ISSN 0146-4116, Automatic Control and Computer Sciences, 2015, Vol. 49, No. 6, pp. 328–339. © Allerton Press, Inc., 2015.
Original Russian Text © A.A. Nazarov, A.N. Moiseev, 2015, published in Avtomatika i Vychislitel'naya Tekhnika, 2015, No. 6, pp. 14–27.

Analysis of the GI/PH/∞ System with High-Rate Arrivals

A. A. Nazarov and A. N. Moiseev

Tomsk State National Research University, pr. Lenina 36, Tomsk, 634050 Russia
e-mail: nazarov.tsu@gmail.com and moiseev.tsu@gmail.com

Received April 7, 2015; in final form, August 13, 2015

Abstract—The work presents an analysis of infinite-server queueing systems with renewal arrival process and phase-type services. Analysis was carried out using the N-dimensional Markov processes. The study was implemented in the asymptotic condition of high intensity of arrivals. It has been shown that, under these conditions, the stationary distribution of the number of customers in the system is Gaussian, and the parameters of this distribution were received. A prelimit analytical expression has also been derived for the dispersion of the number of customers in the system, and a numerical comparison has been carried out with asymptotic values that allow one to determine the domain of applicability of asymptotic results.

Keywords: queueing system, renewal process, infinite-server, asymptotic analysis

DOI: 10.3103/S0146411615060085

1. INTRODUCTION

Infinite-server queueing systems are adequate mathematical models of systems in a wide class of subject areas, such as physics, engineering, telecommunications, data processing, and socioeconomic systems. Therefore, the development of methods of studying them is an actual scientific problem.

A direct analytical study of these mathematical models is only possible for the Poisson arrivals [1] or for systems with deterministic service; therefore, the study of models with other types of arrival processes and service laws is usually implemented in some asymptotic (limit) conditions, which is referred to in the literature as *heavy-traffic* conditions [2].

This work presents an analysis of systems with renewal arrival process, unlimited number of servers, and phase-type services, which is usually denoted as GI/PH/∞ [3]. The analysis was carried out using the classical method of N-dimensional Markov processes; the system of equations concerning the probability distribution of the number of customers in the GI/PH/∞ system was derived. Since the solution of this system of equations is a significant problem then, in the present work, it is performed using the method of asymptotic analysis in limit condition of an unlimited increase of the intensity of arrivals [4] that relates to the class of the aforementioned heavy-traffic conditions. It was determined that the asymptotic distribution is a normal one (Gaussian distribution). The parameters of this distribution were obtained in this work.

The prelimit expression for the second moment (dispersion) of the number of customers in the GI/PH/∞ system. For the relative error of the asymptotic dispersion relative to its corresponding prelimit value, the numerical calculations with different values of the parameters that determine the intensity of arrival process and variations of the lengths of the intervals in it were performed. This allows one to define the domain of applicability of asymptotic results within the framework of the specified accuracy of approximations.

2. DEFINITION OF THE PROBLEM

Consider a queueing system with an unlimited number of servers and high-rate renewal arrival process. The arrival process is defined by the distribution function of the lengths of the intervals between the consecutive arrivals of customers specified in the form of $A(Nx)$ [4]. Here, N is the parameter that determines the intensity of arrivals , which, at $N \to \infty$, justifies the indication that the arrival process has a high intensity. Let us assume that, for a random variable defined by distribution function $A(z)$, the mathematical expectation and dispersion have finite values, and we will denote them as follows:

$$\int_0^\infty z dA(z) = a = \frac{1}{\lambda}, \quad \int_0^\infty (z-a)^2 dA(z) = \sigma^2.$$

Thus, the intensity of arrivals will be $N\lambda$.

The customer coming into the system takes any of the available servers. The service time has a phase-type distribution (PH distribution) given by the irreducible representation $\{\mathbf{v}, \mathbf{Q}\}$ [1, 5]. This means the following. The Markov chain $k(t)$ with state space $\{0, 1, 2, ..., K\}$ is considered. At the beginning of the service for this chain, the initial state is chosen from the set $\{1, 2, ..., K\}$ based on a given probabilistic row vector $\mathbf{v}$. Next, in the chain $k(t)$, the states are changed according to the specified subgenerator $\mathbf{Q}$ until the chain reaches the absorbing state 0, the intensity of transitions in which are specified by the vector

$$\mathbf{q_0} = -\mathbf{Q}\mathbf{e}. \tag{1}$$

Here, $\mathbf{e}$ is the column vector, consisting of units. The service time is interpreted as the time when the described Markov chain $k(t)$ will switch in the absorbing state. In these queueing systems, all intermediate states of the respective Markov chain from the set $\{1, 2, ..., K\}$ is called the *service phases*.

Since it is believed that the matrix $\mathbf{Q} + \mathbf{q_0}\mathbf{v}$ is irreducible, the average value b of the service time specified in the mentioned above manner is defined by the equality [1]

$$b = -\mathbf{v}\mathbf{Q}^{-1}\mathbf{e}.$$

It is also known [1] that the distribution function $B(x)$ of the service time for PH distribution has the form

$$B(x) = 1 - \mathbf{v}e^{\mathbf{Q}x}\mathbf{e}.$$

The task is posed of finding the asymptotic at $N \to \infty$ and the probability distribution of the number of customers in the system in the stationary mode of operation.

3. THE DERIVATION OF THE KOLMOGOROV SYSTEM OF EQUATIONS

Using $i_k(t)$, let us denote the number of customers being serviced in the system at time of point t in the kth phase ($k = \overline{1,K}$) in the vector form $\mathbf{i}(t) = (i_1(t)\ \dots\ i_K(t))$. Suppose also that $z(t)$ is the length of the time interval from the moment t until the next customer arrives. Then $(K+1)$-dimensional stochastic process $\{\mathbf{i}(t), z(t)\}$ will be the Markov process. For the probability distribution given in the $P(\mathbf{i}, z, t) = P\left\{\mathbf{i}(t) = \mathbf{i},\ \ z(t) < \frac{z}{N}\right\}$, one can write the equality

$$P(\mathbf{i}, z - N\Delta t, t + \Delta t) = [P(\mathbf{i}, z, t) - P(\mathbf{i}, N\Delta t, t)]\prod_{k=1}^{K}(1 + i_k Q_{kk}\Delta t) + \sum_{k=1}^{K} P(\mathbf{i} - \mathbf{e}_k, N\Delta t, t)\nu_k A(z)$$
$$+ \sum_{k=1}^{K} P(\mathbf{i} + \mathbf{e}_k, z, t)(i_k + 1)Q_{k0}\Delta t + \sum_{k=1}^{K}\sum_{l \neq k} P(\mathbf{i} + \mathbf{e}_k - \mathbf{e}_l, z, t)(i_k + 1)Q_{kl}\Delta t + o(\Delta t).$$

Here $\mathbf{e}_k$ is the kth component of which is equal to unit, and the remainder components are zero; i_k, ν_k, Q_{k0}, and Q_{kl} are the elements of the vectors $\mathbf{i}$, $\mathbf{v}$, $\mathbf{q}_0$ and the matrix $\mathbf{Q}$, respectively.

