = \sup_{u,v:\ \min\{u;v\}=x} \min\{\lambda_P(u);\lambda_Q(v)\},$$

$$\lambda_{\max\{P;Q\}}(x) = (\lambda_P \vee \lambda_Q)(x) = \sup_{u,v:\ \max\{u;v\}=x} \min\{\lambda_P(u);\lambda_Q(v)\},$$

for all $x \in A \subset \mathbb{R}$, where $\wedge$ and $\vee$ are symbols of the minimum and maximum operations, respectively.

For the convenience of further presentation, we introduce the fuzzy quantity $\delta^{[1]}$ on the set $J \subset [0,1]$, which is equivalent to the ordinary number $\delta \in J$, whose membership function is determined by the equality:

$$\lambda_{\delta^{[1]}}(u) = \begin{cases} 1, & u = \delta, \\ 0, & u \ne \delta. \end{cases}$$

Obviously, if $p < q$, then $p^{[1]} \wedge q^{[1]} = p^{[1]}$, $p^{[1]} \vee q^{[1]} = q^{[1]}$, i.e., the minimum and maximum operations on fuzzy numbers are consistent with similar operations on ordinary numbers.

The *fuzzy binary relation* (type-1) is defined on set A using the membership function $\mu : A \times A \to [0, 1]$. The number $\mu(x, y) \in [0,1]$ is interpreted as the degree of confidence that the element x is in this relation with the element y.

A fuzzy relation with membership function $\mu(\cdot,\cdot)$ is called:

- *irreflexive*, if $\mu(x,x) = 0$ for all $x \in A$;
- *transitive*, if $\mu(x,z) \ge \min\{\mu(x,y);\mu(y,z)\}$ for all $x, y, z \in A$;
- *asymmetrical*, if $\mu(x,y) > 0 \Rightarrow \mu(y,x) = 0$ for all $x, y \in A$.

Any irreflexive and transitive fuzzy binary relation is asymmetric. Additional information from the theory of fuzzy sets and relations (type-1) can be found, for example, in [13, 14].

A *fuzzy set of the second order* (*type-2 fuzzy set*) X is determined by the membership function $\lambda_X(\cdot,\cdot)$ of two variables $x \in A$ and $u \in J \subset [0,1]$ taking values in the unit segment $[0,1]$. This function for each element x and every number $u = u^* \in J$, which is called the *primary* degree of membership, gives the so-called *secondary* degree of membership, i.e., the number $\lambda_X(x,u^*) \in [0,1]$ expresses the degree of confidence that the degree of membership of the element x to the set X is equal to u^*. The nonempty set J can be both finite and infinite, in particular, it is possible that $J = [0,1]$. Because $\lambda_X(x,\cdot)$ at every fixed x specifies some numerical function of $u \in J$ with values in a unit segment $[0,1]$, a type-2 fuzzy set can be considered as a mapping of each element $x \in A$ to some function (fuzzy number) defined on J with values in a unit segment. Recall that a type-1 fuzzy set is characterized by mapping of each element x∈ A to some number from a unit segment that expresses the degree of membership of this element to the given fuzzy set. Thus, in the transition from a type-1 fuzzy set to a type-2 fuzzy set, it becomes possible to describe the degree of membership of each element more flexibly not with a number, but by using a function. This provides ample opportunity to take the nature and degree of fuzziness into account when operating with fuzzy data in solving a variety of applied problems.

Let there be two type-2 fuzzy sets X and Y in A with membership functions $\lambda_X(\cdot,\cdot)$ and $\lambda_Y(\cdot,\cdot)$, respectively. The operations of union and intersection of these sets, taking the principle of expansion of L. Zadeh into account, are defined in terms of membership functions as follows [15]:

$$\lambda_{X\cup Y}(x,u) = \lambda_X \sqcup \lambda_Y = \lambda_X(x,u) \vee \lambda_Y(x,u) = \sup_{p,q:\max\{p,q\}=u} \min\{\lambda_X(x,p),\lambda_Y(x,q)\},$$

$$\lambda_{X\cap Y}(x,u) = \lambda_X \sqcap \lambda_Y = \lambda_X(x,u) \wedge \lambda_Y(x,u) = \sup_{p,q:\min\{p,q\}=u} \min\{\lambda_X(x,p),\lambda_Y(x,q)\},$$

for all $x \in A$ and $u \in J$, where the symbols $\sqcup$ and $\sqcap$ denote the so-called join and meet operations, applied respectively, to the membership functions of type-2 fuzzy sets.

