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Nonclassical Relaxation Oscillations in Neurodynamics

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Abstract—A modification of the well-known FitzHugh–Nagumo model from neuroscience has been proposed. This model is a singularly perturbed system of ordinary differential equations with a fast variable and a slow variable. The existence and stability of a nonclassical relaxation cycle in this system have been studied. The slow component of the cycle is asymptotically close to a discontinuous function, while the fast component is a δ-like function. A one-dimensional circle of unidirectionally coupled neurons has been considered. The existence of an arbitrarily large number of traveling waves for this chain has been shown. In order to illustrate the increase in the number of stable traveling waves, numerical methods were involved.

Keywords: FitzHugh–Nagumo neuron model, relaxation cycle, asymptotics, stability, buffering

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1. MAIN RESULT

The suggested method of simulating neuron activity is based on idea from [1, 2] about the replacement of a biological neuron with an equivalent generator of electric oscillations. Namely, let us consider a self-generator, the block design of which is shown in Fig. 1. Assume that this generator uses some virtual nonlinear element N, the volt-ampere characteristics $i = f(u)$ is shown in Fig. 2. Two points are essential here. First, in contrast to the tunnel diode characteristic, the function $f(u)$ now has only one extremum (maximum) and approximates to a finite positive limit as $u \to +\infty$. Second, all intersection points of diagrams of functions $i = f(u)$ and $i = (E - u)/R$ (the set of these points is definitely not empty) belong to the incident region of the characteristic $i = f(u)$.

In order to derive the mathematical model of our generator, let us refer to the Ohm and Kirchhoff laws, which imply the following correlations for voltages u_0, u_1 and current i (see Fig. 1):

$$i = C\frac{du_1}{dt} + f(u_1), \quad L\frac{di}{dt} = u_0 - u_1, \quad E - u_0 = Ri. \tag{1}$$

Then, excluding variable u_0 from (1) and assuming that $u_1 = u$, we get the following system for components u, i:

$$C\frac{du}{dt} = i - f(u), \quad L\frac{di}{dt} = E - u - Ri. \tag{2}$$

Finally, in the assumption of the smallness of $\varepsilon = R^2 C/L$, after normalizing $Rt/L \to t$, $v = Ri$ and redesignations $a = E$, $g(u) = Rf(u)$, system of equations (2) is transformed into

$$\varepsilon\dot{u} = v - g(u), \quad \dot{v} = a - u - v, \tag{3}$$

where $0 < \varepsilon \ll 1$, $a = \text{const} > 0$.

Let us consider the properties of function $g(u) \in C^\infty(\mathbb{R})$ from (3) separately. According to the requirements on characteristic $i = f(u)$ described above, we will assume that it satisfies the following restrictions.

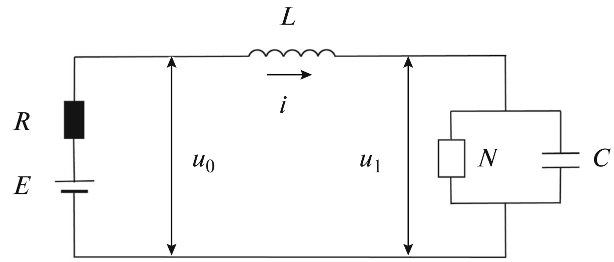

Fig. 1.

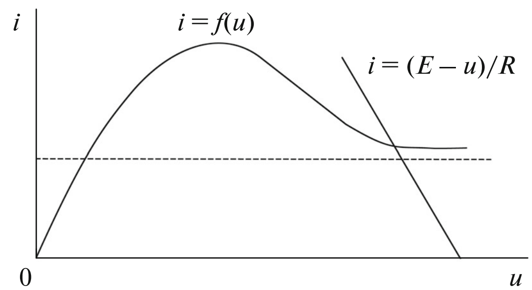

Fig. 2.

Condition 1.1. *There is* $u = u_* > 0$ *such that*

$$g(0) = 0,\ g'(u) > 0 \text{ at } u \in (-\infty, u_*),\ g'(u) < 0 \text{ at } u \in (u_*, +\infty),$$
$$g'(u_*) = 0,\ g''(u_*) < 0,\ a - u_* - g(u_*) > 0. \tag{4}$$

Condition 1.2. *Assume that, as* $u \to +\infty$, *the asymptotic representation*

$$g(u) = \alpha_0 + \sum_{k=1}^{\infty} \frac{\alpha_k}{u^k}, \quad \alpha_0 > 0, \tag{5}$$

takes place, which remains true during differentiation by u any number of times.

An approximate diagram of the function $g(u)$ has the same shape as for case of $f(u)$ (see Fig. 2); the following representation can be taken as a specific one:

$$g(u) = c_1 u \exp(-u) + c_2(1 - \exp(-u)), \quad c_1, c_2 = \text{const} > 0. \tag{6}$$

It can be easily seen here that condition 1.2 is not fulfilled; at the same time, in expansion (5), coefficient α_0 equals to c_2 and all the rest α_k, $k \geq 1$ are zero. Further value u_* taking place in (4) is set by equality $u_* = 1 + c_2/c_1$ and requirement $a - u_* - g(u_*) > 0$ is equivalent to condition

$$[a - u - c_1 u \exp(-u) - c_2(1 - \exp(-u))]\,|_{u=1+c_2/c_1} > 0 \tag{7}$$

on parameters a, c_1, c_2.

Suggested system (3) represents some modification of the well-known FiztHugh–Nagumo model [3] which is denoted as

$$\varepsilon \dot{u} = v + u - u^3/3 + c, \quad \dot{v} = a - u - bv, \tag{8}$$

where $0 < \varepsilon \ll 1$; $a, b = \text{const} > 0$; $c = \text{const} \in \mathbb{R}$. Disadvantages of system (8) include the circumstance that oscillations of component $u = u(t)$ in it do not quite correspond to oscillations of the membrane potential of a real neuron. Actually, the latter are characterized by the presence of short-term and quite high splashes (spikes) that alternate with regions of a slow change in the membrane potential. In the case of system (8), as $\varepsilon \to 0$ and at suitable selection of parameters a, b, and c, the so-called classical relaxation oscillation [4], which does not possess the required properties, is realized.

New system (3) is free of the pointed disadvantage, since it accepts a stable nonclassical relaxation cycle or pulse type cycle. According to the terminology accepted in [5], we will call this cycle $(u,v) = (u_*(t,\varepsilon), v_*(t,\varepsilon))$ of this system of period $T_*(\varepsilon)$, the component of which $v_*(t,\varepsilon)$ converges point by point to some discontinuous function, $T_*(\varepsilon)$ approximates to finite limit $T_* > 0$ and component $u_*(t,\varepsilon)$ changes over time as a δ-like function.

Let us introduce some designations before formulating a strict result about the existence and stability of pulse type cycle. For this purpose, we assume that

$$v_* = g(u_*),\ \ v_{**} = 2\alpha_0 - v_*,\ \ u_{**} = \varphi(v)\big|_{v=v_{**}},\ \ x_{int}^* = v_* - v_{**},\ \ x_{\max}^* = v_* - \alpha_0, \tag{9}$$

where $u = \varphi(v)$, $v \in (-\infty, v_*]$ is the only root of equation $g(u) = v$ from the interval $(-\infty, u_*]$. It is necessary to note that, below, due to the properties (4), (5), inequalities $x_{\text{int}}^* > 0$, $x_{\max}^* > 0$, $u_{**} < u_*$ take place and, additionally, knowingly the positive magnitude

$$T_* = \int\limits_{u_{**}}^{u_*} \frac{g'(u)du}{a - u - g(u)}. \tag{10}$$

Besides constants in (9), (10), below, we will need functions $u_*(t)$, $v_*(t)$, where $u_*(t) = \varphi(v_*(t))$ and $v_*(t)$ is defined from the Cauchy problem

$$\dot{v} = a - \varphi(v) - v,\ \ v|_{t=0} = v_{**}. \tag{11}$$

It can be seen that, as a continuation of the segment $0 \le t \le T_*$ to the whole axis *t,* by the T_*-periodicity law, these functions turn out to be discontinuous at points $t = kT_*$, $k \in \mathbb{Z}$.

