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The Kuramoto–Sivashinsky Equation. A Local Attractor Filled with Unstable Periodic Solutions

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Abstract—A periodic boundary value problem is considered for one of the first versions of the Kuramoto–Sivashinsky equation, which is widely known in mathematical physics. Local bifurcations in a neighborhood of spatially homogeneous equilibrium points are studied in the case when they change stability. It is shown that the loss of stability of homogeneous equilibrium points leads to the occurrence of a two-dimensional local attractor on which all solutions are periodic functions of time, except for one spatially inhomogeneous state. The spectrum of frequencies of this family of periodic solutions fills the entire number line, and all of them are unstable in the sense of Lyapunov's definition in the metric of the phase space (the space of initial conditions) of the corresponding initial boundary value problem. As the phase space, a Sobolev functional space natural for this boundary value problem is chosen. Asymptotic formulas are given for periodic solutions filling the two-dimensional attractor. To analyze the bifurcation problem, analysis methods for an infinite-dimensional dynamical system are used: the integral (invariant) manifold method combined with the methods of the Poincaré normal form theory and asymptotic methods. Analyzing the bifurcations for the periodic boundary value problem is reduced to analyzing the structure of the neighborhood of the zero solution to the homogeneous Dirichlet boundary value problem for the equation under consideration.

Keywords: the Kuramoto–Sivashinsky equation, periodic boundary value problem, local bifurcations, stability, attractor, asymptotic formulas

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INTRODUCTION

One of the most traditional versions of the Kuramoto–Sivashinsky equation (KSE) is dealt with in this paper [1–3]. The periodic boundary value problem (BVP) for this equation is studied. Several papers have been devoted to this problem [4–10]. The problem of local bifurcations has been considered. Most of these papers were based on reducing the BVP to a finite-dimensional dynamical system. As a rule, one version of the Galerkin method was used for such a reduction (see, for example, [5]). Thereafter, the resulting system of ordinary differential equations was analyzed, and local bifurcations for it were studied. Generally, such an analysis led to the establishment of conditions under which either Andronov–Hopf or Turing–Prigogine bifurcations occur, that is, the conditions were established under which cycles or spatially inhomogeneous solutions can be found for a finite-dimensional dynamical system.

An analysis of this problem without using the Galerkin method or other methods for reducing the problem to its finite-dimensional analog revealed the possibility of bifurcation of a two-dimensional local attractor the solutions on which are periodic functions of time, and all these solutions are unstable in the sense of Lyapunov's classical definition. These results were obtained based on strict mathematically grounded methods for analyzing infinite-dimensional dynamical systems. The possibility of such a bifurcation was also noted earlier when studying other dynamical systems [11, 12]. The results presented below formed the basis for the authors' report at the conference "New Trends in Nonlinear Dynamics," which took place from October 5 through October 7, 2017 [13].

1. STATEMENT OF THE MATHEMATICAL PROBLEM

We consider periodic BVP

$$w_\tau + \alpha w_{\xi\xi\xi\xi} + \beta w_{\xi\xi} + 2\gamma w w_\xi = 0, \tag{1}$$

$$w(\tau, \xi + 2H) = w(\tau, \xi), \tag{2}$$

where $\alpha > 0, H > 0, \beta, \gamma \in R$. The differential equations in partial derivatives (1) is usually called the Kuramoto–Sivashinsky equation (see, for example, [1–3]). Changes

$$\xi = \frac{xH}{\pi},\ \tau = \frac{1}{\alpha}\left(\frac{H}{\pi}\right)^4 t,\ w = \left(\frac{\alpha}{\gamma}\left(\frac{\pi}{H}\right)^3\right)u$$

make it possible to rewrite the BVP in a normalized form, thus reducing the number of parameters in the equation. From now on, we will study BVP

$$u_t + u_{xxxx} + bu_{xx} + (u^2)_x = 0, \tag{3}$$

$$u(t, x + 2\pi) = u(t, x), \tag{4}$$

where $b = \frac{\beta}{\alpha}\left(\frac{H}{\pi}\right)^2$. It is assumed in this equation that $(u^2)_x = 2uu_x$.

We note certain properties typical of the solutions to BVPs (3) and (4).