Further, the inclusion relation for these sets is introduced by the formula:

$$X \subset Y \Leftrightarrow X \cap Y = X \text{ and } X \cup Y = Y. \qquad (1)$$

In the event that for any $x, y \in A$ membership functions $\lambda_X(x,\cdot)$ and $\lambda_Y(y,\cdot)$ are normal and convex on a convex (or finite) set J, on the right side of the above equivalence, the conjunctive conjunction "and" can be replaced by "or."

Following [15], for the membership functions of type-2 fuzzy sets, we indicate the method of their partial ordering based on the membership functions:

$$\lambda_X \sqsubseteq \lambda_Y \Leftrightarrow [\lambda_X \sqcap \lambda_Y] = \lambda_X \text{ and } [\lambda_X \sqcup \lambda_Y] = \lambda_Y. \qquad (2)$$

From (1) and (2) it follows that:

$$X \subset Y \Leftrightarrow \lambda_X \sqsubseteq \lambda_Y \Leftrightarrow [\lambda_X(x,u) \wedge \lambda_Y(x,u)] = \lambda_X(x,u) \text{ and } [\lambda_X(x,u) \vee \lambda_Y(x,u)] = \lambda_Y(x,u).$$

We note that in the case where the membership functions λ_X and λ_Y are normal and convex, one of the two equalities involved in the right-hand side of the last equivalence can be omitted.

The membership function $\neg\lambda_X$ of a complement $\bar{X}$ of a type-2 fuzzy set X with membership function λ_X is defined as follows: $\lambda_{\bar{X}}(x,u) = \neg\lambda_X(x,u) = \lambda_X(x, 1-u)$ for all $u \in J$, $x \in A$.

The operations of union, intersection, and complement introduced by the above method over type-2 fuzzy sets possess the properties of commutativity, associativity, idempotency, de Morgan's rules and the law of involution, but do not obey the laws of distributivity, identity, and complementarity [15].

A *type-2 fuzzy relation* on the set A is determined by the membership function $\mu(\cdot,\cdot,\cdot)$ with three arguments $x, y \in A$ and $u \in J \subset [0,1]$. For each fixed pair of elements $x, y \in A$ and a number $u = u^* \in J$ the value $\mu(x, y, u^*) \in [0,1]$ indicates the secondary degree of confidence with which elements x and y are in this relation with a primary degree of certainty $u = u^*$. Thus, using a type-2 fuzzy relation, each pair of elements $x, y \in A$ is associated with a certain fuzzy value, which, in particular, may turn out to be a fuzzy number if it possesses the properties of normality and convexity.

The type-2 fuzzy relation on the set A with membership function $\mu(\cdot,\cdot,\cdot)$ is called [16]:

- *irreflexive*, if $\mu(x,x,u) = \lambda_{0^{[1]}}(u)$ for all $x \in A$, $u \in J$;
- *asymmetrical* if for all $x, y \in A$, $u \in J$ inequality $\mu(x,y,u) \neq \lambda_{0^{[1]}}(u)$ entails equality $\mu(y,x,u) = \lambda_{0^{[1]}}(u)$;
- *transitive*, if $\mu(x,z,u) \sqsupseteq [\mu(x,y,u) \wedge \mu(y,z,u)]$ is fulfilled for all $x, y, z \in A$, $u \in J$.