Theorem 1.1. *Assume that conditions* 1.1, 1.2 *are fulfilled. Then, there exist a small enough* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon \le \varepsilon_0$, *system* (3) *has an exponentially orbital stable relaxation cycle*

$$\Gamma_*(\varepsilon) = \{(u,v) : u = u_*(t,\varepsilon),\ v = v_*(t,\varepsilon),\ 0 \le t \le T_*(\varepsilon)\} \tag{12}$$

of period $T_*(\varepsilon)$. *At the same time*, $u_*(0,\varepsilon) \equiv u_* + 1$. *The limit equalities are true for this cycle as follows*:

$$\lim_{\varepsilon\to 0} T_*(\varepsilon) = T_*, \tag{13}$$

$$\lim_{\varepsilon\to 0} \int\limits_0^{t_*(\varepsilon)} u_*(t,\varepsilon)dt = x_{\text{int}}^*,\ \ \lim_{\varepsilon\to 0} \max_t(\sqrt{\varepsilon}u_*(t,\varepsilon)) = x_{\max}^*, \tag{14}$$

$$\lim_{\varepsilon\to 0} \max_{\delta_1 \le t \le T_*(\varepsilon)-\delta_2} \left(\left|u_*(t,\varepsilon) - u_*(t)\right| + \left|v_*(t,\varepsilon) - v_*(t)\right|\right) = 0. \tag{15}$$

Here $t_*(\varepsilon) = O(\sqrt{\varepsilon})$ *is the first positive root of equation* $u_*(t,\varepsilon) = u_* + 1$ *and constants* $\delta_1, \delta_2 \in (0, T_*/2)$ *are randomly fixed.*

Figure 3 presents a visual comprehension of the properties of the relaxation cycle of the pulse type, which shows the dependences of its components on time. The aforeentioned diagrams were obtained by numeric integration of system (3) in case (6), (7) at $\varepsilon = 0.005$, $a = 12$, $c_1 = 10$, $c_2 = 3$ (a solid line shows a diagram of $u_*(t+c,\varepsilon)$, dashed line, diagram of $v_*(t+c,\varepsilon)$, where $c \in \mathbb{R}$ – some phase shift).

The proof of theorem 1.1 presented in the next chapter is based on some additional constructions. In order to describe them, let us randomly fix segment Ω, which belongs to the set $(g(u_* + 1), +\infty)$ such that v_* is an internal point of Ω. Below, we introduce the family of solutions $(u,v) = (u(t,v_0,\varepsilon), v(t,v_0,\varepsilon))$, $t \ge 0$ of system (3) with the initial conditions

$$u|_{t=0} = u_* + 1,\ \ v|_{t=0} = v_0 \in \Omega, \tag{16}$$

and designate the first and second positive roots of the equation

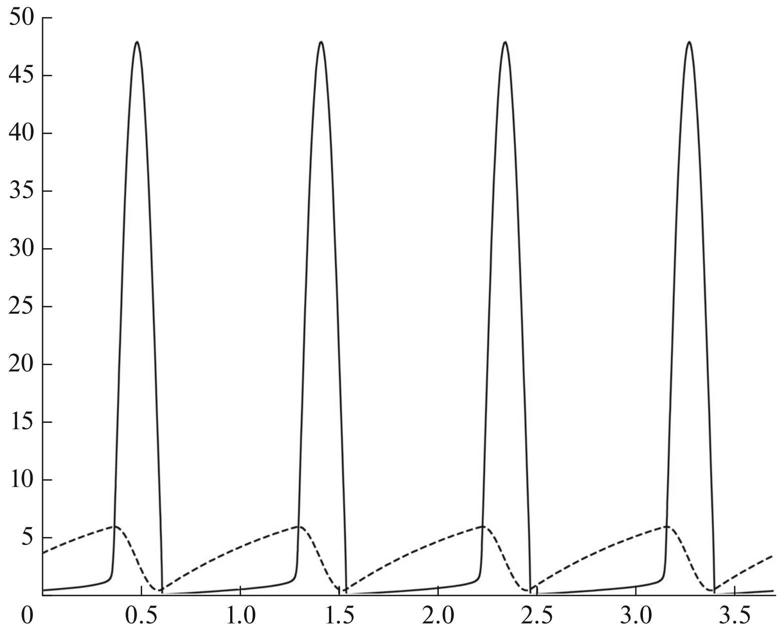

Fig. 3.

$$u(t, v_0, \varepsilon) = u_* + 1 \tag{17}$$

as $t_1(v_0, \varepsilon)$ and $t_2(v_0, \varepsilon)$, correspondingly (if they exist). Finally, let us set the Poincare succession mapping $\Pi_\varepsilon : \Omega \to \mathbb{R}$ using the equality

$$\Pi_\varepsilon(v_0) = v(t, v_0, \varepsilon)\,|_{t=t_2(v_0,\varepsilon)}. \tag{18}$$

The further plan is as follows. First of all, we will show that mapping (18) transforms segment Ω into itself and is a contracting segment. Then, we will make sure that the stable periodic solution (12) of system (3), which corresponds to the only stable point of mapping Π_ε possesses the required asymptotic properties (13)–(15), where constants T_*, x^*_{int}, $x^*_{\max}$ and functions $u_*(t)$, $v_*(t)$ are set by equalities (9)–(11).

2. PROOF OF THEOREM 1.1

Let us consider the curve

$$\Gamma(\varepsilon) = \{(u, v) : u = u(t, v_0, \varepsilon), \quad v = v(t, v_0, \varepsilon), \ 0 \le t \le t_2(v_0, \varepsilon)\}, \tag{19}$$

where $u(t, v_0, \varepsilon)$, $v(t, v_0, \varepsilon)$ are the components of the solution of the Cauchy problem (3), (16). Our nearest goal is to obtain the asymptotic representations for it in different segments of t.

We will start analyzing curve (19) from region $\Gamma_1(\varepsilon)$ corresponding to values $0 \le t \le \tau_1(v_0, \varepsilon)$ where $t = \tau_1(v_0, \varepsilon)$ is the first positive root of the equation

$$u(t, v_0, \varepsilon) = \varepsilon^{-3/8}. \tag{20}$$

Since the moment of time $\tau_1(v_0, \varepsilon)$ is asymptotically small and in the considered interval of the change in the t component $u(t, v_0, \varepsilon)$ monotonously grows from $u_* + 1$ to $\varepsilon^{-3/8}$, we will call region $\Gamma_1(\varepsilon)$ as the takeoff segment. Figure 4 shows approximate shape of $\Gamma_1(\varepsilon)$.

In the statement formulated below, the variable $u \in [u_* + 1, \varepsilon^{-3/8}]$ is taken instead of time t as the parameter in $\Gamma_1(\varepsilon)$.