First, $u(t, x) = \text{const}$ is a solution to this problem. Second, if $u(t, x)$ is its solution, then

$$M_0(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t, x)dx = c\ \ (c \in R).$$

Indeed, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} u_{xxx}dx = 0, \frac{1}{2\pi}\int_{-\pi}^{\pi} u_{xx}dx = 0, \frac{1}{2\pi}\int_{-\pi}^{\pi} (u^2)_x dx = 0.$

Therefore, relations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u_t dx = \frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} udx = 0,\ \text{ hold, that is, } \frac{1}{2\pi}\int_{-\pi}^{\pi} udx = c.$$

In BVPs (3) and (4), we put

$$u(t, x) = c + v(t, x),\ \ c \in R,\ \ \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t, x)dx = 0.$$

This change allows us to reduce BVPs (3) and (4) to auxiliary BVP

$$v_t = Av - (v^2)_x, \tag{5}$$

$$v(t, x + 2\pi) = v(t, x), M_0(v) = 0. \tag{6}$$

In BVPs (5) and (6), the linear differential operator (LDO) $A = A(c)$ is defined by equality

$$A(c)v = -v_{xxxx} - bv_{xx} - 2cv_x$$

and depends on the parameter $c \in R$. We emphasize that the value of c is arbitrary. LDO $A(c)$, which is defined on sufficiently smooth functions $p(x)$ satisfying conditions (6), is the generating operator of an analytic semigroup of linear bounded operators in Hilbert space $H_0 : f(x) \in H_0$ if

$$(1)\ f(x + 2\pi) = f(x),\ \ (2)\ f(x) \in L_2(-\pi, \pi),\ \ (3)\ M_0(f) = 0.$$

We let H_k denote the Hilbert space consisting of 2π-periodic functions $f(x)$ having generalized derivatives up to order k that belong to space $L_2(-\pi, \pi)$. A norm in H_k can be defined by equality

$$\|f\|^2_{H_k} = \|f\|^2_{L_2} + \|f'\|^2_{L_2} + \ldots + \|f^{(k)}\|^2_{L_2},\ \ \|f\|^2_{L_2} = \int_{-\pi}^{\pi} f^2(x)dx.$$

Finally, $H_{k,0} \subset H_k$ and consists of functions $f(x) \in H_k$ such that $M_0(f) = 0$. If BVP (3), (4) is supplemented with initial condition

$$u(0, x) = f(x) \in H_{4,0}, \tag{7}$$

combined (initial boundary value) problem (5)–(7) will be locally correctly solvable and its solutions form local semiflow

$$f(x) \to f_t(x) \to u(t,x) \in H_{4,0} \quad \text{for any} \quad t > 0.$$

Note that BVP (5), (6) has a zero-equilibrium point. In particular, the problems of the behavior of solutions to auxiliary BVP (5), (6) with the initial conditions $f(x) \in Q(r) \subset H_{4,0}$. as $t \to \infty$ will be considered in this study. Here, $Q(r)$ denotes a ball of radius r centered at zero of the phase space of solutions to BVP (5), (6) (that is, the ball centered at zero of the Hilbert space $H_{4,0}$)

2. LINEARIZED BOUNDARY VALUE PROBLEM

In this section, we consider a linearized version of BVP (5), (6), that is, BVP

$$v_t = Av, \quad v = v(t,x), \tag{8}$$

$$v(t, x + 2\pi) = v(t,x), M_0(v) = 0, \tag{9}$$

where LDO $A = A(c)$ has been defined in the previous section.

We can verify in a standard manner that BVP

$$-p^{(\mathrm{IV})} - bp'' - 2cp' = \lambda p, \quad p(x + 2\pi) = p(x), \quad M_0(p) = 0$$

has nontrivial solutions

$$p(x) = p_n(x) = \exp(inx), \quad n = \pm 1, \pm 2, \ldots,$$

if $\lambda_n = \lambda_n(c) = -n^4 + bn^2 + i\sigma n, n = \pm 1, \pm 2, \ldots,$ and $\sigma = -2c$.

We emphasize that $\tau_n = \mathrm{Re}\lambda_n(c) = -n^4 + bn^2$ and does not depend on c. In our case, quantity c plays the role of a parameter, and only $\mathrm{Im}\lambda_n(c)$ depends on c. We also note that $\lim\limits_{n\to\infty} \tau_n = -\infty$. The family of eigenfunctions of the LDO under consideration $\exp(inx), n = \pm 1, \pm 2, \pm 3, \ldots,$ forms a complete orthogonal system in separable Hilbert space H_0. These remarks imply the assertion.