Note that in the definition of a type-2 fuzzy transitive relation, the minimum operation $\wedge$ has the same meaning as before, namely:

$$\mu_1(x,y,u) \wedge \mu_2(x,y,u) = \sup_{p,q:\min\{p,q\}=u} \min\{\mu_1(x,y,p), \mu_2(x,y,q)\}.$$

As indicated above, in the case of type-1 fuzzy relations, the simultaneous fulfillment of the properties of irreflexivity and transitivity leads to the asymmetricity of this relation. A similar situation takes place for type-2 fuzzy relations.

Lemma 1. *Any irreflexive and transitive binary type-2 fuzzy relation for which the values of the membership function are normal fuzzy values is asymmetric.*

The proof of this and all subsequent statements is given in the Appendix.

2. A FUZZY SET OF NONDOMINATED VARIANTS

We introduce a crisp set X of feasible variants of an arbitrary nature. From this set, the decision maker should make the best choice. We will assume that on the set X some asymmetric binary preference relation is given $\succ_X$, which should help this decision maker to make the best choice. When this relation is crisp, the record $x \succ_X x'$ means that variant x is preferred over x' (x dominates x'), i.e., when choosing from these two variants, the decision maker will choose the first one and will not choose the second variant. In this case, deleting all dominated variants from X leads to the set of nondominated variants $Ndom(X)$. If a $\succ_X$ is a type-1 fuzzy relation, then the dominance of one variant x over another x' is carried out with a degree of confidence $\mu_{\succ_X}(x,x') \in [0,1]$, and the degree of nondominance of x can be expressed as a number $1 - \mu_{\succ_X}(x,x') \in [0,1]$. The set defined by the formula

$$\lambda_{Ndom(X)}(x) = \inf_{x' \in X}(1 - \mu_{\succ_X}(x,x')) = 1 - \sup_{x' \in X} \mu_{\succ_X}(x,x') \text{ for all } x \in X$$

is called a *fuzzy set of nondominated variants* [13]; it plays an important role in the issues of multi-criteria choice in the presence of fuzzy information.

In choice problems, the decision-maker's preference relation, as a rule, is not completely known. Moreover, information about it is often fragmentary and vague (fuzzy). If the preference relation is a type-1 fuzzy relation, then the fuzziness is expressed by assigning to each pair of possible variants a certain number within the segment $[0,1]$. Moreover, the larger this number is, the higher the degree of confidence in the dominance of one variant over the other is. However, the ambiguity of the decision maker's preference relationship may turn out to be so significant that one number for its expression will not be enough. Similar situations arise when, instead of a number, the degree of dominance is itself a fuzzy value of some type, in particular, a linguistic variable. In this case, it is necessary to consider the decision-maker's preference relation as a type-2 fuzzy relation with all the ensuing consequences.

Let $\mu_{\succ_X}(\cdot,\cdot,\cdot)$ be a membership function of an asymmetric type-2 fuzzy preference relation of a decision maker. The number $\mu_{\succ_X}(x', x, 1-u) \in [0,1]$ expresses a secondary degree of confidence that the variant x dominates x' with primary confidence u. Then, the number $\mu_{\succ_X}(x', x, 1-u) \in [0,1]$ is an indicator of a secondary degree of confidence that x is not dominated by variant x'. Thus, we arrive at the following formula for determining the membership function $\lambda_{Ndom(X)}(\cdot,\cdot)$ of the fuzzy set of nondominated variants $Ndom(X)$ in the case of a type-2 fuzzy preference relation $\succ_X$ with the membership function $\mu_{\succ}(\cdot,\cdot,\cdot)$:

$$\lambda_{Ndom(X)}(x,u) = \inf_{x' \in X} \mu_{\succ_X}(x', x, 1-u)$$
$$= \inf_{x' \in X}\{\neg\mu_{\succ_X}(x', x, u)\} \text{ for all } x \in X,\, u \in J.$$

Here, under the sign of the exact lower bound a membership function $\mu_{\succ_X}(x', x, \cdot)$ of some fuzzy value occurs rather than a number; therefore, the concept of the exact lower bound should be clarified. In this case, one should take the fact that the set X can contain not only a finite, but also an infinite number of elements into account.