Lemma 2.1. *Takeoff segment* $\Gamma_1(\varepsilon)$ *is set by the equality*

$$\Gamma_1(\varepsilon) = \{(u, v) : v = \tilde{v}(u, v_0, \varepsilon),\ u_* + 1 \le u \le \varepsilon^{-3/8}\}, \tag{21}$$

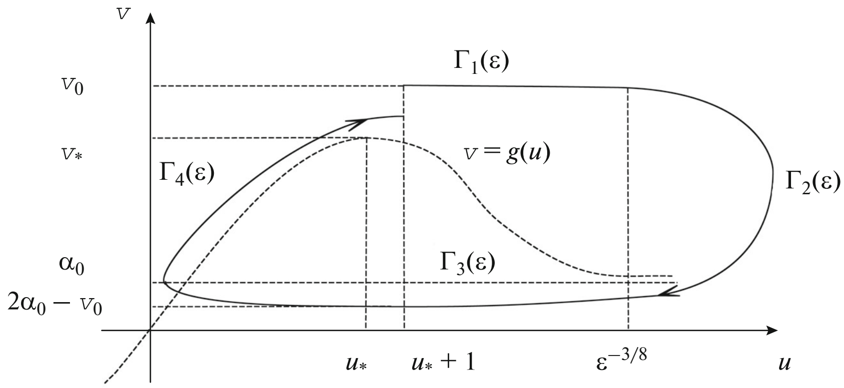

Fig. 4.

where the function $\tilde{v}(u, v_0, \varepsilon)$ *as* $\varepsilon \to 0$ *accepts the following asymptotic representations uniform by* $v_0 \in \Omega$, $u \in [u_* + 1, \varepsilon^{-3/8}]$

$$\tilde{v}(u, v_0, \varepsilon) = v_0 + \varepsilon \int_{u_*+1}^{u} \frac{a - s - v_0}{v_0 - g(s)} ds + O(\varepsilon^2 u^4), \tag{22}$$

$$\frac{\partial \tilde{v}}{\partial v_0}(u, v_0, \varepsilon) = 1 + \varepsilon \frac{\partial}{\partial v_0} \left(\int_{u_*+1}^{u} \frac{a - s - v_0}{v_0 - g(s)} ds \right) + O(\varepsilon^2 u^4), \tag{23}$$

$$\frac{\partial \tilde{v}}{\partial u}(u, v_0, \varepsilon) = \varepsilon \frac{a - u - v_0}{v_0 - g(u)} + O(\varepsilon^2 u^3). \tag{24}$$

Proof. Obviously (3), (16) imply that, on curve $\Gamma_1(\varepsilon)$ variable v as a function of u must satisfy the Cauchy problem

$$\frac{dv}{du} = \varepsilon F(u, v), \quad v|_{u=u_*+1} = v_0 \tag{25}$$

with the right-hand side

$$F(u, v) = \frac{a - u - v}{v - g(u)}. \tag{26}$$

Let us perform some preliminary discussion in order to define asymptotic properties of this problem.

Let us fix segment $\Omega' \subset (g(u_* + 1), +\infty)$, which includes segment Ω. Properties (4), (5) imply that, in this case,

$$v - g(u) > 0, \;\; v - \alpha_0 > 0 \;\; \forall u \geq u_* + 1, \;\; \forall v \in \Omega'. \tag{27}$$

Then, let us consider the set

$$\Sigma = \{(u, v) : u_* + 1 \leq u \leq \varepsilon^{-3/8}, \; v \in \Omega'\}. \tag{28}$$

Based on explicit view (26) of function F and correlations (4), (5), (27), it can easily be seen that

$$|F(u, v)| \leq M_1 u, \;\; |F'_v(u, v)| \leq M_2 u, \;\; |F''_{vv}(u, v)| \leq M_3 u \;\; \forall (u, v) \in \Sigma. \tag{29}$$

Here and below, M, M_1, M_2, etc. stand for various universal positive constants whose precise values are not essential. In particular constants M_1, M_2, M_3, in (29) do not depend on u, v, ε, but do depend on selection of segment Ω' from (28).

Besides set (28) we will also need Banach space X. Its elements are represented by functions $v(u, v_0)$, which are continuous by $(u, v_0) \in [u_* + 1, \varepsilon^{-3/8}] \times \Omega$ including partial derivative $\partial v / \partial v_0$. The norm in X is set by the equality

$$\|v\| = \max_{(u, v_0) \in [u_* + 1, \varepsilon^{-3/8}] \times \Omega} \left(|v(u, v_0)| + \left| \frac{\partial v}{\partial v_0}(u, v_0) \right| \right).$$

Let us refer directly to Cauchy problem (25) and will seek its solution $v = \tilde{v}(u, v_0, \varepsilon)$ as a stationary point of operator

$$\mathcal{L}(v)(u, v_0) = v_0 + \varepsilon \int_{u_*+1}^{u} F(s, v(s, v_0)) ds, \tag{30}$$

defined in a set of functions

$$\left\{ v(u, v_0) : v(u, v_0) \in \Omega', \left| \frac{\partial v}{\partial v_0}(u, v_0) \right| \le 2 \right\} \subset X \tag{31}$$

and taking values in X. A simple check shows that, for any elements $v(u, v_0)$, $v_1(u, v_0)$, $v_2(u, v_0)$ from definition domain (31) of operator (30), the following estimations are true:

$$|\mathcal{L}(v)(u, v_0) - v_0| \le \frac{\varepsilon}{2} M_1 u^2, \quad \left| \frac{\partial}{\partial v_0} L(v)(u, v_0) - 1 \right| \le \varepsilon M_2 u^2,$$

$$|\mathcal{L}(v_1)(u, v_0) - \mathcal{L}(v_2)(u, v_0)| \le \varepsilon M_2 \int_{u_*+1}^{u} s |v_1(s, v_0) - v_2(s, v_0)| ds,$$

$$\left| \frac{\partial}{\partial v_0} \mathcal{L}(v_1)(u, v_0) - \frac{\partial}{\partial v_0} \mathcal{L}(v_2)(u, v_0) \right| \le \varepsilon M_2 \int_{u_*+1}^{u} s \left| \frac{\partial v_1}{\partial v_0}(s, v_0) - \frac{\partial v_2}{\partial v_0}(s, v_0) \right| ds$$

$$+ \ 2\varepsilon M_3 \int_{u_*+1}^{u} s |v_1(s, v_0) - v_2(s, v_0)| ds \quad \forall u \in [u_* + 1, \varepsilon^{-3/8}], \ \forall v_0 \in \Omega,$$

where M_1, M_2, M_3 are constants from (29). This obviously implies the following facts that take place for all small enough $\varepsilon > 0$.

First, operator $\mathcal{L}$ transforms set (31) into itself and is a contracting operator (with contraction constant of order about $\varepsilon^{1/4}$). Second, the following asymptotic representations are uniform by $u \in [u_* + 1, \varepsilon^{-3/8}]$ and $v_0 \in \Omega$ are true for its stationary point $\tilde{v}(u, v_0, \varepsilon)$, the existence and singleness of which are guaranteed by the principle of contracting mappings as follows:

$$\tilde{v}(u, v_0, \varepsilon) = v_0 + O(\varepsilon u^2), \quad \frac{\partial \tilde{v}}{\partial v_0}(u, v_0, \varepsilon) = 1 + O(\varepsilon u^2), \varepsilon \to 0. \tag{32}$$

Finally, let us refer to the equality

$$\tilde{v}(u, v_0, \varepsilon) = v_0 + \varepsilon \int_{u_*+1}^{u} F(s, \tilde{v}(s, v_0, \varepsilon)) ds, \tag{33}$$

as well as to equalities obtained from (33) based on differentiation by u, v_0. Then, let us substitute them into right parts of equations (32) and repeatedly expand the obtained expressions by ε. As a result, we get the required asymptotic representations (22)–(24). Lemma 2.1 has been proved.