Based on this equality, for the stationary distribution $P(\mathbf{i}, z, t) = P(\mathbf{i}, z)$, noting $\left.\frac{\partial P(\mathbf{i}, z)}{\partial z}\right|_{z=0} = \frac{\partial P(\mathbf{i}, 0)}{\partial z}$, one can write the Kolmogorov system of equations

$$N\frac{\partial P(\mathbf{i}, z)}{\partial z} - N\frac{\partial P(\mathbf{i}, 0)}{\partial z} + P(\mathbf{i}, z)\sum_{k=1}^{K} i_k Q_{kk} + N\sum_{k=1}^{K}\frac{\partial P(\mathbf{i} - \mathbf{e}_k, 0)}{\partial z}\nu_k A(z)$$
$$+ \sum_{k=1}^{K} P(\mathbf{i} + \mathbf{e}_k, z)(i_k + 1)Q_{k0} + \sum_{k=1}^{K}\sum_{\substack{l=1 \\ l \neq k}}^{K} P(\mathbf{i} + \mathbf{e}_k - \mathbf{e}_l, z)(i_k + 1)Q_{kl} = 0 \tag{2}$$

for all nonnegative values of **i** and z (here, if at least one component of vector **i** is negative, it is assumed that $P(\mathbf{i}, z) = 0$).

For the N-dimensional partial characteristic functions

$$H(\mathbf{u}, z) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_K=0}^{\infty} P(i_1, \ldots, i_K, z) \exp\left\{ j \sum_{k=1}^{K} u_k i_k \right\},$$

where **u** is a vector with components u_k and $j = \sqrt{-1}$ – is the imaginary unit, from Eq. (2), we get the equality

$$N \frac{\partial H(\mathbf{u}, z)}{\partial z} - N \frac{\partial H(\mathbf{u}, 0)}{\partial z} - j \sum_{k=1}^{K} \frac{\partial H(\mathbf{u}, z)}{\partial u_k} Q_{kk} + N A(z) \sum_{k=1}^{K} e^{j u_k} \frac{\partial H(\mathbf{u}, 0)}{\partial z} \nu_k$$
$$- j \sum_{k=1}^{K} \sum_{\substack{l=1 \\ l \neq k}}^{K} e^{-j u_k} e^{j u_l} \frac{\partial H(\mathbf{u}, z)}{\partial u_k} Q_{kl} - j \sum_{k=1}^{K} \frac{\partial H(\mathbf{u}, z)}{\partial u_k} e^{-j u_k} Q_{k0} = 0,$$

which, because of (1), we rewrite in the form of the following equation:

$$\frac{\partial H(\mathbf{u}, z)}{\partial z} + \frac{\partial H(\mathbf{u}, 0)}{\partial z} \left[A(z) \sum_{k=1}^{K} \nu_k e^{j u_k} - 1 \right] - \frac{j}{N} \sum_{k=1}^{K} \frac{\partial H(\mathbf{u}, z)}{\partial u_k} e^{-j u_k} \sum_{l=1}^{K} Q_{kl} \left(e^{j u_l} - 1 \right) = 0. \quad (3)$$

In a prelimit situation, at finite values of N, the analytical solution of this equation is unlikely possible; therefore, we will find it to have an asymptotic solution at $N \to \infty$ using the method for asymptotic analysis [6].

4. ASYMPTOTIC ANALYSIS

Denoting

$$\frac{1}{N} = \varepsilon, \quad \mathbf{u} = \varepsilon \mathbf{w}, \quad H(\mathbf{u}, z) = F(\mathbf{w}, z, \varepsilon), \quad (4)$$

we rewrite Eq. (3) as

$$\frac{\partial F(\mathbf{w}, z, \varepsilon)}{\partial z} + \frac{\partial F(\mathbf{w}, 0, \varepsilon)}{\partial z} \left[A(z) \sum_{k=1}^{K} \nu_k e^{j \varepsilon w_k} - 1 \right] - j \sum_{k=1}^{K} \frac{\partial F(\mathbf{w}, z, \varepsilon)}{\partial w_k} e^{-j \varepsilon w_k} \sum_{l=1}^{K} Q_{kl} \left(e^{j \varepsilon w_l} - 1 \right) = 0. \quad (5)$$

Let us formulate the following statement.

Theorem 1. The limit at the $\varepsilon \to 0$ value $F(\mathbf{w}, z)$ of the solution $F(\mathbf{w}, z, \varepsilon)$ of Eq. (5) has the form

$$F(\mathbf{w}, z) = R(z) \exp\left\{ \sum_{k=1}^{K} j w_k x_k \right\},$$

where the function $R(z)$ has the form

$$R(z) = \lambda \int_0^z (1 - A(\tau))\, d\tau, \quad (6)$$

and the vector **x** with components x_k is determined by the equality

$$\mathbf{x} = -\lambda \mathbf{v} \mathbf{Q}^{-1}. \quad (7)$$

The proof of this theorem is given in the Appendix.

Because the values x_k make sense of the normalized by value N average value of the number of customers served in the system on the kth phase, then the normalized total average value κ_1 is

$$\kappa_1 = \mathbf{x}\mathbf{e} = -\lambda \mathbf{v} \mathbf{Q}^{-1} \mathbf{e} = \lambda \left(-\mathbf{v} \mathbf{Q}^{-1} \mathbf{e} \right) = \lambda b, \quad (8)$$

which naturally coincides with the total average value of the number of customers in the system and in the prelimit situation, which can be found by applying Little's theorem [7].