It is proposed to use the following definition of an *exact lower bound* for an arbitrary family $\lambda_i(x,\cdot)$, $i \in I$, $card\, I \le \infty$ of fuzzy values defined on the set $J \subset [0,1]$, namely:

$$(\inf_{i \in I} \lambda_i)(x,u) = \sup_{u_i \in J:\inf_{i \in I} u_i = u} \inf_{i \in I} \lambda_i(x, u_i)$$
$$= \bigwedge_{i \in I} \lambda_i(x, u_i) \text{ for all } x \in X,\, u \in J.$$

It is easy to understand that in this way the introduced operation of the exact lower bound in the case of a finite number of fuzzy quantities coincides with the operation of the minimum $\wedge$ over the fuzzy values described above.

Similarly, one can introduce the definition of an *exact upper bound* of arbitrary family $\lambda_i(x,\cdot)$, $i \in I$, $card\, I \le \infty$ of fuzzy values by equality:

$$(\sup_{i \in I} \lambda_i)(x,u) = \sup_{u_i \in J:\sup_{i \in I} u_i = u} \inf_{i \in I} \lambda_i(x, u_i)$$
$$= \bigvee_{i \in I} \lambda_i(x, u_i) \text{ for all } x \in X,\, u \in J.$$

The following properties (de Morgan rules) of the introduced operations are valid:

$$\neg[\bigvee_{i \in I} \lambda_i(x,u_i)] = \bigwedge_{i \in I}[\neg\lambda_i(x,u_i)],$$
$$\neg[\bigwedge_{i \in I} \lambda_i(x,u_i)] = \bigvee_{i \in I}[\neg\lambda_i(x,u_i)]. \quad (3)$$

The proof of equalities (3) is given in the Appendix.

Assumption. *We will assume that all fuzzy values that are the values of the membership functions of fuzzy sets and relations that participate further are normal and convex, and in the case of an infinite set J, are also upper semicontinuous.*

Recall that the fuzzy quantity $\lambda(\cdot)$ is called upper semicontinuous if for any number $\alpha \in (0,1]$ the set of the form $\{u \mid \lambda(u) \ge \alpha\}$ is closed.

Taking the indicated properties of the operations of the exact lower and exact upper bounds in the new notation into account, the membership function of a type-2 fuzzy set of nondominated variants can be represented as follows:

$$\lambda_{Ndom(X)}(x,u) = \bigwedge_{x' \in X} \mu_{\succ_X}(x', x, 1-u)$$
$$= \neg[\bigvee_{x' \in X} \mu_{\succ_X}(x', x, u)] \text{ for all } x \in X,\, u \in J.$$

In the case where the fuzzy preference relation is a type-1 fuzzy relation, the fuzzy set of nondominated variants introduced above coincides with that introduced in [13].

2. FUZZY CHOICE AXIOMS. THE EDGEWORTH–PARETO PRINCIPLE

Recall that the symbol X denotes an abstract set of feasible variants. We will assume that this set is crisp. In the general case, the best choice is a whole set, which will be denoted $C(X)$ below and will be called *the set of selected variants* $C(X) \subset X$. The solution to the choice problem consists in finding this set. On the set X an asymmetrical binary preference relation $\succ_X$ of the DM with a membership function $\mu_{\succ_X}$ is defined. It may turn out to be crisp or type-1 as well as type-2 fuzzy relation; information about it is fragmentary and vague. It is obvious that the type of the fuzzy set of selected variants coincides with the type of the preference relation, so that in the case of a type-1 (type-2) fuzzy relation, the set $C(X)$ is also a fuzzy set of the same type.