Lemma 1. The solutions to BVP (8), (9) are asymptotically stable if $b < 1$ and unstable if $b > 1$. If $b = 1$, the solutions are stable.

The zero solution to nonlinear BVP (5), (6) is asymptotically stable for $b < 1$, and it is unstable for $b > 1$. With $b = 1$, the critical case concerning the stability of the zero solution to the BVP is realized. For this b, we have $\lambda_{\pm}(c) = \pm i\sigma, \sigma = -2c$. This pair of eigenvalues is associated with eigenfunctions $\exp(\pm ix)$. For the other eigenvalues of LDO $A(c)$, inequalities $\mathrm{Re}\lambda_n(c) \le -12, n = \pm 2, \pm 3, \ldots,$ hold.

3. THE MAIN RESULT

We consider nonlinear BVPs (3), (4) and (5), (6) with $b = 1 + \gamma\varepsilon$. In this section, it is convenient to assume that $\gamma = \frac{1}{12}$. This choice of γ is due to the convenience when stating the main result.

Theorem 1. There exists positive constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ BVP (5), (6) for any $c \in R$ has unique stable limit cycle $l(c, \varepsilon)$ belonging to sufficiently small neighborhood $Q(r)$. All solutions with sufficiently small initial conditions approach $l(c, \varepsilon)$ in the sense of the norm of the phase space with an exponential rate. For solutions forming this cycle, we have asymptotic formula

$$v_p(t,x,c,\varepsilon) = \varepsilon^{1/2}\sin(x + \sigma t + \varphi_0) - \frac{\varepsilon}{12}\sin(2x + 2\sigma t + 2\varphi_0) + \frac{\varepsilon^{3/2}}{288}\sin(3x + 3\sigma t + 3\varphi_0) + o(\varepsilon^{3/2}), \tag{10}$$

where $\sigma = -2c, \varphi_0 \in R$.

With $c = 0$, we obtain not a cycle but a one-dimensional invariant manifold filled with spatially inhomogeneous equilibrium points of BVP (5), (6). In the case of general position ($c \neq 0$), the Andronov–Hopf bifurcation theorem is applicable. When analyzing BVP (5), (6), we should take into account its characteristic feature lying in the fact that Eq. (5) depends on parameter c. This problem has a cycle for any $c \neq 0$, and only the period depends on $c: T = T(c) = \frac{\pi}{|c|}$. Theorem 1 can be proven in a rather standard manner. The derivation of (10), as well as the above-mentioned fact about the nature of the dependence on parameter c, require detailed explanation. This will be done in the next section (that is, in Section 4).

Obviously, equality

$$u_p(t,x,c,\varphi_0,\varepsilon) = c + v_p(t,x,c,\varphi_0,\varepsilon) \tag{11}$$

defines a two-parameter family of periodic solutions to BVP (3), (4) if $b = 1 + \frac{\varepsilon}{12}$. The two-parameter family of solutions (11) forms two-dimensional invariant manifold $V_2(\varepsilon)$, which geometrically is the direct product of cycle $l(c,\varepsilon)$ and a line. All other solutions from its small neighborhood approach $V_2(\varepsilon)$ at an exponential rate with an exponent that does not depend on c (recall that $\mathrm{Re}\lambda_n(c)$ does not depend on c). Therefore, $V_2(\varepsilon)$ is a local attractor for solutions to BVP (3), (4).

On the other hand, all solutions $u_p(t,x,c,\varepsilon) \in V_2(\varepsilon)$ are individually unstable in the norm of space H_4. To see this, we consider two distinct solutions from the family of periodic solutions (11), that is, $u_p(t,x,c_1,\varphi_1,\varepsilon)$ and $u_p(t,x,c_2,\varphi_2,\varepsilon)(c_1 \neq c_2)$, and single out the leading parts in the asymptotic representations for these two solutions:

$$u_p(t,x,c_1,\varphi_1,\varepsilon) = w_p(t,x,c_1,\varphi_1,\varepsilon) + o(\varepsilon),\ \ u_p(t,x,c_2,\varphi_2,\varepsilon) = w_p(t,x,c_2,\varphi_2,\varepsilon) + o(\varepsilon),$$

$$w_1(t,x) = w_p(t,x,c_1,\varphi_1,\varepsilon) = c_1 + \varepsilon^{1/2}\sin(x + \sigma_1 t + \varphi_1),\ \ \sigma_1 = -2c_1,$$

$$w_2(t,x) = w_p(t,x,c_2,\varphi_2,\varepsilon) = c_2 + \varepsilon^{1/2}\sin(x + \sigma_2 t + \varphi_2),\ \ \sigma_2 = -2c_2.$$