Let us perform a more detailed study of the number of customers in the GI/PH/∞ system. In the original Eq. (3), we will replace

$$H(\mathbf{u}, z) = H_2(\mathbf{u}, z)\exp\left\{\sum_{k=1}^{K} ju_k Nx_k\right\};$$

then, for the function $H_2(\mathbf{u}, z)$, we will obtain the equation

$$\begin{gathered}\frac{\partial H_2(\mathbf{u}, z)}{\partial z} + \frac{\partial H_2(\mathbf{u}, 0)}{\partial z}\left[A(z)\sum_{k=1}^{K} v_k e^{ju_k} - 1\right] + H_2(\mathbf{u}, z)\sum_{k=1}^{K} x_k e^{-ju_k}\sum_{l=1}^{K} Q_{kl}\left(e^{ju_l} - 1\right) \\ - \frac{j}{N}\sum_{k=1}^{K}\frac{\partial H_2(\mathbf{u}, z)}{\partial u_k} e^{-ju_k}\sum_{l=1}^{K} Q_{kl}\left(e^{ju_l} - 1\right) = 0.\end{gathered} \tag{9}$$

Here, $H_2(\mathbf{u}, z)$ is a partial function of the N-dimensional characteristic function of values of the centered registered random processes $i_k(t) - Nx_k$.

Applying the method of asymptotic analysis in Eq. (9) similar to (4), we will execute the replacements

$$\frac{1}{N} = \varepsilon^2, \quad \mathbf{u} = \varepsilon\mathbf{w}, \quad H_2(\mathbf{u}, z) = F_2(\mathbf{w}, z, \varepsilon).$$

Then, we get the equation for the function $F_2(\mathbf{w}, z, \varepsilon)$ as follows:

$$\begin{gathered}\frac{\partial F_2(\mathbf{w}, z, \varepsilon)}{\partial z} + \frac{\partial F_2(\mathbf{w}, 0, \varepsilon)}{\partial z}\left[A(z)\sum_{k=1}^{K} v_k e^{j\varepsilon w_k} - 1\right] + F_2(\mathbf{w}, z, \varepsilon)\sum_{k=1}^{K} x_k e^{-j\varepsilon w_k}\sum_{l=1}^{K} Q_{kl}\left(e^{j\varepsilon w_l} - 1\right) \\ - j\varepsilon\sum_{k=1}^{K}\frac{\partial F_2(\mathbf{w}, z, \varepsilon)}{\partial w_k} e^{-j\varepsilon w_k}\sum_{l=1}^{K} Q_{kl}\left(e^{j\varepsilon w_l} - 1\right) = 0.\end{gathered} \tag{10}$$

The following theorem is true.

Theorem 2. Limit value of $\Phi_2(\mathbf{w})$ at $\varepsilon \to 0$ and $z \to \infty$ of the solution $F_2(\mathbf{w}, z, \varepsilon)$ of Eq. (10) has the form

$$\Phi_2(\mathbf{w}) = \exp\left\{-\frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} w_k G_{kl} w_l\right\}, \tag{11}$$

where the matrix $\mathbf{G} = \{G_{kl}\}$ is the solution of the equation

$$\mathbf{GQ} + \mathbf{Q}^T\mathbf{G} = \mathrm{diag}\{\mathbf{x}\}\mathbf{Q} + \mathbf{Q}^T\,\mathrm{diag}\{\mathbf{x}\} - \lambda\left(\lambda^2\sigma^2 - 1\right)\mathbf{v}^T\mathbf{v}. \tag{12}$$

The normalized dispersion κ_2 of the number of customers in the GI/PH/∞ system is defined by the equality

$$\kappa_2 = \mathbf{e}^T\mathbf{Ge}. \tag{13}$$

For this, the next theorem is true.

Theorem 3. The normalized dispersion κ_2 of the number of customers in the GI/PH/∞ system is equal to

$$\kappa_2 = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\beta,$$

where parameter β is determined by the equality

$$\beta = \int_0^\infty\left(\mathbf{v}e^{\mathbf{Q}z}\mathbf{e}\right)^2 dz = \int_0^\infty[1 - B(z)]^2\,dz. \tag{14}$$

The proofs of Theorems 2 and 3 are given in the Appendix.

Thus, for GI/PH/∞ system in the conditions of the high intensity of arrivals the distribution of the number of customers in the system is approximated by Gaussian distribution $N\kappa_1$ and dispersion $N\kappa_2$, where the parameters κ_1 and κ_2 are defined by the equalities $\kappa_1 = \lambda b$, $\kappa_2 = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\beta$ and depend only on the first and second moments of the lengths of the intervals in the arrival process, as well as the average value of b and of the parameter β from formula (14) for the service time, which indicates to the ability to save these parameters and for arbitrary functions $B(x)$, the distribution of the service time in the GI/GI/∞ systems. Indeed, in works [8, 9] the studies of the GI/GI/∞ system with the high-rate arrival process were carried out using the technique of separation of the first jump [1, 10] and the dynamic sifting technique [6], respectively. It is shown that, under the conditions of high-intensity arrivals, the asymptotic probability distribution of the number of customers in the system is Gaussian. The characteristic function of this distribution has the form

$$h(w) = \exp\left\{ jwN\lambda b + \frac{(jw)^2}{2} N \left(\lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right) \int_0^\infty [1 - B(t)]^2 \, dt \right) \right\}, \tag{15}$$

and, taking into account the normalization, fully coincides with the results obtained above.

5. THE DOMAIN OF APPLICABILITY OF THE ASYMPTOTIC RESULTS

Since the found Gaussian distribution and its dispersion are asymptotic ones, then the task of defining the domain of applicability of asymptotic results for approximating the prelimit characteristics at finite N arises, i.e., it is necessary to determine for which values of parameter N the error of Gaussian approximation can be considered permissible.

For systems with the unlimited number of servers, it is fairly easy to find the first moment of the number of customers in the system. For this purpose, it is sufficient to apply Little's theorem and write

$$\kappa_1 = \lambda b,$$

where κ_1 is the average value of the number of customers in the system with an unlimited number of servers, λ is the intensity of arrivals, and b is the average value of service time. Thus, the average number of customers in the prelimit and asymptotic cases coincide.