Finalizing the consideration of the takeoff segment, it is necessary to note that values $t \in [0, \tau_1(v_0, \varepsilon)]$ correspond to it, where $\tau_1(v_0, \varepsilon)$ is the first positive root of Eq. (20). Formulas (22)–(24) imply that the following asymptotic representation, which is uniform by $v_0 \in \Omega$, is true for this root:

$$\tau_1(v_0,\varepsilon) \overset{\text{def}}{=} \varepsilon \int\limits_{u_*+1}^{\varepsilon^{-3/8}} \frac{ds}{\tilde{v}(s,v_0,\varepsilon) - g(s)} = \frac{\varepsilon^{5/8}}{v_0 - \alpha_0} + O(\varepsilon^{7/8}). \tag{34}$$

Now let us consider another segment $\Gamma_2(\varepsilon)$ of curve (19) that corresponds to values $t \in [\tau_1(v_0,\varepsilon),\ \tau_2(v_0,\varepsilon)]$, where $\tau_2(v_0,\varepsilon)$ is the second positive root of Eq. (20). This segment is characterized by the fact that it lies entirely in semi-plane $\{(u,v) : u \geq \varepsilon^{-3/8}\}$ and, when moving by it for about time $\sqrt{\varepsilon}$, point (u, v) first leaves line $u = \varepsilon^{-3/8}$, then returns back to it again (see Fig. 4). In connection with this, we will call this segment a turn segment and take variable v as a parameter on it.

Lemma 2.2. *The following equality takes place for turn segment* $\Gamma_2(\varepsilon)$ *as follows*:

$$\Gamma_2(\varepsilon) = \{(u,v) : u = \sqrt{x(v,v_0,\varepsilon)/\varepsilon},\ \bar{v}(v_0,\varepsilon) \leq v \leq \bar{\bar{v}}(v_0,\varepsilon)\}, \tag{35}$$

where

$$x(v,v_0,\varepsilon)\,|_{v=\bar{v}(v_0,\varepsilon),\bar{\bar{v}}(v_0,\varepsilon)} = \varepsilon^{1/4}, \quad x(v,v_0,\varepsilon) > \varepsilon^{1/4} \text{ at } v \in (\bar{v}(v_0,\varepsilon),\bar{\bar{v}}(v_0,\varepsilon)). \tag{36}$$

In addition, as $\varepsilon \to 0$*, the following asymptotic representations uniform by* $v_0 \in \Omega$ *are true for the functions* $\bar{v}(v_0,\varepsilon)$, $\bar{\bar{v}}(v_0,\varepsilon)$:

$$\bar{v}(v_0,\varepsilon) = 2\alpha_0 - v_0 + \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)} + O(\sqrt{\varepsilon}),\quad \frac{\partial \bar{v}}{\partial v_0}(v_0,\varepsilon) = -1 - \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)^2} + O(\varepsilon^{3/8}), \tag{37}$$

$$\bar{\bar{v}}(v_0,\varepsilon) = v_0 - \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)} + O(\sqrt{\varepsilon}),\quad \frac{\partial \bar{\bar{v}}}{\partial v_0}(v_0,\varepsilon) = 1 + \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)^2} + O(\sqrt{\varepsilon}), \tag{38}$$

and representations uniform by $v_0 \in \Omega$, $v \in [\bar{v}(v_0,\varepsilon),\bar{\bar{v}}(v_0,\varepsilon)]$ *for the function* $x(v,v_0,\varepsilon)$:

$$x(v,v_0,\varepsilon) = \gamma(v,v_0) + O(\sqrt{\varepsilon}),\quad \frac{\partial x}{\partial v}(v,v_0,\varepsilon) = \frac{\partial \gamma}{\partial v}(v,v_0) + O(\varepsilon^{3/8}), \tag{39}$$

$$\frac{\partial x}{\partial v_0}(v,v_0,\varepsilon) = \frac{\partial \gamma}{\partial v_0}(v,v_0) + O(\varepsilon^{3/8}), \tag{40}$$

where $\gamma(v,v_0) = (v_0 - v)(v - 2\alpha_0 + v_0)$.

Proof. First of all, it is necessary to note that the function $v = \bar{\bar{v}}(v_0,\varepsilon)$ in (35) is defined using the equalities

$$\bar{\bar{v}}(v_0,\varepsilon) = \tilde{v}(u,v_0,\varepsilon)\big|_{u=\varepsilon^{-3/8}},\quad \frac{\partial \bar{\bar{v}}}{\partial v_0}(v_0,\varepsilon) = \frac{\partial \tilde{v}}{\partial v_0}(u,v_0,\varepsilon)\bigg|_{u=\varepsilon^{-3/8}}, \tag{41}$$

where $\tilde{v}(u,v_0,\varepsilon)$ is the function from (21). As for formulas (38), they are obtained from (41) after the substitution of correlations (22), (23) and repeated expansion by ε.

In order to find the remaining functions $x(v,v_0,\varepsilon)$, $\bar{v}(v_0,\varepsilon)$ let us refer to system (3), make replacement $u = \sqrt{x/\varepsilon}$ in it, and repeatedly expand its right-hand sides by ε taking into account property (5). As a result, we make sure that $x = x(v,v_0,\varepsilon)$ satisfies Cauchy problem in the form

$$\frac{dx}{dv} = -2(v-\alpha_0) + \Phi(x,v,\varepsilon),\quad x|_{v=\bar{\bar{v}}(v_0,\varepsilon)} = \varepsilon^{1/4}. \tag{42}$$

At the same time, we will omit the explicit expression for function $\Phi(x,v,\varepsilon)$, since it will not be needed further. However, it is necessary to note that, at $\forall (x,v) \in \Sigma$, where

$$\Sigma = \{(x,v) : x \geq d\varepsilon^{1/4}, v \in J\},\quad d = \text{const} \in (0,1), \tag{43}$$

$$J = [2\alpha_0 - v_0 + c_1\varepsilon^{1/4}, v_0 - c_2\varepsilon^{1/4}],\quad c_1, c_2 = \text{const} > 0, \tag{44}$$

$$0 < c_1, c_2 < \min_{v_0 \in \Omega} \frac{1}{2(v_0-\alpha_0)}, \tag{45}$$

the following estimations take place:

$$|\Phi(x,v,\varepsilon)| \le M_1\sqrt{\frac{\varepsilon}{x}},\quad \left|\Phi'_x(x,v,\varepsilon)\right| \le M_2\frac{\sqrt{\varepsilon}}{x^{3/2}}. \tag{46}$$

Properties (46) indicate that problem (42) represents a regular perturbation of the system

$$\frac{dx}{dv} = -2(v-\alpha_0),\quad x|_{v=\bar{\bar{v}}(v_0,\varepsilon)} = \varepsilon^{1/4}.$$

In this way, the solution $x(v,v_0,\varepsilon)$ to this problem at values $v \in J$ for which

$$(x(v,v_0,\varepsilon),v) \in \Sigma, \tag{47}$$

accepts asymptotic representations

$$x(v,v_0,\varepsilon) = \varepsilon^{1/4} + (\bar{\bar{v}}(v_0,\varepsilon)-\alpha_0)^2 - (v-\alpha_0)^2 + O(\sqrt{\varepsilon}), \tag{48}$$

$$\begin{gathered}\frac{\partial x}{\partial v}(v,v_0,\varepsilon) = -2(v-\alpha_0) + O(\varepsilon^{3/8}),\\ \frac{\partial x}{\partial v_0}(v,v_0,\varepsilon) = 2(\bar{\bar{v}}(v_0,\varepsilon)-\alpha_0)\frac{\partial\bar{\bar{v}}}{\partial v_0}(v_0,\varepsilon) + O(\varepsilon^{3/8}).\end{gathered} \tag{49}$$

Formula (48) should be explained in more detail. Actually, properties (46) directly imply that

$$x(v,v_0,\varepsilon) = \varepsilon^{1/4} + (\bar{\bar{v}}(v_0,\varepsilon)-\alpha_0)^2 - (v-\alpha_0)^2 + O(\varepsilon^{3/8}).$$

Further substituting the obtained equality into the right-hand side of the integral equation

$$x(v,v_0,\varepsilon) = \varepsilon^{1/4} + (\bar{\bar{v}}(v_0,\varepsilon)-\alpha_0)^2 - (v-\alpha_0)^2 + \int\limits_{\bar{\bar{v}}(v_0,\varepsilon)}^{v}\Phi(x(s,v_0,\varepsilon),s,\varepsilon)ds$$

and taking estimations (46) into account again, we obtained the specified asymptotic representation (48).