Put $\Delta w = w_1 - w_2$. Then, we have

$$\|\Delta w\|_{H_4}^2 = \|\Delta w\|_{L_2}^2 + \|(\Delta w)_x\|_{L_2}^2 + \|(\Delta w)_{xx}\|_{L_2}^2 + \|(\Delta w)_{xxx}\|_{L_2}^2 + \|(\Delta w)_{xxxx}\|_{L_2}^2.$$

Direct calculations show that

$$\|\Delta w\|_{L_2}^2 = 2\pi(c_1 - c_2)^2 + 4\varepsilon\pi\sin^2\left(\frac{(\sigma_1 - \sigma_2)t + \varphi_2 - \varphi_1}{2}\right).$$

Calculations for other terms are similar; finally, we arrive at

$$\|\Delta w\|_{H_4}^2 = 2\pi(c_1 - c_2)^2 + 20\varepsilon\pi\sin^2\left((c_2 - c_1)t + \frac{\varphi_2 - \varphi_1}{2}\right).$$

Without loss of generality, we can assume that $c_2 > c_1(c_2 - c_1 > 0)$. We put

$$t_k = \frac{\left(\frac{\pi}{2} + \pi k - \frac{\varphi_2 - \varphi_1}{2}\right)}{c_2 - c_1},\ \ \lim_{k\to\infty} t_k = \infty.$$

With t_k chosen in this way, inequality

$$\|\Delta w\|_{H_4}^2 \geq 20\varepsilon\pi \ \text{ or } \|w\|_{H_4} \geq 2\sqrt{5\pi\varepsilon}$$

holds. Therefore, for sufficiently small $\varepsilon(\varepsilon \in (0,\varepsilon_0))$, inequality

$$\left\|u_p(t_k,x,c_1,\varphi_1,\varepsilon) - u_p(t_k,x,c_2,\varphi_2,\varepsilon)\right\|_{H_4} \geq 2\sqrt{5\pi\varepsilon}$$

is valid. On the other hand, with sufficiently small $\Delta c = c_2 - c_1$ and $\Delta\varphi = \varphi_1 - \varphi_2$, we have

$$\left\|u_p(t_k,x,c_1,\varphi_1,\varepsilon) - u_p(t_k,x,c_2,\varphi_2,\varepsilon)\right\|_{H_4} < \delta,$$

where δ is an arbitrarily small positive constant. The last remark proves that any solution from family (11) is unstable.

4. AUXILIARY BOUNDARY VALUE PROBLEM

In this section, we consider another auxiliary BVP

$$w_t = B(\varepsilon)w - (w^2)_y,\ \ w = w(t,y), \tag{12}$$

$$w(t,0) = w(t,\pi) = w_{yy}(t,0) = w_{yy}(t,\pi) = 0, \tag{13}$$

which is also of independent interest. Here, $y \in [0, \pi]$ and the LDO is $B(\varepsilon)g(y) = -g^{(IV)}(y) - \left(1 + \frac{\varepsilon}{12}\right)g''(y)$, whose domain of definition contains sufficiently smooth functions $g(y)$ satisfying boundary conditions

$$g(0) = g(\pi) = g''(0) = g''(\pi) = 0.$$

The spectrum of LDO $B(\varepsilon)$ consists of a countable set of eigenvalues $\lambda_n(\varepsilon) = -n^4 + \left(1 + \frac{\varepsilon}{12}\right)n^2$, which are associated with eigenfunctions $e_n(y) = \sin ny$. All eigenvalues of this LDO are real and of multiplicity one. In particular, $\lambda_1(\varepsilon) = \frac{\varepsilon}{12}$, that is, $\lambda_1(0) = 0$. Nonlinear BVP (12), (13) has a zero equilibrium point for which a case close to the critical case of the simple zero eigenvalue is realized. In a neighborhood of the zero solution to BVP (12), (13), there exists one-dimensional invariant manifold $M_1(\varepsilon)$. Other solutions in the vicinity of the zero solution to BVP (12), (13), approach $M_1(\varepsilon)$ with time at an exponential rate. The dynamics of solutions on $M_1(\varepsilon)$ is governed by scalar first-order equation (the normal form (NF))

$$\dot{z} = \varepsilon\Psi(z, \varepsilon) = \varepsilon\psi(z) + o(\varepsilon), \quad z = z(t). \tag{14}$$