In this regard, we will analyze the domain of applicability of obtained asymptotic results based on the calculation of relative error of asymptotic dispersion defined by the equality

$$\kappa_2 = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\beta, \tag{16}$$

in relation to similar dispersion for prelimit case. For this, at first, we will find the prelimit value of dispersion.

As shown above, the partial characteristic function $H(\mathbf{u}, z)$ of $(K + 1)$-dimensional stochastic process $\{\mathbf{i}(t), z(t)\}$ is a solution of Eq. (3), which we will rewrite as

$$\frac{\partial H(\mathbf{u},z)}{\partial z} + \frac{\partial H(\mathbf{u},0)}{\partial z}\left[A(z)\sum_{k=1}^{K} \nu_k e^{ju_k} - 1 \right] - j\sum_{k=1}^{K} \frac{\partial H(\mathbf{u},z)}{\partial u_k} e^{-ju_k} \sum_{l=1}^{K} Q_{kl}\left(e^{ju_l} - 1\right) = 0, \tag{17}$$

where, for convenience, the scalar factor $1/N$ is written in the values of the elements of the matrix $\mathbf{Q}$.

If, in Eq. (17), we assume that $\mathbf{u} = \mathbf{0}$ and denote

$$H(\mathbf{0}, z) = R(z),$$

then, for function $R(z)$, we will obtain Eq. (26), which has the form

$$R(z) = \lambda \int_0^z (1 - A(x))\, dx.$$

It is known that the initial moments of a random variable are defined with the values of derivatives of the characteristic function in zero. Using this property, we will differentiate Eq. (17) with respect to u_m $(m = \overline{1,K})$ and get the equalities

$$\frac{\partial^2 H(\mathbf{u},z)}{\partial u_m \partial z} + \frac{\partial^2 H(\mathbf{u},0)}{\partial u_m \partial z}\left[A(z)\sum_{k=1}^{K} \nu_k e^{ju_k} - 1\right] + \frac{\partial H(\mathbf{u},0)}{\partial z} A(z) j e^{ju_m} \nu_m$$
$$- j\sum_{k=1}^{K} \frac{\partial^2 H(\mathbf{u},z)}{\partial u_m \partial u_k} e^{-ju_k} \sum_{l=1}^{K} Q_{kl}\left(e^{ju_l} - 1\right) - \frac{\partial H(\mathbf{u},z)}{\partial u_m} e^{-ju_m} \sum_{l=1}^{K} Q_{ml}\left(e^{ju_l} - 1\right) \tag{18}$$
$$+ \sum_{k=1}^{K} \frac{\partial H(\mathbf{u},z)}{\partial u_k} e^{-ju_k} Q_{km} e^{ju_m} = 0.$$

Substituting $\mathbf{u} = \mathbf{0}$ into this equation and denoting

$$\left.\frac{\partial H(\mathbf{u},z)}{\partial u_m}\right|_{\mathbf{u}=\mathbf{0}} = j a_m(z), \text{ at that } \left.\frac{\partial H(\mathbf{u},0)}{\partial u_m}\right|_{\mathbf{u}=\mathbf{0}} = \lambda.$$

We will obtain the system of equations

$$a_m'(z) + a_m'(0)\left[A(z) - 1\right] + \lambda A(z)\nu_m + \sum_{k=1}^{K} a_k(z) Q_{km} = 0.$$

Regarding the components $a_m(z)$ of row vector $\mathbf{a}(z) = \{a_1(z), \ldots, a_K(z)\}$, which, due to this system, satisfies the equation

$$\mathbf{a}'(z) + \mathbf{a}'(0)\left[A(z) - 1\right] + \lambda A(z)\mathbf{v} + \mathbf{a}(z)\mathbf{Q} = \mathbf{0}. \tag{19}$$

At $z \to \infty$, denoting $\mathbf{a}(\infty) = \mathbf{a}$, for (19), we will obtain an equation relative to vector $\mathbf{a}$ as follows:

$$\lambda \mathbf{v} + \mathbf{a}\mathbf{Q} = \mathbf{0},$$

the solution of which has the form

$$\mathbf{a} = -\lambda \mathbf{v}\mathbf{Q}^{-1}. \tag{20}$$

The components a_m of vector $\mathbf{a}$ make sense of the average value of the number of customers serviced on mth PH phase of the service in the system servers in the stationary mode of operation.

The average value k_1 of the number of customers in the system (the first moment) is determined by the equality

$$\kappa_1 = \mathbf{a}\mathbf{e} = -\lambda \mathbf{v}\mathbf{Q}^{-1}\mathbf{e} = \lambda\left(-\mathbf{v}\mathbf{Q}^{-1}\mathbf{e}\right) = \lambda b,$$

which naturally coincides with the results obtained from Little's theorem.

To find the value of the second moment it is necessary to define the vector $\mathbf{a}'(0)$ of Eq. (19). For this, in Eq. (19), we will perform the Laplace–Stieltjes transform, which denotes

$$\int_0^\infty e^{-\alpha z} dA(z) = A^*(\alpha), \quad \int_0^\infty e^{-\alpha z} d\mathbf{a}(z) = \mathbf{a}^*(\alpha).$$

Then, we will obtain the equality

$$\mathbf{a}^*(\alpha)\left(\alpha \mathbf{I} + \mathbf{Q}\right) = \mathbf{a}'(0)\left[1 - A^*(\alpha)\right] - \lambda A^*(\alpha)\mathbf{v}. \tag{21}$$

We will assume that the matrix $\mathbf{Q}$ is just a simple negative real characteristic numbers α_l, the eigenvectors (columns) for which we denote as $\mathbf{X}_l$ $(l = \overline{1,K})$. So, as

$$\left(\alpha_l \mathbf{I} + \mathbf{Q}\right)\mathbf{X}_l = \mathbf{0},$$

then due to (21), the equalities are satisfied as follows:

$$\mathbf{a}^*(\alpha_l)\left(\alpha_l \mathbf{I} + \mathbf{Q}\right)\mathbf{X}_l = 0 = \mathbf{a}'(0)\left[1 - A^*(\alpha_l)\right]\mathbf{X}_l - \lambda A^*(\alpha_l)\mathbf{v}\mathbf{X}_l, \quad l = \overline{1,K}.$$

Therefore, the vector $\mathbf{a}'(0)$ is the solution of the system of equations

$$\mathbf{a}'(0)\mathbf{X}_l = \lambda \frac{A^*(\alpha_l)}{1 - A^*(\alpha_l)} \mathbf{v}\mathbf{X}_l, \quad l = \overline{1, K}. \tag{22}$$

Denoting the row vector with components in $\mathbf{y}$

$$y_l = \lambda \frac{A^*(\alpha_l)}{1 - A^*(\alpha_l)} \mathbf{v}\mathbf{X}_l,$$

and the matrix, the columns of which are eigenvectors $\mathbf{X}_l$, in $\mathbf{X}$, we get the solution of the systems (22) in the form

$$\mathbf{a}'(0) = \mathbf{y}\mathbf{X}^{-1}. \tag{23}$$

The vectors $\mathbf{a}$ of (20) and $\mathbf{a}'(0)$ of (23) will be used below to determine the value of the second moment of the number of customers in the GI/PH/∞ system.