It is necessary to note that correlations (48), (49) are transformed to the required form (39), (40). In order to confirm this, one should substitute already established asymptotic formulas for $\bar{\bar{v}}(v_0,\varepsilon)$ (see (38)) into (48), (49) and repeatedly expand the results by ε.

Now let us ask a question about the exact values of v at which prior condition (47) is fulfilled. In connection with this, one should note that the explicit form of function $\gamma(v,v_0)$ and prior information (48), (49) mentioned above imply the existence of such a root $v = \bar{v}(v_0,\varepsilon)$ of equation $x(v,v_0,\varepsilon) = \varepsilon^{1/4}$ that, first, are true at equalities (37) uniformly by $v_0 \in \Omega$; second, the following estimate required in (36) takes place in the interval $\bar{v}(v_0,\varepsilon) < v < \bar{\bar{v}}(v_0,\varepsilon)$ as follows:

$$x(v,v_0,\varepsilon) > \varepsilon^{1/4}. \tag{50}$$

In addition, due to the selection of c_1, c_2 (see (45)) points $v = \bar{v}(v_0,\varepsilon)$, $v = \bar{\bar{v}}(v_0,\varepsilon)$ belong to segment (44).

Inequality (50) and estimate $d < 1$ (see (43)) guarantee the fulfillment of condition (47) on the set $[\bar{v}(v_0,\varepsilon),\bar{\bar{v}}(v_0,\varepsilon)] \subset J$. Moreover, the decrease in the constant d in (43) if necessary may lead to the fulfillment of this condition on the whole segment J. In this way at $v \in J$ both equalities (39), (40) and derived correlations (37), (50) gain validity. Lemma 2.2 is proved.

Finalizing the study of turn segment $\Gamma_2(\varepsilon)$, it is necessary to note that the second positive root $\tau_2(v_0,\varepsilon)$ of Eq. (20) mentioned above is set by the equality

$$\tau_2(v_0,\varepsilon) \overset{\text{def}}{=} \tau_1(v_0,\varepsilon) + \sqrt{\varepsilon}\int\limits_{\bar{v}(v_0,\varepsilon)}^{\bar{\bar{v}}(v_0,\varepsilon)}\frac{ds}{\sqrt{x(s,v_0,\varepsilon)} - \sqrt{\varepsilon}(a-s)}. \tag{51}$$

The following asymptotic representation uniform by $v_0 \in \Omega$, $v \in J$ takes place due to formulas (37)–(39):

$$\frac{\sqrt{\varepsilon}}{\sqrt{x(v,v_0,\varepsilon)} - \sqrt{\varepsilon}(a-v)} = \frac{\sqrt{\varepsilon}}{\sqrt{\gamma(v,v_0)}} + O\left(\frac{\varepsilon}{\gamma^{3/2}(v,v_0)}\right). \tag{52}$$

Then, substituting equalities (34), (52) into (51) and taking into account

$$\varepsilon \int\limits_{\bar{v}(v_0,\varepsilon)}^{\bar{\bar{v}}(v_0,\varepsilon)} \frac{ds}{\gamma^{3/2}(s,v_0)} = O(\varepsilon^{7/8}),$$

we finally get

$$\tau_2(v_0,\varepsilon) = \sqrt{\varepsilon} \int\limits_{2\alpha_0 - v_0}^{v_0} \frac{ds}{\sqrt{\gamma(s,v_0)}} + O(\varepsilon^{5/8}) = \sqrt{\varepsilon}\pi + O(\varepsilon^{5/8}). \tag{53}$$

The next segment $\Gamma_3(\varepsilon)$ of curve (19) corresponds to values of t from the time interval $[\tau_2(v_0,\varepsilon), t_1(v_0,\varepsilon)]$, where $t_1(v_0,\varepsilon)$ is the first positive root of Eq. (17) (its existence should also be proved). Then, it will be shown that the length of this segment is asymptotically small and, at the same time component $u(t,v_0,\varepsilon)$, monotonously decreases from $\varepsilon^{-3/8}$ to value $u_* + 1$ (see Fig. 4). In this way, segment $\Gamma_3(\varepsilon)$ can be called a *return segment*. As in the case with $\Gamma_1(\varepsilon)$, we will take variable u to be a parameter in it.

Lemma 2.3. *Representation similar to* (21) *is true for return segment* $\Gamma_3(\varepsilon)$

$$\Gamma_3(\varepsilon) = \{(u,v) : v = \tilde{\tilde{v}}(u,v_0,\varepsilon),\ u_* + 1 \le u \le \varepsilon^{-3/8}\}. \tag{54}$$

Here, function $\tilde{\tilde{v}}(u,v_0,\varepsilon)$ *is such that, first,*

$$\tilde{\tilde{v}}(u,v_0,\varepsilon)\big|_{u=\varepsilon^{-3/8}} = \bar{v}(v_0,\varepsilon); \tag{55}$$

second, as $\varepsilon \to 0$*, the following asymptotic equalities that are uniform by* $v_0 \in \Omega$, $u \in [u_* + 1,\ \varepsilon^{-3/8}]$ *take place*:

$$\tilde{\tilde{v}}(u,v_0,\varepsilon) = 2\alpha_0 - v_0 + O(\varepsilon^{1/4}),\quad \frac{\partial \tilde{\tilde{v}}}{\partial v_0}(u,v_0,\varepsilon) = -1 + O(\varepsilon^{1/4}),\quad \frac{\partial \tilde{\tilde{v}}}{\partial u}(u,v_0,\varepsilon) = O(\varepsilon u). \tag{56}$$

Proof. Since the proof of lemma 2.3 is mostly similar to that of lemma 2.1, let us omit some technical details.

In this case, function $v = \tilde{\tilde{v}}(u,v_0,\varepsilon)$ in (54), (55) is defined from the similar Cauchy problem (25)

$$\frac{dv}{du} = \varepsilon F(u,v),\quad v|_{u=\varepsilon^{-3/8}} = \bar{v}(v_0,\varepsilon), \tag{57}$$

where $F(u,\ v)$ is function (26). To study it, we need the set

$$\Sigma = \{(u,v) : u_* + 1 \le u \le \varepsilon^{-3/8}, v \in \Omega''\}, \tag{58}$$

similar to (28), where Ω'' is the image of segment Ω' under the influence of mapping $v_0 \to 2\alpha_0 - v_0$. Correlations (4), (5) imply that inequalities $v - g(u) < 0$, $v - \alpha_0 < 0$ similar to (27) take place on set (58) and consequently estimations in the form (29) hold true.

The following discussion repeats corresponding part of proof for lemma 2.1. Namely, in this case, the following operator is considered instead of (30) in the same Banach space X:

$$\mathcal{L}(v)(u,v_0) = \bar{v}(v_0,\varepsilon) + \varepsilon \int\limits_{\varepsilon^{-3/8}}^{u} F(s, v(s,v_0))ds.$$

Based on estimates (29) and formulas (37), we make sure that, first, this operator maps a similar set of functions (30)

$$\left\{ v(u,v_0) : v(u,v_0) \in \Omega'', \left| \frac{\partial v}{\partial v_0}(u,v_0) \right| \le 2 \right\} \subset X$$

into itself and is a contracting operator. Second, the following asymptotic equalities uniform by $v_0 \in \Omega$, $u \in [u_* + 1, \varepsilon^{-3/8}]$ take place for its stationary point $v = \tilde{\tilde{v}}(u,v_0,\varepsilon)$:

$$\tilde{\tilde{v}}(u,v_0,\varepsilon) = \bar{v}(v_0,\varepsilon) + O(\varepsilon^{1/4}),\quad \frac{\partial \tilde{\tilde{v}}}{\partial v_0}(u,v_0,\varepsilon) = \frac{\partial \bar{v}}{\partial v_0}(v_0,\varepsilon) + O(\varepsilon^{1/4}), \frac{\partial \tilde{\tilde{v}}}{\partial u}(u,v_0,\varepsilon) = O(\varepsilon u),$$

which imply the required formulas (56) due to (37). Lemma 2.3 is proved.