We will be interested in the case of a type-2 fuzzy preference relation, since simpler cases were mentioned in the Introduction. Let us denote by $\lambda_{C(X)}(\cdot,\cdot)$

the membership function of an unknown type-2 fuzzy set of selected variants, and the membership function of a type-2 fuzzy preference relation, which is adhered by the decision maker in the selection process, as $\mu_{\succ_X}(\cdot,\cdot,\cdot)$. We recall that the function $\lambda_{C(X)}(\cdot,\cdot)$ is not known in advance; it is to be found in the choice process. Let us formulate the following assumption.

Axiom 1 (exclusion of dominated variants). *For all $x, x' \in X$ and any $u \in J$ the inclusion is fulfilled:*

$$\lambda_{C(X)}(x,u) \sqsubseteq \mu_{\succ_X}(x',x,1-u) = \neg\mu_{\succ_X}(x',x,u). \quad (4)$$

This inclusion means that the degree of membership to the unknown set of selected variants "does not exceed" the degree of membership with which the variant x is not dominated by variant x'. In the case of a crisp preference relation, Axiom 1 takes the form of the axiom of exclusion of dominated variants, according to which a variant not chosen in a pair should not be selected from the entire set X [17].

Let us add to our model one more main object (i.e., in addition to X and $\succ_X$), the numerical vector criterion $f(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_m(\cdot))$ defined on the set X. For this case, the Pareto axiom is formulated as follows.

Pareto axiom. *For any $x, x' \in X$ from inequalities $f_i(x') \geq f_i(x)$, $i = 1,2,\dots,m$; $f(x') \neq f(x)$ it follows that* $\mu_{\succ_X}(x',x,\cdot) = 1^{[1]}(\cdot)$.

This axiom fixes the decision maker's interest in maximizing each of the given numerical criteria.

We introduce the Pareto set

$$P_f(X) = \{x \in X | f_i(x') \geq f_i(x),\ i = 1,2,\dots,m;\ x' \in X \Rightarrow f(x') = f(x)\},$$

and after $\lambda_{P_f(X)}(x,\cdot)$ we denote its membership function, which takes the value $1^{[1]}(\cdot)$ at Pareto optimal points $x \in P_f(X)$ and $0^{[1]}(\cdot)$ at all other points of the set X.

The Edgeworth–Pareto principle. *Under the assumption that Axiom 1 and the Pareto Axiom hold for any type-2 fuzzy set of selected variants $C(X)$, the inclusion*

$$C(X) \subset P_f(X), \quad (5)$$

holds, which in terms of membership functions takes the form

$$\lambda_{C(X)}(x,u) \sqsubseteq \lambda_{P_f(X)}(x,u) \quad (6)$$

for all $x \in X$, $u \in J$.

Comment. In the case of a finite set J the condition of upper semicontinuity of the membership function in the formulation of the Edgeworth–Pareto principle, which is present in it due to the earlier assumption, is redundant.

The principle formulated here in the particular case of crisp preference relations coincides with the one that was established earlier. The examples given in [17] show that if at least one of the two axioms involved in it is removed from the conditions guaranteeing the fulfillment of the Edgeworth–Pareto principle, then we can give examples showing that in this case this principle may be violated. It is easy to understand that due to the indicated direct connection between the fuzzy and crisp cases, the same examples retain their meaning in the case of the type-2 fuzzy preference relation considered here.

The Pareto axiom in the formulation of the Edgeworth–Pareto principle can be replaced by a weaker condition, but this requires the transitivity of the preference relation.

Let us move on to a corresponding consideration. For this purpose, on the set $Y = f(X) \subset \mathbb{R}^m$ we introduce the binary preference relation $\succ_Y$ induced by the relation $\succ_X$ according to the following rule:

$y \succ_Y y' \Leftrightarrow x \succ_X x'$ for all $x \in \tilde{x}, x' \in \tilde{x}'$; $\tilde{x}, \tilde{x}' \in \tilde{X}$, where $y = f(x)$, $y' = f(x')$ and $\tilde{X}$ is the set of equivalence classes generated by the equivalence relation $x \approx x' \Leftrightarrow f(x) = f(x')$ on the set X.