In equality (18), we will pass to the limit at $z \to \infty$, which denotes $H(\mathbf{u}, \infty) = H(\mathbf{u})$, and we obtain

$$\frac{\partial^2 H(\mathbf{u},0)}{\partial u_m \partial z} \sum_{k=1}^{K} v_k \left(e^{ju_k} - 1\right) + j\frac{\partial H(\mathbf{u},0)}{\partial z} e^{ju_m} v_m - j\sum_{k=1}^{K} \frac{\partial^2 H(\mathbf{u})}{\partial u_m \partial u_k} e^{-ju_k} \sum_{l=1}^{K} Q_{kl} \left(e^{ju_l} - 1\right)$$
$$- \frac{\partial H(\mathbf{u})}{\partial u_m} e^{-ju_m} \sum_{l=1}^{K} Q_{ml} \left(e^{ju_l} - 1\right) + \sum_{k=1}^{K} \frac{\partial H(\mathbf{u})}{\partial u_k} e^{-ju_k} Q_{km} e^{ju_m} = 0.$$

To find the second moment of the number of customers in the system, we will differentiate this equality with respect to u_n and denote the sum of all components that are equal to zero at $\mathbf{u} = \mathbf{0}$ as $g(\mathbf{u})$. As a result, we obtain

$$j\frac{\partial^2 H(\mathbf{u},0)}{\partial u_m \partial z} v_n + j\frac{\partial^2 H(\mathbf{u},0)}{\partial u_n \partial z} v_m + \sum_{k=1}^{K} \frac{\partial^2 H(\mathbf{u})}{\partial u_m \partial u_k} Q_{kn} - j\frac{\partial H(\mathbf{u})}{\partial u_m} Q_{mn}$$
$$+ \sum_{k=1}^{K} \frac{\partial^2 H(\mathbf{u})}{\partial u_n \partial u_k} Q_{km} - j\frac{\partial H(\mathbf{u})}{\partial u_n} Q_{nm} + g(\mathbf{u}) = 0.$$

In this equality, we set $\mathbf{u} = \mathbf{0}$ and denote

$$\left.\frac{\partial^2 H(\mathbf{u})}{\partial u_m \partial u_k}\right|_{\mathbf{u}=\mathbf{0}} = j^2 A_{mk}.$$

Then, we get the following system of equations for A_{mk}:

$$\sum_{k=1}^{K} A_{mk} Q_{kn} + \sum_{k=1}^{K} A_{nk} Q_{km} = a_m Q_{mn} + a_n Q_{nm} - a'_m(0) v_n - a'_n(0) v_m,$$

which we rewrite in the matrix form

$$\mathbf{A}\mathbf{Q} + \mathbf{Q}^T \mathbf{A} = \mathrm{diag}\{\mathbf{a}\}\mathbf{Q} + \mathbf{Q}^T \mathrm{diag}\{\mathbf{a}\} - \mathbf{v}^T \mathbf{a}'(0) - \left[\mathbf{a}'(0)\right]^T \mathbf{v}. \tag{24}$$

Here, $\mathbf{A}$ is the matrix of the second mixed initial moments A_{mk}. To pass to the central moments $\mathbf{G}$ in Eq. (24), we will perform the replacement

$$\mathbf{A} = \mathbf{C} + \mathbf{a}^T \mathbf{a}.$$

Taking into account equality (20), we get the following equation for matrix $\mathbf{C}$:

$$\mathbf{C}\mathbf{Q} + \mathbf{Q}^T \mathbf{C} = \mathbf{G},$$

where $\mathbf{G} = \mathrm{diag}\{\mathbf{a}\}\mathbf{Q} + \mathbf{Q}^T \mathrm{diag}\{\mathbf{a}\} + \left[\lambda\mathbf{a} - \mathbf{a}'(0)\right]^T \mathbf{v} + \mathbf{v}^T \left[\lambda\mathbf{a} - \mathbf{a}'(0)\right]$.

Relative error of the dispersion of Gaussian approximation relative to the prelimit one at different values of N and variations in the lengths of the intervals of arrivals V

V \ N	1	5	10	20	30	50	100
5	0.274	0.160	0.112	0.071	0.051	0.033	0.017
2.5	0.160	0.065	0.037	0.019	0.013	0.007	0.004
1	0	0	0	0	0	0	0
0.5	0.084	0.016	0.008	0.004	0.003	0.002	0.001
0.2	0.122	0.023	0.011	0.006	0.004	0.002	0.001
0.01	0.131	0.024	0.012	0.006	0.004	0.002	0.001

Applying the integral form of the solution record of these equations [11],we write the matrix **C** of second partial mixed central moments in the form

$$\mathbf{C} = \int_0^\infty e^{\mathbf{Q}^T t}\mathbf{G}e^{\mathbf{Q}t}dt,$$

then, the dispersion κ_2 of the number of customers in the GI/PH/∞ system in the prelimit case will be equal to

$$\kappa_2 = \mathbf{e}^T\left(\int_0^\infty e^{\mathbf{Q}^T t}\mathbf{G}e^{\mathbf{Q}t}dt\right)\mathbf{e}. \tag{25}$$

The intensity parameter of the arrival process N presents in this expression in the elements of the matrix **Q**, as well as in the vectors **a** and $\mathbf{a}'(0)$, included in the matrix **G**.