It is advisable to seek solutions on $M_1(\varepsilon)$ in form

$$w(t, y, \varepsilon) = \varepsilon^{1/2}w_1(z, y) + \varepsilon w_2(z, y) + \varepsilon^{3/2}w_3(z, y) + o(\varepsilon^{3/2}). \tag{15}$$

Here, $z = z(t)$ are solutions to NF (14), $w_1(z, y) = z\sin y$, and functions $w_k(y, z), k = 2, 3, 4, \ldots$, belong to the following class of functions: $f(z, y) \in W$ if

(1) for a fixed z $f(z, y) \in W_2^4[0, \pi]$.

(2) The function has continuous partial derivatives with respect to z and $f_k(0, y) = 0$.

(3) The function satisfies hinge support boundary conditions (13).

(4) Equalities $\frac{2}{\pi}\int_0^\pi f(z, y)\sin y dy = 0$ hold.

Upon substituting sum (15) into BVP (12), (13) and equating the terms at the same powers of $\varepsilon(\varepsilon, \varepsilon^{3/2}, \ldots)$, we obtain inhomogeneous BVPs to determine the terms of (15):

$$B_0w_2 = \Phi_2(z, y), \quad w_2(z, 0) = w_2(z, \pi) = w_{2yy}(z, 0) = w_{2yy}(z, \pi) = 0, \tag{16}$$

$$B_0w_3 = \Phi_3(z, y), \quad w_3(z, 0) = w_3(z, \pi) = w_{3yy}(z, 0) = w_{3yy}(z, \pi) = 0, \tag{17}$$

where the LDO is $B_0w_k = -w_{yyyy} - w_{yy}, k = 2, 3$. The derivatives of $z(t)$ should be calculated using Eq. (14), that is, the NF. Therefore, we have

$$\Phi_2(z, y) = 2(w_1^2)_y = z^2\sin 2y, \quad \Phi_3(z, y) = 2(w_1w_2)_y + \frac{1}{12}(w_1)_{yy} + \psi(z)\sin y.$$

An analysis of the solvability of BVP (16), (17) has shown that

$$w_2 = -\frac{1}{12}z^2\sin 2y, \quad w_3 = \frac{1}{288}z^3\sin 3y, \quad \psi(z) = \frac{1}{12}(z - z^3).$$

We now consider a truncated version of NF (14), namely, ordinary differential equation

$$\dot{z} = \frac{\varepsilon}{12}(z - z^3),$$

which has three equilibrium points: $S_0 : z = 0$; $S_+ : z = 1$; $S_- : z = -1$. Zero equilibrium point S_0 is unstable, while nontrivial equilibrium points S_+ and S_- are asymptotically stable. The following assertion is true.

Theorem 2. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ BVP (12), (13) has two asymptotically stable equilibrium points

$$S_+ : w_+(y, \varepsilon) = \varepsilon^{1/2}\sin y - q_1\varepsilon\sin 2y + q_2\varepsilon^{3/2}\sin 3y + o(\varepsilon^{3/2}),$$

$$S_- : w_-(y, \varepsilon) = -\varepsilon^{1/2}\sin y - q_1\varepsilon\sin 2y - q_2\varepsilon^{3/2}\sin 3y + o(\varepsilon^{3/2}),$$

where $q_1 = \frac{1}{12}$ and $q_2 = \frac{1}{288}$.

Note that equality

$$w_-(y,\varepsilon) = -w_+(\pi - y,\varepsilon)$$

remains valid; that is, physically, we have the same equilibrium point but in another system of coordinates. Therefore, equilibrium point S_+ is dealt with in the sequel.

We extend function $w_+(y,\varepsilon)$ to interval $[-\pi, 0]$ to be an odd function and then to the entire number line to be a periodic function with a period of 2π. Resulting function $v_+(y,\varepsilon)$ is an equilibrium point of BVP

$$v_t + v_{yyyy} + \left(1 + \frac{1}{12}\varepsilon\right)v_{yy} + (v^2)_y = 0, \tag{18}$$

$$v(t, y + 2\pi) = v(t, y), M_0(v) = 0. \tag{19}$$

When verifying this assertion, we use the fact that Eq. (18) is invariant for odd functions. Indeed, if $v(t, y)$ is an odd function of variable y, then all terms in the left-hand side of Eq. (18) preserve this property.