Obviously, the relationships $\succ_X$ and $\succ_Y$ have the same properties, in particular, both are asymmetric.

Axiom 2 (existence of a transitive extension). *There is a transitive relation $\succ$ in space $\mathbb{R}^m$ that is an extension of the relationship $\succ_Y$ from Y to the specified space.*

We note that Axiom 2, due to the direct connection of relations $\succ_X$ and $\succ_Y$ requires, in particular, the transitivity of the preference relation $\succ_X$. The second requirement in this axiom is the existence of the relation $\succ$, the narrowing of which to Y coincides with $\succ_Y$. Let us denote the membership function of the relation $\succ$ through $\mu_{\succ}(\cdot,\cdot,\cdot)$.

Axiom 3 (consistency of criteria with a preference relation). *Each criterion f_i, $i = 1,2,\dots,m$, is consistent with the preference relation $\succ$ in the sense that for all pairs of vectors $y = (y_1,\dots,y_{i-1},y_i,y_{i+1},\dots,y_m)$ and $y' = (y_1,\dots,y_{i-1},y_i',y_{i+1},\dots,y_m)$ where $y_i > y_i'$, the equality $\mu_{\succ}(y,y',u) = 1^{[1]}(u)$ is true for all $u \in J$.*

It is easy to understand that the Pareto Axiom leads to the fulfillment of Axiom 3, but not vice versa. The opposite will take place if we restrict ourselves to the class of transitive relations. Namely, the following result is true.

Lemma 2. *The fulfillment of Axioms 2 and 3 implies the validity of the Pareto Axiom.*

The following statement follows directly from the Edgeworth–Pareto principle and Lemma 2.

Corollary 1 (The Edgeworth–Pareto principle for transitive relations). *If Axioms 1–3 are fulfilled then for any type-2 fuzzy set of selected variants $C(X)$ with membership function $\lambda_{C(X)}(x,\cdot)$ the following relations hold:* (5) *and* (6).

As established above for a type-2 fuzzy preference relation and a crisp set X the Edgeworth–Pareto principle can be adapted to justify it in the class of multicriteria choice problems with a type-2 fuzzy set of feasible variants $\tilde{X}$ in the set A with membership function$\lambda_{\tilde{X}}(\cdot,\cdot)$. To do this, it is necessary to agree on what is meant by the set of selected variants. By analogy with the case of a type-1 fuzzy set of feasible variants considered in [17], we will assume that the type-2 fuzzy set of selected variants $C(\tilde{X})$ with membership function $\lambda_{C(\tilde{X})}(\cdot,\cdot)$ by definition is the intersection $C(\tilde{X}) = \tilde{X} \cap C(X)$ where $C(X)$ is a type-2 fuzzy set of feasible variants with a membership function $\lambda_{C(X)}(\cdot,\cdot)$ found under the assumption that the set of feasible variants is crisp. Then, the main relations (5--6) of the Edgeworth–Pareto principle take the form:

$$C(\tilde{X}) = \tilde{X} \cap P_f(X)$$
$$\lambda_{C(\tilde{X})}(x,u) \sqsubseteq \left[\lambda_{\tilde{X}}(x,u) \wedge \lambda_{P_f(X)}(x,u)\right]$$
$$\text{for all } x \in A, u \in J.$$

It should not be forgotten that all fuzzy values that participate in the membership functions of fuzzy sets and relations are assumed to be normal and convex, and, in the case of an infinite set J, also upper semicontinuous.

CONCLUSIONS

The Edgeworth–Pareto principle, according to which the "best" solutions to a multicriteria choice problem must be Pareto-optimal, is extended to the class of problems, when the decision maker's preference relation is a type-2 fuzzy binary relation. The concept of a fuzzy set of nondominated variants for a type-2 fuzzy relation is introduced. It has been established that the application of this principle is correct if the decision maker accepts two axioms of a "reasonable" choice: exclusion and Pareto. In this case, the transitivity of the preference relation is not assumed. Rejection of at least one of the above axioms can lead to the fact that the "best" choice may be outside the Pareto set. It has been shown that the Edgeworth–Pareto principle remains valid under a certain weakening of the Pareto axiom; however, in this case, transitivity of the preference relation is necessary. In addition, this principle is extended to the class of multicriteria choice problems with a type-2 fuzzy set of feasible variants.