To define the domain of applicability of asymptotic results, we present the table of relative errors of the asymptotic dispersion (16) regarding prelimit (25). The calculations were carried out for the function $A(x)$ that corresponds to the gamma distribution with the fixed parameter of form, but different coefficients of variation V. The intensity of arrivals is equal to N, and the average service time is equal to 1. The coefficient of variations for the service time is fixed and equal to 0.636.

It can be seen from the table that, if the coefficient of variation $V = 1$, i.e., when arrivals is Poisson, the asymptotic formulas yield an accurate result. If we accept that the permissible relative error is 5%, then the asymptotic results can be considered acceptable if $N > 30$ for the coefficient of variation $V > 1$ and at $N \geq 5$ for $V < 1$.

6. CONCLUSIONS

The work presents the study of queueing system with high-rate renewal arrival process, an unlimited number of servers, and phase-type services. The study was carried out by methods of N-dimensional Markov processes and asymptotic analysis. It is shown that, at an asymptotic condition of the increasing intensity of arrivals, the stationary distribution of the number of customers in the system is the Gaussian distribution, the parameters of this distribution were obtained. A comparison of the study result with the results of [8, 9] for systems with arbitrary service obtained by the technique of separation the first jump and the dynamic sifting technique, respectively, was performed. It was determined that all three approaches yield identical results. Thus, in the hands of the researcher, there are three tools for studying the queueing systems with an unlimited number of servers and renewal arrival process. However, the classic approach of N-dimensional Markov processes is unfortunately only applied to systems with exponential service and, as is shown in the work, for systems with phase-type service. The second approach consists of the technique of separation of the first jump, which extends the domain of applicability to the systems with arbitrary (recurrent) service, as well as multi-stage systems and the networks with renewal arrival process [12, 13]. The third method consists of the screened process technique, which allows one to study not only the systems with renewal arrivals, but also with other types of arrival process, including MAP and the semi-Markov processes [14], as well as the multi-stage systems and the queueing networks with an unlimited number of servers with specified types of arrival process [15–17].

Based on a comparison of the asymptotic value of dispersion with prelimit (exact) value, a numerical analysis of the domain of applicability of asymptotic results was carried out in the work (the mathematical expectations in the limit and prelimit cases fully coincide). It has been shown that the asymptotic formulas yield acceptable results at an arrivals intensity that exceeds the intensity of service in more than 30 times and in some cases and at small intensities that exceed the intensity of service by only five times.

APPENDIX

Proof of Theorem 1

Denote $R(z) = H(\mathbf{u}, z)$ and substitute in (5) $\mathbf{u} = \mathbf{0}$, from which we get the equation

$$R'(z) + R'(0)\left[A(z) - 1\right] = 0. \tag{26}$$

The solution has the form $R(z) = \lambda\int_0^z (1 - A(x))\,dx$, which coincides with (6) where

$$\lambda = 1\Big/\int_0^\infty x\,dA(x) = 1\Big/\int_0^\infty [1 - A(x)]\,dx.$$

Here, the equality $R'(0) = \lambda$ holds.

Denoting $\lim_{\varepsilon\to 0} F(\mathbf{w}, z, \varepsilon) = F(\mathbf{w}, z, \varepsilon)$,we will accomplish the passage to the limit in the Eq. (5) at $\varepsilon \to 0$. We will get the equation

$$\frac{\partial F(\mathbf{w}, z)}{\partial z} + \frac{\partial F(\mathbf{w}, 0)}{\partial z}\left[A(z) - 1\right] = 0,$$

which has the same form as (26); hence, one can represent $F(\mathbf{w}, z)$ in the form

$$F(\mathbf{w}, z) = \Phi(\mathbf{w})R(z). \tag{27}$$

To find the function $\Phi(\mathbf{w})$, we will accomplish the passage to the limit in Eq. (5) at $z \to \infty$, and we will get the equality

$$\frac{\partial F(\mathbf{w}, 0, \varepsilon)}{\partial z}\sum_{k=1}^{K} \nu_k\left(e^{j\varepsilon w_k} - 1\right) - j\sum_{k=1}^{K}\frac{\partial F(\mathbf{w}, \infty, \varepsilon)}{\partial w_k} e^{-j\varepsilon w_k}\sum_{l=1}^{K} Q_{kl}\left(e^{j\varepsilon w_l} - 1\right) = 0.$$

Performing this equality, the simple transformations at $\varepsilon \to 0$ can be written as

$$\frac{\partial F(\mathbf{w}, 0)}{\partial z}\sum_{k=1}^{K} \nu_k w_k - j\sum_{k=1}^{K}\frac{\partial F(\mathbf{w}, \infty)}{\partial w_k}\sum_{l=1}^{K} Q_{kl} w_l = 0.$$

Here, substituting the product (27), we will obtain

$$R'(0)\sum_{k=1}^{K} \nu_k w_k - j\sum_{k=1}^{K}\frac{\partial\Phi(\mathbf{w})/\partial w_k}{\Phi(\mathbf{w})}\sum_{l=1}^{K} Q_{kl} w_l = 0. \tag{28}$$

Write the function $\Phi(\mathbf{w})$ in the form

$$\Phi(\mathbf{w}) = \exp\left\{\sum_{k=1}^{K} j w_k x_k\right\},$$

then, due to (28), one can write the equality

$$\lambda\sum_{k=1}^{K} \nu_k w_k + \sum_{k=1}^{K} x_k \sum_{l=1}^{K} Q_{kl} w_l = 0,$$

from which it follows that the values x_k are the solution of the system of equations

$$\lambda\nu_l + \sum_{k=1}^{K} x_k Q_{kl} = 0.$$

In the matrix form, this system has the form

$$\lambda \mathbf{v} + \mathbf{x}\mathbf{Q} = \mathbf{0}.$$

From this it follows, that the equality (7) is fulfilled. The theorem is proved.