In addition to the established lemma, it is necessary to note that, as $\varepsilon \to 0$, the first positive root $t = t_1(v_0, \varepsilon)$ of Eq. (17) accepts an asymptotic representation uniform by $v_0 \in \Omega$ due to (53), (56) as follows:

$$t_1(v_0,\varepsilon) \stackrel{\text{def}}{=} \tau_2(v_0,\varepsilon) + \varepsilon \int_{\varepsilon^{-3/8}}^{u_*+1} \frac{ds}{\tilde{\tilde{v}}(s,v_0,\varepsilon) - g(s)} = \tau_2(v_0,\varepsilon) + O(\varepsilon^{5/8}) = \sqrt{\varepsilon}\pi + O(\varepsilon^{5/8}). \quad (59)$$

We will call the remaining segment $\Gamma_4(\varepsilon)$ of curve (19), which completely belongs to the semi-plane $\{(u,v) : u \le u_* + 1\}$ (see Fig. 4), a *segment of classical relaxation oscillations*. This name is motivated by the circumstance that its asymptotics is well known and is thoroughly described in monographs [4, 5], the results of which have already become classical. Thus, here, we will consider only minimal required information about asymptotic behavior of components $u(t,v_0,\varepsilon)$, $v(t,v_0,\varepsilon)$ of the solution of the Cauchy problem (3), (16) at $t \ge t_1(v_0,\varepsilon)$.

At the beginning, a fall to a stable segment of slow motions curve $v = g(u)$ takes place during a time of about $\varepsilon\ln(1/\varepsilon)$. Meanwhile, point $(u,v) = (u(t,v_0,\varepsilon), v(t,v_0,\varepsilon))$, which moves in an asymptotically small neighborhood (about $\varepsilon^{1/4}$) of the segment $\{(u,v) : v = 2\alpha_0 - v_0, \varphi(2\alpha_0 - v_0) \le u \le u_* + 1\}$, which gets into the $\varepsilon^{1/4}$-neighborhood of point $(u,v) = (\varphi(2\alpha_0 - v_0), 2\alpha_0 - v_0)$ (here, $\varphi(v)$ is the function from (9), (11)).

After falling, the so-called slow motion phase begins, which lasts during a time of about 1. More precisely on any segment $\delta_1 \le t \le T(v_0) - \delta_2$, where

$$T(v_0) = \int_{\varphi(2\alpha_0 - v_0)}^{u_*} \frac{g'(u)du}{a - u - g(u)}, \quad 0 < \delta_1, \delta_2 < \min_{v_0 \in \Omega}(T(v_0)/2), \quad (60)$$

and the limit equality

$$\lim_{\varepsilon\to 0} \max_{\substack{\delta_1 \le t \le T(v_0) - \delta_2 \\ v_0 \in \Omega}} \left(|u(t,v_0,\varepsilon) - u(t,v_0)| + \left| \frac{\partial u}{\partial v_0}(t,v_0,\varepsilon) - \frac{\partial u}{\partial v_0}(t,v_0) \right| \right.$$
$$\left. + |v(t,v_0,\varepsilon) - v(t,v_0)| + \left| \frac{\partial v}{\partial v_0}(t,v_0,\varepsilon) - \frac{\partial v}{\partial v_0}(t,v_0) \right| \right) = 0 \quad (61)$$

is fulfilled.

Here, $u(t,v_0) = \varphi(v(t,v_0))$, and $v(t,v_0)$ is the solution of the Cauchy problem similar to (11)

$$\dot{v} = a - \varphi(v) - v, \quad v|_{t=0} = 2\alpha_0 - v_0. \quad (62)$$

At the end of the slow motion segment phase point $(u,v) = (u(t,v_0,\varepsilon), v(t,v_0,\varepsilon))$ gets into an asymptotically small (about $\varepsilon^{2/3}$) neighborhood of the segment $\{(u,v) : u_* \le u \le u_* + 1, v = v_*\}$. The time of passing this segment is asymptotically small.

Combining lemmas 2.1–2.3 with results from [4, 5] briefly described above, we come to the following conclusions. First, the second positive root $t = t_2(v_0,\varepsilon)$ of Eq. (17) exists and accepts an asymptotic that is uniform by $v_0 \in \Omega$

$$t_2(v_0,\varepsilon) = T(v_0) + O(\varepsilon^{1/4}), \quad (63)$$

where $T(v_0)$ is a function from (60). Second, to map (18), we have

$$\max_{v_0\in\Omega} \left(|\Pi_\varepsilon(v_0) - v_*| + |\Pi'_\varepsilon(v_0)| \right) \le M\varepsilon^{2/3} \quad (64)$$

(here, prime means derivative by v_0).

Let us make an intermediate conclusion. Estimate (64) attests that mapping (18) transfers segment Ω into itself and is a contracting segment. Therefore, it accepts the single stationary point

$$v_0 = v_*(\varepsilon), \quad v_*(\varepsilon) = v_* + O(\varepsilon^{2/3}), \quad (65)$$

which, in initial system (3), corresponds to exponentially orbital stable cycle (12) of the period

$$T_*(\varepsilon) = t_2\,(v_0,\varepsilon)\big|_{v_0=v_*(\varepsilon)}\,. \tag{66}$$

As for limit equalities (13), (15) and second limit correlation from (14), they are obvious consequences of formulas (35), (39), (60)–(66).

In order to finalize the proof of theorem 1.1, one must make sure that the limit equality from (14) is true. In connection with this, we need the following statement.

Lemma 2.4. *As* $\varepsilon \to 0$*, the following asymptotic representation is uniformly fulfilled by* $v_0 \in \Omega$:

$$\int\limits_0^{t_1(v_0,\varepsilon)} u(t,v_0,\varepsilon)dt = 2(v_0-\alpha_0) + O(\sqrt{\varepsilon}). \tag{67}$$

Proof. Let us designate integrals of function $u(t,v_0,\varepsilon)$ by time intervals $0 \le t \le \tau_1(v_0,\varepsilon)$, $\tau_1(v_0,\varepsilon) \le t \le \tau_2(v_0,\varepsilon)$ and $\tau_2(v_0,\varepsilon) \le t \le t_1(v_0,\varepsilon)$ as $I_k(v_0,\varepsilon)$, $k = 1, 2, 3$, correspondingly. According to formulas (21)–(24), for the first one we have

$$I_1(v_0,\varepsilon) = \varepsilon\int\limits_{u_*+1}^{\varepsilon^{-3/8}} \frac{u\,du}{\tilde{v}(u,v_0,\varepsilon)-g(u)} = \varepsilon\int\limits_{u_*+1}^{\varepsilon^{-3/8}} u\left(\frac{1}{v_0-g(u)} + O(\varepsilon u^2)\right)du = \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)} + O(\sqrt{\varepsilon}). \tag{68}$$