We now consider family of functions $v_p(t, x, c, \varepsilon) = v_+(x + \sigma t + \varphi_0, \varepsilon)$, where $\sigma = -2c$; for each c and φ_0, this family gives a periodic solution to auxiliary BVP (5), (6) with the preservation of stability of equilibrium point S_+ in the sense that periodic solution $v_p(t, x, c, \varepsilon)$ is orbitally asymptotically stable. The last fragment of this section completes the proof of formula (10). We also note that equilibrium point $w_-(y,\varepsilon)$ undoubtedly leads to the same family of periodic solutions. Equilibrium points $v_+(y,\varepsilon)$ and $v_-(y,z)$ of BVP (18), (19) are associated with different representatives of the family of periodic solutions (10). They differ in phase value.

In conclusion, we note that solution $v_p(t, x, c, \varepsilon)$ has a period of $T = 2\pi/|c|$ if $c \neq 0$ and quantity T assumes any value in R_+.

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REFERENCES

1. Kuramoto, Y., *Chemical Oscillations, Waves and Turbulence,* Berlin: Springer, 1984.
2. Sivashinsky, G.I., Weak turbulence in periodic flows, *Phys. D: Nonlinear Phenom.,* 1985, vol. 17, no. 2, pp. 243–255.
3. Akhmediev, N. and Ankevich, A., *Dissipativnye solitony* (Dissipative Solitons), Moscow: Fizmatlit, 2008.
4. Armbruster, D., Guckenheimer, J., and Holmes, P., Kuramoto–Sivashinsky dynamics on the center-unstable manifold, *SIAM J. Appl. Math.,* 1989, vol. 49, no. 3, pp. 676–691.
5. Kevrekidis, I.G., Nicolaenko, B., and Scovel, J.C., Back in the saddle again: A computer assisted study of the Kuramoto–Sivashinsky equation, *SIAM J. Appl. Math.,* 1990, vol. 50, no 3, pp. 760–790.
6. Nicolaenko, B., Scheurer, B., and Temam, R., Some global dynamical properties of the Kuramoto–Sivashinsky equations: Nonlinear stability and attractors, *Phys. D: Nonlinear Phenom.,* 1985, vol. 16, no. 2, pp. 155–183.
7. Changpin, Li and Zhonghua, Y., Bifurcation of two-dimensional Kuramoto–Sivashinsky equation, *Appl. Math. JCU,* 1998, vol. 13, no. 3, pp. 263–270.
8. Kulikov, A.N. and Kulikov, D.A., Formation of wave-like nanostructures on the surface of flat substrates during ion bombardment, *Zh. Vychisl. Mat. Mat. Fiz.,* 2012, vol. 52, no. 5, pp. 800–814.
9. Kulikov, A.N. and Kulikov, D.A., Bifurcations of spatially inhomogeneous solutions in two boundary value problems for the generalized Kuramoto–Sivashinsky equation, *Vestn. Mosk. Inzh.-Fiz. Inst.,* 2014, vol. 3, no. 4, pp. 408–415.
10. Kulikov, A.N. and Kulikov, D.A., Inhomogeneous solutions for a modified Kuramoto–Sivashinsky equation, *J. Math. Sci.,* 2016, vol. 219, no. 2, pp. 173–183.
11. Kulikov, A.N., Attractors of two boundary-value problems for a modified non-linear telegraph equation, *Nelineinaya Din.,* 2008, vol. 4, no. 1, pp. 57–68.
12. Glyzin, S.D. and Kolesov, A.Yu., An example of an attractor consisting of periodic Lyapunov-unstable periodic trajectories, *Model. Anal. Inf. Syst.,* 2008, vol. 15, no. 2, pp. 94–95.
13. Kulikov, A.N. and Kulikov, D.A., The Kuramoto–Sivashinsky equation: The existence of an attractor all solutions on which are unstable, *Tezisy dokladov mezhdunarodnoi nauchnoi konferentsii Novye tendentsii v nelineinoi dinamike* (Abstracts of the International Scientific Conference New Trends in Nonlinear Dynamics), Yaroslavl, 2017, pp. 50–51.

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