The Theorem 2 Proving

Let us write the solution of the equation $F_2(\mathbf{w}, z, \varepsilon)$ (10) in the form of the following expansion:

$$F_2(\mathbf{w}, z, \varepsilon) = \Phi(\mathbf{w})\left\{R(z) + \sum_{k=1}^{K} j\varepsilon w_k f_k(z)\right\} + O(\varepsilon^2). \tag{29}$$

In Eq. (10), expanding the exponents we will get the equality

$$\frac{\partial F_2(\mathbf{w}, z, \varepsilon)}{\partial z} + \frac{\partial F_2(\mathbf{w}, 0, \varepsilon)}{\partial z}\left[A(z) - 1 + A(z)\sum_{k=1}^{K} v_k j\varepsilon w_k\right] + F_2(\mathbf{w}, z, \varepsilon)\sum_{k=1}^{K} x_k \sum_{l=1}^{K} Q_{kl} j\varepsilon w_l = O(\varepsilon^2),$$

which, based on (28), we can rewrite as

$$\frac{\partial F_2(\mathbf{w}, z, \varepsilon)}{\partial z} + \frac{\partial F_2(\mathbf{w}, 0, \varepsilon)}{\partial z}\left[A(z) - 1 + A(z)\sum_{k=1}^{K} v_k j\varepsilon w_k\right] + \lambda F_2(\mathbf{w}, z, \varepsilon)\sum_{k=1}^{K} v_k j\varepsilon w_k = O(\varepsilon^2).$$

Here, substituting expansion (29), it is easy to show that the functions $f_k(z)$ satisfy the equations

$$f_k'(z) + f_k'(0)\left[A(z) - 1\right] + \lambda v_k\left[A(z) - R(z)\right] = 0,$$

therefore,

$$f_k'(0) - \lambda f_k(\infty) = \lambda^2 v_k \int_0^{\infty}\left[A(z) - R(z)\right] dz.$$

It can be shown that

$$\int_0^{\infty}\left[A(z) - R(z)\right] dz = \frac{\lambda^2\sigma^2 - 1}{2\lambda},$$

therefore, for the functions $f_k(z)$, the following equality holds:

$$f_k'(0) - \lambda f_k(\infty) = \frac{1}{2}\lambda\left(\lambda^2\sigma^2 - 1\right) v_k. \tag{30}$$

In Eq. (10) passing to the limit at $z \to \infty$, and expanding the exponents with an accuracy to ε^2, we get

$$\frac{\partial F_2(\mathbf{w}, 0, \varepsilon)}{\partial z}\sum_{k=1}^{K} v_k\left[j\varepsilon w_k + \frac{(j\varepsilon w_k)^2}{2}\right] + F_2(\mathbf{w}, \infty, \varepsilon)\sum_{k=1}^{K} x_k\left(1 - j\varepsilon w_k\right)\sum_{l=1}^{K} Q_{kl}\left[j\varepsilon w_l + \frac{(j\varepsilon w_l)^2}{2}\right]$$
$$- j\varepsilon\sum_{k=1}^{K}\frac{\partial F_2(\mathbf{w}, \infty, \varepsilon)}{\partial w_k}\sum_{l=1}^{K} Q_{kl} j\varepsilon w_l = O(\varepsilon^3).$$

Here, substituting expansion (29) and performing simple conversions, we get the equality

$$\sum_{k=1}^{K} w_k\left[f_k'(0) - \lambda f_k(\infty)\right]\sum_{l=1}^{K} v_l w_l - \sum_{k=1}^{K} x_k w_k \sum_{l=1}^{K} Q_{kl} w_l - \sum_{k=1}^{K}\frac{\partial\Phi_2(\mathbf{w})/\partial w_k}{\Phi_2(\mathbf{w})}\sum_{l=1}^{K} Q_{kl} w_l = 0,$$

which, based on (30), can be rewritten as

$$-\sum_{k=1}^{K}\frac{\partial\Phi_2(\mathbf{w})/\partial w_k}{\Phi_2(\mathbf{w})}\sum_{l=1}^{K} Q_{kl} w_l + \frac{1}{2}\lambda\left(\lambda^2\sigma^2 - 1\right)\sum_{k=1}^{K} v_k w_k \sum_{l=1}^{K} v_l w_l - \sum_{k=1}^{K} x_k w_k \sum_{l=1}^{K} Q_{kl} w_l = 0.$$

Here, substituting expression (11) for function $\Phi_2(\mathbf{w})$, we obtain the equality

$$\sum_{m=1}^{K} w_m \sum_{k=1}^{K} G_{mk} \sum_{n=1}^{K} Q_{kn} w_n + \sum_{n=1}^{K} w_n \sum_{k=1}^{K} G_{kn} \sum_{m=1}^{K} Q_{km} w_m + \lambda\left(\lambda^2\sigma^2 - 1\right) \sum_{m=1}^{K} v_m w_m \sum_{n=1}^{K} v_n w_n$$
$$- \sum_{m=1}^{K} x_m w_m \sum_{n=1}^{K} Q_{mn} w_n - \sum_{n=1}^{K} x_n w_n \sum_{m=1}^{K} Q_{nm} w_m = 0,$$

from which one can write K^2 equalities for all $m, n = \overline{1, K}$ as follows:

$$\sum_{k=1}^{K} G_{mk} Q_{kn} + \sum_{k=1}^{K} G_{kn} Q_{km} + \lambda\left(\lambda^2\sigma^2 - 1\right) v_m v_n - x_m Q_{mn} - x_n Q_{nm} = 0.$$

In the matrix form, we obtain the equation regarding matrix $\mathbf{G}$ in the form

$$\mathbf{GQ} + \mathbf{Q}^T\mathbf{G} = \mathrm{diag}\{\mathbf{x}\}\mathbf{Q} + \mathbf{Q}^T \mathrm{diag}\{\mathbf{x}\} - \lambda\left(\lambda\sigma^2 - 1\right)\mathbf{v}^T\mathbf{v}.$$

Coincides with (12). The theorem is proved.