Further correlations (35)–(40) lead to the equalities

$$\begin{aligned} I_2(v_0,\varepsilon) &= \int\limits_{\bar{v}(v_0,\varepsilon)}^{\bar{\bar{v}}(v_0,\varepsilon)} \frac{\sqrt{x(v,v_0,\varepsilon)}}{\sqrt{x(v,v_0,\varepsilon)}-\sqrt{\varepsilon}(a-v)}dv = \int\limits_{\bar{v}(v_0,\varepsilon)}^{\bar{\bar{v}}(v_0,\varepsilon)} \left(1 + O\left(\frac{\sqrt{\varepsilon}}{\sqrt{\gamma(v,v_0)}}\right)\right)dv \\ &= \bar{\bar{v}}(v_0,\varepsilon) - \bar{v}(v_0,\varepsilon) + O(\sqrt{\varepsilon}) = 2(v_0-\alpha_0) - \frac{\varepsilon^{1/4}}{v_0-\alpha_0} + O(\sqrt{\varepsilon}), \end{aligned} \tag{69}$$

and formulas (54)–(56) imply that

$$\begin{aligned} I_3(v_0,\varepsilon) &= \varepsilon\int\limits_{u_*+1}^{\varepsilon^{-3/8}} \frac{u\,du}{g(u)-\tilde{\bar{v}}(u,v_0,\varepsilon)} = \varepsilon\int\limits_{u_*+1}^{\varepsilon^{-3/8}} u\left(\frac{1}{g(u)-2\alpha_0+v_0} + O(\varepsilon^{1/4})\right)du \\ &= \frac{\varepsilon^{1/4}}{2(v_0-\alpha_0)} + O(\sqrt{\varepsilon}). \end{aligned} \tag{70}$$

Finally, combining obtained correlations (68)–(70), we get the required asymptotic representation (67). Lemma 2.4 has been proved.

Returning to proof of theorem 1.1, it is necessary to note that moment of time $t_*(\varepsilon)$ in (14) is set by formula $t_*(\varepsilon) = t_1(v_0,\varepsilon)\,|_{v_0=v_*(\varepsilon)}$ and, due to (59), (65), it is on the order of $\sqrt{\varepsilon}$. As for the first limit equality (14), it is an evident result of formulas (65), (67). Theorem 1.1 has been completely proved.

3. SELF-OSCILLATIONS IN A CONTINUAL RING CHAIN OF UNIDIRECTIONALLY COUPLED NEURONS

Let us firstly consider a discrete ring chain of unidirectionally coupled neurons assuming that each separate neuron is described by system (3). As a result we have system

$$\varepsilon\dot{u}_j = v_j - g(u_j) + \frac{\mu m}{2\pi}(u_{j+1}-u_j), \quad \dot{v}_j = a - u_j - v_j, \quad j = 1,\dots,m, \tag{71}$$

where $u_{m+1} = u_1$, $\mu > 0$ is the coupling parameter. Then, at condition $m \gg 1$, we approximate magnitude $2\pi j/m$ by continuous index $x \in [0,2\pi] (\operatorname{mod} 2\pi)$ and replace component $m(u_{j+1}-u_j)/2\pi$ in (71) with derivative $\partial u/\partial x$. As a result, we get the boundary value problem

$$\varepsilon\frac{\partial u}{\partial t} = v - g(u) + \mu\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial t} = a - u - v,$$
$$u(t, x + 2\pi) \equiv u(t, x), \quad v(t, x + 2\pi) \equiv v(t, x), \tag{72}$$

which represents a mathematical model of continual ring chain of unidirectionally coupled neurons.

We will consider special periodic solutions of problem (72), where the so-called running waves are denoted as

$$u = u_n(\xi, \varepsilon, \mu), \quad v = v_n(\xi, \varepsilon, \mu), \quad \xi = \omega_n(\varepsilon, \mu)t - nx. \tag{73}$$

Here $\omega_n(\varepsilon, \mu) > 0$, $n \in \mathbb{N}$ and 2π-periodic by ξ functions $u_n(\xi, \varepsilon, \mu)$, $v_n(\xi, \varepsilon, \mu)$ satisfy system

$$(\varepsilon\omega_n(\varepsilon, \mu) + n\mu)\frac{du}{d\xi} = v - g(u), \quad \omega_n(\varepsilon, \mu)\frac{dv}{d\xi} = a - u - v. \tag{74}$$

Established theorem 1.1 allows one to deal with the issue of the existence of running waves (73), (74). For this purpose, let us consider the auxiliary system

$$\tilde{\varepsilon}\frac{du}{d\xi} = v - g(u), \quad \omega\frac{dv}{d\xi} = a - u - v, \tag{75}$$

assuming that $0 < \tilde{\varepsilon} \ll 1$ and parameter $\omega > 0$ is of order of unit and changes within a finite segment. Applying the mentioned theorem to system (75), we make sure that, for all fairly small values of $\tilde{\varepsilon} > 0$, it accepts the periodic solution

$$u = u(\xi, \tilde{\varepsilon}, \omega), \quad v = v(\xi, \tilde{\varepsilon}, \omega), \quad u(0, \tilde{\varepsilon}, \omega) \equiv u_* + 1 \tag{76}$$

of period $T(\tilde{\varepsilon}, \omega)$. At the same time,

$$T(\tilde{\varepsilon}, \omega) = \omega T_* + O(\tilde{\varepsilon}^{1/4}), \quad \tilde{\varepsilon} \to 0, \tag{77}$$

where T_* is constant (10). In turn, asymptotic representation (77) implies that equation

$$T(\tilde{\varepsilon}, \omega)\,|_{\tilde{\varepsilon} = \varepsilon\omega + n\mu} = 2\pi \tag{78}$$

for defining frequency ω has at least one solution

$$\omega = \omega_n(\varepsilon, \mu), \quad \omega_n(\varepsilon, \mu) = 2\pi/T_* + O((\varepsilon + \mu)^{1/4}), \quad \varepsilon, \mu \to 0.$$

It is necessary to add that the triplet of functions $\omega_n(\varepsilon, \mu)$, $u_n(\xi, \varepsilon, \mu)$, $v_n(\xi, \varepsilon, \mu)$, where

$$u_n(\xi, \varepsilon, \mu) = u(\xi, \tilde{\varepsilon}, \omega)\big|_{\tilde{\varepsilon} = \varepsilon\omega_n(\varepsilon, \mu) + n\mu, \omega = \omega_n(\varepsilon, \mu)},$$
$$v_n(\xi, \varepsilon, \mu) = v(\xi, \tilde{\varepsilon}, \omega)\big|_{\tilde{\varepsilon} = \varepsilon\omega_n(\varepsilon, \mu) + n\mu, \omega = \omega_n(\varepsilon, \mu)},$$

is the sought value; i.e., it turns correlations (74) into correct equalities.

The analysis performed above leads to the following statement.

Theorem 3.1. *By any natural N small enough values $\varepsilon_N > 0$, $\mu_N > 0$ can be found such that, at any $0 < \varepsilon \le \varepsilon_N$, $0 < \mu \le \mu_N$ boundary value problem* (72) *accepts running waves* (73), (74) *with numbers n* = 1, ..., *N.*

It is worth noting that, in the case

$$\mu = \varepsilon d, \quad d = \text{const} > 0, \tag{79}$$

boundary problem (72) may also have running waves in the form

$$u = u_n(\xi, \varepsilon), \quad v = v_n(\xi, \varepsilon), \quad \xi = \omega_n(\varepsilon)t + nx. \tag{80}$$

Here, as earlier $\omega_n(\varepsilon) > 0$, $n \in \mathbb{N}$, and 2π-periodic by ξ functions $u_n(\xi, \varepsilon)$, $v_n(\xi, \varepsilon)$ are defined from the system similar to (74)

$$\varepsilon(\omega_n(\varepsilon) - nd)\frac{du}{d\xi} = v - g(u), \quad \omega_n(\varepsilon)\frac{dv}{d\xi} = a - u - v. \tag{81}$$

Functions $\omega_n(\varepsilon)$, $u_n(\xi, \varepsilon)$, $v_n(\xi, \varepsilon)$ from (80) are defined by the same scheme as described above. Namely, let us consider auxiliary system similar to (75)

$$\varepsilon(\omega - nd)\frac{du}{d\xi} = v - g(u), \quad \omega\frac{dv}{d\xi} = a - u - v, \tag{82}$$

where parameters $\omega, d > 0$ are on the order of unit and are coupled by the inequality $nd < \omega$. Theorem 1.1 guarantees the existence of a periodic solution of system (82) similar to (76)

$$u = u_n(\xi, \varepsilon, \omega), \quad v = v_n(\xi, \varepsilon, \omega), \quad u_n(0, \varepsilon, \omega) \equiv u_* + 1 \tag{83}$$

of period

$$T_n(\varepsilon, \omega) = \omega T_* + O(\varepsilon^{1/4}), \quad \varepsilon \to 0 \tag{84}$$

at any small enough value of $\varepsilon > 0$.