APPENDIX

Proof of Lemma 1. Let us denote by $\mu(x,y,\cdot)$ the membership function of a type-2 fuzzy binary relation defined on the set A. According to the condition for all $x,y \in X$ the fuzzy value $\mu(x,y,\cdot)$ given on $J \subset [0,1]$, is normal.

First, let us establish that for all $x,y \in A$ and $u \in J$ from equality $[\mu(x,y,u) \wedge \mu(y,x,u)] = \lambda_{0^{[1]}}(u)$ it follows that $\mu(x,y,u) = \lambda_{0^{[1]}}(u)$ or $\mu(y,x,u) = \lambda_{0^{[1]}}(u)$. If for all $x,y \in A$ and $u \in J$ $\mu(x,y,u) = \lambda_{0^{[1]}}(u)$ or for all $x,y \in A$, $u \in J$ $\mu(y,x,u) = \lambda_{0^{[1]}}(u)$, then, due to the normality of the participating membership functions, the statement that is being proved is obvious. Otherwise, there are elements $x,y \in A$ and numbers $u_1,u_2 \in J \cap (0,1]$ for which $\mu(x,y,u_1) > 0$ and $\mu(y,x,u_2) > 0$. In this case, we have

$$[\mu(x,y,\min\{u_1,u_2\}) \wedge \mu(y,x,\min\{u_1,u_2\})]$$
$$\geq \min\{\mu(x,y,u_1),\mu(y,x,u_2)\} > 0,$$

which is incompatible with the initial condition $\mu(x,y,u) \wedge \mu(y,x,u) = \lambda_{0^{[1]}}(u)$ for all $u \in J$.

Let us proceed directly to the proof of the lemma. Let a type-2 fuzzy binary relation on the set A with membership function μ be irreflexive and transitive. Due to the irreflexivity of this relationship, for any $x \in A$ and $u \in J$ we have $\mu(x,x,u) = \lambda_{0^{[1]}}(u)$. On the basis of transitivity for all $x,y \in A$ and $u \in J$ the relation $\mu(x,x,u) = \lambda_{0^{[1]}}(u) \sqsupseteq [\mu(x,y,u) \wedge \mu(y,x,u)]$ holds using the normality of all fuzzy values that participate in the last relation, we obtain $[\mu(x,y,u) \wedge \mu(y,x,u)] = \lambda_{0^{[1]}}(u)$. Therefore, based on what was proved above, for any $x,y \in A$ and $u \in J$ the equality $\mu(y,x,u) = \lambda_{0^{[1]}}(u)$ follows from the fulfillment of $\mu(x,y,u) \neq \lambda_{0^{[1]}}(u)$.

The asymmetricity of the relationship μ is established.

Proof of equalities (3). Let us establish the validity of the first of equalities (3) (the second is verified in a similar way). We have:

$$\neg[\mathop{\wedge}_{i\in I}\lambda_i(x,u_i)] = \sup_{\inf_{i\in I} u_i = 1-u}\ \inf_{i\in I}\lambda_i(x,u_i)$$
$$= \sup_{1-\inf_{i\in I} u_i = u}\ \inf_{i\in I}\lambda_i(x,u_i)$$
$$= \sup_{\sup_{i\in I}(1-u_i)=u}\ \inf_{i\in I}[\neg\lambda_i(x,1-u_i)] = \mathop{\vee}_{i\in I}[\neg\lambda_i(x,u_i)].$$