Proof of Theorem 3

Solution G for Eq. (12) in the form of

$$\mathbf{G} = \mathrm{diag}\{\mathbf{x}\} + \lambda\left(\lambda^2\sigma^2 - 1\right)\mathbf{G}_1. \tag{31}$$

Substituting this into (12), we get the matrix equation $\mathbf{G}_1$ as follows:

$$\mathbf{G}_1\mathbf{Q} + \mathbf{Q}^T\mathbf{G}_1 = -\mathbf{v}^T\mathbf{v}. \tag{32}$$

As already noted above, the equation refers to the class of matrix equations

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}.$$

In the theorem in [11], the only solution to this equation is the matrix $\mathbf{X}$ type

$$\mathbf{X} = -\int_0^{\infty} e^{\mathbf{A}t}\mathbf{C}e^{\mathbf{B}t}dt.$$

Therefore, solution $\mathbf{G}_1$ of Eq. (32) can be written as

$$\mathbf{G}_1 = \int_0^{\infty} e^{\mathbf{Q}^T t}\mathbf{v}^T\mathbf{v}e^{\mathbf{Q}t}dt.$$

Due to the resulting equality and Eqs. (13) and (31) can be written as

$$\kappa_2 = \mathbf{e}^T\mathbf{Ge} = \mathbf{e}^T\left[\mathrm{diag}\{\mathbf{x}\} + \lambda\left(\lambda^2\sigma^2 - 1\right)\mathbf{G}_1\right]\mathbf{e} = \mathbf{e}^T \mathrm{diag}\{\mathbf{x}\}\mathbf{e} + \lambda\left(\lambda^2\sigma^2 - 1\right)\mathbf{e}^T\mathbf{G}_1\mathbf{e}$$
$$= \mathbf{xe} + \lambda\left(\lambda^2\sigma^2 - 1\right)\mathbf{e}^T\int_0^{\infty} e^{\mathbf{Q}^T t}\mathbf{v}^T\mathbf{v}e^{\mathbf{Q}t}dt\mathbf{e}.$$

Hence, taking into account (8), we get

$$\kappa_2 = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\int_0^{\infty}\left(\mathbf{e}^T e^{\mathbf{Q}^T t}\mathbf{v}^T\right)\left(\mathbf{v}e^{\mathbf{Q}t}\mathbf{e}\right)dt = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\int_0^{\infty}\left(\mathbf{v}e^{\mathbf{Q}t}\mathbf{e}\right)^T\left(\mathbf{v}e^{\mathbf{Q}t}\mathbf{e}\right)dt$$
$$= \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\int_0^{\infty}\left(\mathbf{v}e^{\mathbf{Q}t}\mathbf{e}\right)^2 dt = \lambda b + \lambda\left(\lambda^2\sigma^2 - 1\right)\beta,$$

where parameter β is defined by equality (14). The theorem is proved.

REFERENCES

1. Bocharov, P.P. and Pechinkin, A.V., *Teoriya massovogo obsluzhivaniya: Uchebnik* (Queueing Theory: Textbook), Moscow: RUDN, 1995.
2. Whitt, W., On the heavy-traffic limit theorem for GI/G/∞ queues, *J. Adv. Appl. Prob.,* 1982, vol. 14, pp. 171–190.
3. Kendall, D.G., Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chains, *Ann. Math. Stat.,* 1953, vol. 24, no. 3, pp. 338–354.
4. Moiseev, A. and Nazarov, A., Investigation of high intensive general flow, *Proc. IV Int. Conf. Problems of Cybernetics and Informatics (PCI'2012),* Baku, 2012, pp. 161–163.
5. Klimenok, V.I. and Shumchenya, V.S., Stationary characteristics of a queuing system with a backup device, in *Teoriya veroyatnostei, sluchainye protsessy, matematicheskaya statistika i prilozheniya: materialy Mezhdunar. nauch. konf.* (Proc. Int. Sci. Conf. Probability Theory, Stochastic Processes, Mathematical Statistics, and Applications), Minsk, 2015, pp. 82–87.
6. Nazarov, A.A. and Moiseeva, S.P., *Metod asimptoticheskogo analiza v teorii massovogo obsluzhivaniya* (Asymptotic Analysis of Queuing Theory), Tomsk: NTL, 2006.
7. Gnedenko, B.V. and Kovalenko, I.N., *Vvedenie v teoriyu massovogo obsluzhivaniya* (Introduction to Queuing Theory), Moscow: LKI, 2007, ed. 4.
8. Nazarov, A. and Moiseev, A., The asymptotic analysis of the system GI|GI|∞ by methods of dynamic screening and separation of the first jump, *Proc. of Int. Conf. Distributed Computer and Communication Networks: Control, Computation, Communications (DCCN-2013),* Moscow, 2013, pp. 410–417.
9. Moiseev, A.N. and Nazarov, A.A., Research of the queuing system HIGH|GI|∞, *Vestn. Tomsk. Gos. Univ., Upr., Vychisl. Tekh. Inf.,* 2013, no. 2 (23), pp. 75–83.
10. Korolyuk, V.S., *Stokhasticheskie modeli sistem* (Stochastic Models of Systems), Kiev: Nauk. dumka, 1989.
11. Bellman, R., *Introduction to Matrix Analysis (Classics in Applied Mathematics),* Society for Industrial and Applied Mathematics, 1987, ed. 2.
12. Moiseev, A., The first jump separation technique for the tandem queueing system $GI/(GI/\infty)^K$, in *Information Technologies and Mathematical Modelling,* Dudin, A., Ed., Springer, 2014, pp. 287–300.
13. Nazarov, A. and Moiseev, A., *Investigation of the Queueing Network $GI\text{-}(GI|\infty)^K$ by Means of the First Jump Equation and Asymptotic Analysis,* Vishnevsky, V., et al., Eds., 2013, pp. 229–240.
14. Moiseev, A. and Nazarov, A., Asymptotic analysis of the infinite-server queueing system with high-rate semi-Markov arrivals, *Proc. of the IEEE International Congress on Ultra Modern Tele-Communications and Control Systems (ICUMT 2014),*St. Petersburg, 2014, pp. 507–513.
15. Moiseev, A.N. and Nazarov, A.A., Asymptotic analysis of a multistage queuing system with a high-rate renewal arrival process, *Optoelectronics, Instrumentation and Data Processing,* 2014, vol. 50, no. 2, pp. 163–171.
16. Nazarov, A.A. and Moiseev, A.N., Analysis of an open non-Markovian $GI-(GI|\infty)^K$ queueing network with high-rate renewal arrival process, *Problems of Information Transmission,* 2013, vol. 49, no. 2, pp. 167–178.
17. Nazarov, A.A. and Moiseev, A.N., Asymptotic analysis of the open non-Markov queuing network $HIPP\text{-}(GI|\infty)^K$, *Materialy mezhdunar. nauch. konf. "Sovremennye veroyatnostnye metody analiza, proektirovaniya i optimizatsii informatsionno-telekommuni-katsionnykh setei"* (Proc. Int. Sci. Conf. Modern Probabilistic Methods of Analysis, Design, and Optimization of Information and Telecommunication Networks), Minsk, 2013, pp. 132–140.

Translated by M. Kromin