Then, (84) apparently implies that the equation $T_n(\varepsilon, \omega) = 2\pi$, which is similar to (78), has at least one solution as follows:

$$\omega = \omega_n(\varepsilon) = 2\pi/T_* + O(\varepsilon^{1/4}), \quad \varepsilon \to 0. \tag{85}$$

As for functions $u_n(\xi, \varepsilon)$, $v_n(\xi, \varepsilon)$ in (80), (81), they are obtained from (83) after the substitution of correlation (85). In this way, the following statement is established.

Theorem 3.2. *Assume that at some natural N inequality* $Nd < 2\pi/T_*$ *is fulfilled. Then at condition* (79) *and at all small enough* $\varepsilon > 0$ *boundary value problem* (72) *accepts running waves* (80), (81) *with numbers n* = 1, ..., *N.*

The question about the stability of running waves in metrics of the phase space $(u, v) \in W_2^1 \times W_2^1$ (W_2^1 – Sobolev space of 2π-periodic functions) is reduced to studying the spectrum of some boundary value problems. In the case of running waves (73), (74), the procedure for deriving corresponding spectral problem concludes in the following. Let us proceed to running spatial variable $\xi = \omega_n(\varepsilon, \mu)t - nx$ in problem (72), then linearize it at the equilibrium state $u = u_n(\xi, \varepsilon, \mu)$, $v = v_n(\xi, \varepsilon, \mu)$. Then, let us substitute equalities $u = h_1(\xi)\exp(\lambda t)$, $v = h_2(\xi)\exp(\lambda t)$, $\lambda \in C$ into the obtained linear system. As a result, in order to define $h_1(\xi)$, $h_2(\xi)$, λ, we have

$$\begin{gathered}\varepsilon\lambda h_1 + (\varepsilon\omega_n(\varepsilon, \mu) + n\mu)\frac{dh_1}{d\xi} = h_2 - g'(u_n(\xi, \varepsilon, \mu))h_1, \\ \lambda h_2 + \omega_n(\varepsilon, \mu)\frac{dh_2}{d\xi} = -h_1 - h_2, \quad h_j(\xi + 2\pi n) \equiv h_j(\xi), \quad j = 1, 2.\end{gathered} \tag{86}$$

A study of the arrangement of Eigen values of λ in problem (86), as well as a similar boundary value problem for running waves (80), (81), represents a separate and still unsolved problem. Therefore, we will confine with results of numeric analysis of boundary value problem (72), which show the principal possibility of the existence of stable running waves.

When performing the corresponding numeric experiment, in contrast to (71), we approximate the first derivative by x with symmetric difference operator, i.e., we assume that

$$\begin{gathered}u_j(t) = u(t, x)\,|_{x=2\pi j/m}, \quad u_j(t) = u(t, x)\,|_{x=2\pi j/m}, \\ \left.\frac{\partial u}{\partial x}(t, x)\right|_{x=2\pi j/m} \approx \frac{m}{4\pi}(u_{j+1}(t) - u_{j-1}(t)), \quad j = 1, \ldots, m.\end{gathered}$$

Finally, we obtain the following system for variables $u_j(t)$, $v_j(t)$ after the replacement $\mu/2\pi \to \mu$:

$$\varepsilon\dot{u}_j = v_j - g(u_j) + \frac{m}{2}\mu(u_{j+1} - u_{j-1}), \quad \dot{v}_j = a - u_j - v_j, \quad j = 1, \ldots, m, \tag{87}$$

where $u_0 = u_m$, $u_{m+1} = u_1$.

A numerical analysis of system (87) was performed by the fourth order Runge–Kutta method with a constant step of $h = 10^{-4}$ under conditions (6), (7) and, at values of parameters $m = 21$, $a = 15$, $c_1 = 3$, $c_2 = 1$, $\varepsilon = \mu = 0.01$ (software package Tracer 3.70 developed by D. S. Glyzin was applied). It has been established that, at the given set of parameters, it has seven stable periodic solutions of running wave type and four stable two-dimensional invariant tori. Figures 5–15 show projections of the given attractors to

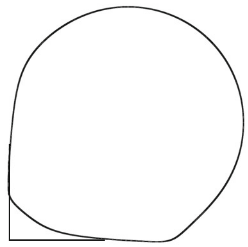

Fig. 5.

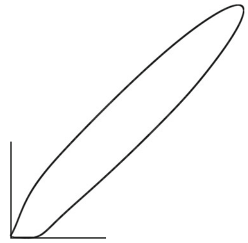

Fig. 6.

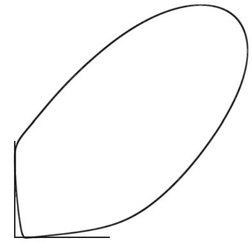

Fig. 7.

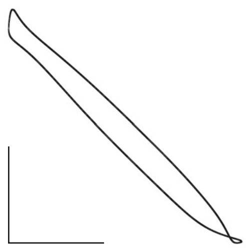

Fig. 8.

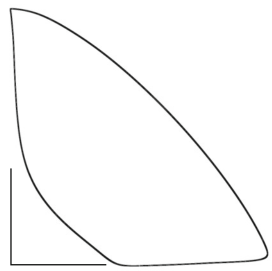

Fig. 9.

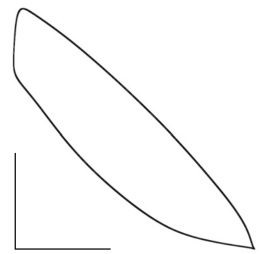

Fig. 10.

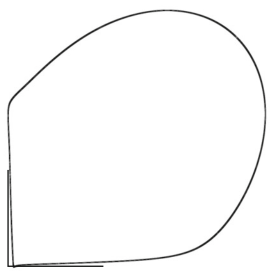

Fig. 11.

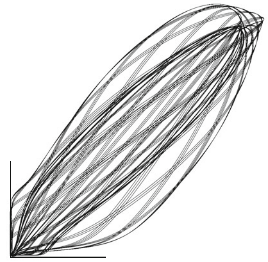

Fig. 12.

Fig. 13.

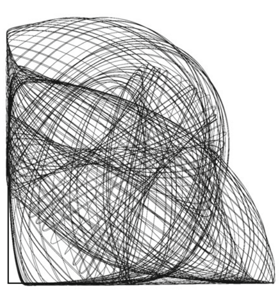

Fig. 14.

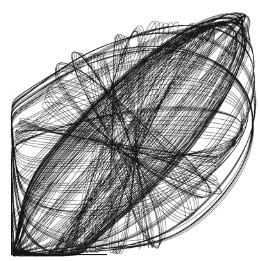

Fig. 15.

plane (u_1, u_{18}) (the first seven images correspond to cycles and the remaining four correspond to invariant tori).

In conclusion, it is necessary to note that the suggested new mathematical model of the functioning of a separate neuron is quite substantial. In fact, its basic idea of nonclassical relaxation oscillations allows one to achieve the required shape of the oscillations of the membrane potential, which is typical for a real biological object. In addition, boundary value problem (72), which corresponds to system (3), demonstrates nontrivial dynamics and namely buffering phenomenon. In connection with this, it is necessary to mention that buffering (coexistence of any predefined number of attractors) is typical for neuron systems, which is indicated by the results of works [6–9].

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