Proof of the Edgeworth–Pareto principle. Choosing an arbitrary set $C(X)$, first we will prove the inclusion $C(X)\subset Ndom(X)$. In accordance with Axiom 1, relation (4) holds. Applying the infimum operation $\mathop{\wedge}_{x'\in X}$ to both sides of it, taking into account the de Morgan rule and upper semicontinuity of the participating fuzzy values, we obtain

$$\lambda_{C(X)}(x,u) \sqsubseteq \mathop{\wedge}_{x'\in X}[\neg\mu_{\succ_X}(x',x,u)]$$
$$= \neg[\mathop{\vee}_{x'\in X}\mu_{\succ_X}(x',x,u)] = \lambda_{Ndom(X)}(x,u),$$
$$\text{for all } x\in X, u\in J$$

i.e., inclusion $C(X)\subset Ndom(X)$.

Taking the transitivity of the relation $\subset$ into account it remains to verify the validity of the inclusion $Ndom(X)\subset P_f(X)$. Recall that the sets in both parts of this inclusion are type-2 fuzzy; therefore, to verify it, one should establish, for example, the equality $[\lambda_{Ndom(X)}\vee\lambda_{P_f(X)}] = \lambda_{P_f(X)}$. The Pareto set is crisp, so for all $x\in X$ $\lambda_{P_f(X)}(x,\cdot) = 0^{[1]}(x,\cdot)$ or $\lambda_{P_f(X)}(x,\cdot) = 1^{[1]}(x,\cdot)$. Consider the first case, i.e., $\lambda_{P_f(X)}(x,\cdot) = 0^{[1]}(x,\cdot)$. Here, $x\notin P_f(X)$, which means that there is such a variant $x'\in X$, that $f_i(x')\geq f_i(x)$, $i=1,2,\ldots,m$; $f(x')\neq f(x)$. Hence, in accordance with the Pareto axiom, we obtain $\mu_{\succ_X}(x',x,\cdot) = 1^{[1]}(\cdot)$. Therefore $\lambda_{Ndom(X)}(x,\cdot) = 0^{[1]}(x,\cdot)$ and the equality $[\lambda_{Ndom(X)}(x,\cdot)\vee\lambda_{P_f(X)}(x,\cdot)] = \lambda_{P_f(X)}(x,\cdot)$ becomes apparent since $0^{[1]}$ is the "smallest" of all fuzzy numbers.

In the second case, i.e., when $\lambda_{P_f(X)}(x,\cdot) = 1^{[1]}(x,\cdot)$, equality $[\lambda_{Ndom(X)}(\cdot)\vee\lambda_{P_f(X)}(\cdot)] = \lambda_{P_f(X)}(\cdot)$ also holds because $1^{[1]}$ is the "largest" of all fuzzy numbers.

Proof of Lemma 2. Let us arbitrarily choose the variants $x,x'\in X$ that satisfy the inequalities $f_i(x')\geq f_i(x)$, $i=1,2,\ldots,m$; $f(x')\neq f(x)$. It is necessary to show that $\mu_{\succ_X}(x',x,\cdot) = 1^{[1]}(\cdot)$. We denote $y = f(x') - f(x)$ and introduce vectors $y^k = f(x) + \sum_{i=1}^{k} y_i e^i$, $k=0,1,\ldots,m$ where e^i is a unit vector of space $\mathbb{R}^m$ and $y^0 = f(x)$. Obviously, $y^m = f(x')$. Using the consistency and transitivity of the relationship $\succ$ on the whole space $\mathbb{R}^m$, we have

$$\mu_{\succ}(y^m,y^0,u) \sqsupseteq \mathop{\wedge}_{i=1}^{m}\mu_{\succ}(y^i,y^{i-1},u) = \mathop{\wedge}_{i=1}^{m}1^{[1]}(u) = 1^{[1]}(u)$$

for all $u\in J$. Hence, $\mu_{\succ}(f(x'),f(x),\cdot) = 1^{[1]}(\cdot)$, which is equivalent to the required equation $\mu_{\succ_X}(x',x,\cdot) = 1^{[1]}(\cdot)$, as $x,x'\in X$.

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